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Polynomial Dynamic Output-feedback Controllers for Positive Polynomial Fuzzy Systems with Time Delay

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Abstract: This paper investigates the stability and stabilization problems of positive polynomial fuzzy uncertain interval models with time delay. Using the polynomial output feedback fuzzy control strategy, the design problem is firstly studied by considering the unmeasured states and measurable premise variables. Then, the control design problem is extended for both the unmeasurable states and unmeasurable premise variables. To study the considered analysis and design problems, a line-integral polynomial fuzzy Lyapunov function with polynomial terms depending of the estimated states is proposed. For each case, the proposed design conditions of the polynomial dynamic output feedback fuzzy controllers guarantying both the stability and the positivity of the resulting closed loop systems, are solved using the Sum of Squares (SOS) approach with tacking into account the positivity of the error signals. Finally, simulation examples are given to show the effectiveness of the proposed approaches.

Keywords: Dynamic output-feedback controllers, line-integral polynomial fuzzy Lyapunov function, positive polynomial fuzzy model, Sum of Squares, uncertain interval systems.

1. INTRODUCTION

Positive Systems (PS) have been extensively studied by a large number of authors [1–3]. In comparison with general dynamical systems, the special characteristic of PS is the positivity of their states variables [1,2]. Much attention has been paid to the development of new analysis and design approaches of positive systems [4–6].

For the stability issue of Positive Linear Systems (PLS) with time delay, several works have been proposed in the literature. Thus, in [7–10], stability analysis has been investigated using the LMI approach. By contrast, in [11,12], the authors have considered Linear Programming (LP) based methods to study stability and control design for both continuous and discrete PLS.

However, many real-world positive systems such as biological systems, electrical circuits, etc, are nonlinear in nature, so that the above-mentioned results are no longer suitable. Thereby, it is essential to investigate the positive nonlinear systems.

Thanks to the Takagi-Sugeno (T-S) fuzzy model [13], a large class of nonlinear systems could be represented by a set of local linear models that are smoothly connected by membership functions [14,15]. In the past few decades, many researchers paid close attention to the fuzzy systems and a great deal of results have been obtained [5,16–19]. Recent developments on polynomial system theory, especially the Sum of Squares (SOS) theory [20] have provided a feasible solution to ensure the global nonnegativity for polynomial functions. Sufficient conditions for the positivity of a polynomial system are formulated as a sum of squared polynomials. The existence of an SOS decomposition is equivalent to a Semi-Definite Programming (SDP) feasible problem. Many toolboxes such as SOSTOOLS [21], have been fully developed to solve the problem.

Relatively speaking, the polynomial fuzzy model is more effective to represent a larger class of nonlinear systems than the T-S fuzzy model [22–28]. However, due to the existence of polynomials, it becomes more complicated to analyze the positivity and stability of positive nonlinear systems. Thus, less attention has been paid to positive polynomial fuzzy-model based control design problems. We can only find little literatures [29–33] studying the problems of the stability and positivity of polynomial fuzzy-model-based control systems with time delay using the LP approach. From another perspective, in the literature aforementioned, it is assumed that system parameters are exactly known and all state variables are available for measurements. However, in practical applications, it is inevitable that uncertainties could affect the system parameters due to some unpredictable factors, e.g.,

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limitation in data acquisition [34], measurement errors, stochastic disturbances from the environments [35], and the individual variability of plants [36]. In fact, the synthesis problems for uncertain positive polynomial fuzzy systems have not been investigated, especially for the output feedback case. This forms the motivation of our study.

In this paper, we investigate the control design problem of Positive Polynomial Fuzzy Models (PPFM) with interval uncertainties and time delay by considering the polynomial fuzzy dynamic output-feedback controllers. Being different from widely used algebraic techniques such as monomial transformation and decomposition, the continuous-time case is treated in a unified polynomial matrix inequality framework. The analysis and design problems are firstly studied by considering the measurable premise variables and then extended to unmeasurable cases. To achieve less conservative convex stability conditions in these two cases, state matrices and input matrices are assumed polynomial matrices and not constant matrices and line-integral polynomial Lyapunov functions, in which the polynomial Lyapunov matrix dependent on estimated states, are employed. Sufficient conditions for the existence of a positive polynomial dynamic output-feedback controllers are established, and the desired polynomial controller matrices can be constructed easily through the solutions of SOS conditions by considering interval uncertainties. Moreover, it is revealed that an unstable positive polynomial fuzzy system cannot be positively stabilized by a certain dynamic output-feedback controller without taking the positivity of the error signals into account. When the positivity of the error signals is considered, an SOS-based synthesis approach is provided to design the stabilizing controllers. Unlike other conditions which may require structural decomposition of positive polynomial matrices, all conditions proposed in this paper are expressed in terms of the system matrices, and can be verified easily. Finally, three design examples are given to illustrate the effectiveness of the different proposed approaches.

2. PRELIMINARIES

The following preliminaries will be used in the derivation of the main results.

Lemma 1 [37]**:** A system with time delay is guaranteed to be positive if the system and the time delay matrices satisfy the conditions that $A(x(t))$ is a Metzler matrix (it means if its off diagonal elements are all non-negative $A(x(t)) = (a_{ml}(x(t)))$, $m, l = \{1, 2, ..., n\}$, $A(x(t))$ is Metzler if $a_{ml}(x(t)) \geq 0$ for all $m \neq l$) and $A_\tau(x(t)) \geq 0$ when $u(t) = 0$.

Lemma 2 [38]**:** For matrices $A, B \in \mathbb{R}^{n\times n}$, if A and B are Metzler and $A \geq B$, then $\mu(A) \geq \mu(B)$. Where $\mu(A)$ is the spectral abscissa of matrix A (max of the real parts of the eigenvalues of A).

Lemma 3 [39]**:** M is a Metzler matrix if and only if there exists a positive scalar α such that: $M + \alpha I > 0$.

3. POSITIVE DYNAMIC OUTPUT-FEEDBACK CONTROLLER WITH MEASURABLE PREMISE VARIABLES

Consider a nonlinear positive uncertain interval system with time delay represented by the following delayed PPFM with r plant rules.

Rule i $(i = 1, 2, ..., r)$**:** If $\theta_1(x(t))$ is μ_{i1} and ... and $\theta_p(x(t))$ is μ_{ip}, then

$$\begin{cases} \dot{x}(t) = A_i(x(t))x(t) + A_{\tau_i}(x(t))x(t-\tau) \\ \qquad\quad + B_i(x(t))u(t), \\ y(t) = C_i x(t), \\ x(t) = \psi(t),\ t \in [-\overline{\tau},\ 0], \end{cases} \tag{1}$$

where $\theta_j(x(t))$ $(j = 1, \cdots, p)$ are the premise variables. μ_{ij} $(i = 1, \cdots, r,\ j = 1, \cdots, p)$ are the fuzzy sets, r denotes the number of Model Rules; $\psi(t)$ is a continuous vector-valued initial function on $[-\overline{\tau}\ \ 0]$; where $x(t) \in \mathbb{R}^{n_x}$ is the state vector, $y(t) \in \mathbb{R}^{n_y}$ is the output vector, $A_i(x(t)) \in \mathbb{R}^{n_x\times n_x}$, $A_{\tau_i}(x(t)) \in \mathbb{R}^{n_x\times n_x}$ and $B_i(x(t)) \in \mathbb{R}^{n_x\times n_u}$ are unknown polynomial matrices with known bounds and $C_i \in \mathbb{R}^{n_y\times n_x}$ are unknown constant matrices with known bounds, fulfilling $A_i(x) \in [\underline{A}_i(x), \overline{A}_i(x)]$, $A_{\tau_i}(x) \in [\underline{A}_{\tau_i}(x), \overline{A}_{\tau_i}(x)]$, $B_i(x) \in [\underline{B}_i(x), \overline{B}_i(x)]$, $C_i \in [\underline{C}_i, \overline{C}_i]$, with $\underline{A}_i(x)$, $\overline{A}_i(x)$, $\underline{A}_{\tau_i}(x)$, $\overline{A}_{\tau_i}(x)$, $\underline{B}_i(x)$, $\overline{B}_i(x)$, $\underline{C}_i$ and $\overline{C}_i$ are given. Moreover, $\underline{A}_i(x)$ is a Metzler matrix, $\underline{A}_{\tau_i}(x)$, $\underline{B}_i(x)$ and $\underline{C}_i$ are non negative matrices.

After defuzzificating model (1), the global uncertain interval system dynamics is given by the following equation

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{r} h_i(x(t))\{A_i(x(t))x(t) + A_{\tau_i}(x(t))x(t-\tau) \\ \qquad\quad + B_i(x(t))u(t)\}, \\ y(t) = \sum_{i=1}^{r} h_i(x(t))C_i x(t), \\ x(t) = \psi(t),\ t \in [-\overline{\tau},\ 0], \end{cases} \tag{2}$$

where $h_i(x(t)) = \frac{\nu_i(x(t))}{\sum_{i=1}^{r}\nu_i(x(t))}$, $\nu_i(x(t)) = \prod_{j=1}^{p}\mu_{ij}(\theta_j(x(t)))$ is the membership function.
It is obvious that fuzzy weighting functions $h_i(x(t))$ satisfy

$$\begin{cases} \sum_{i=1}^{r} h_i(x(t)) = 1, \\ 0 \leq h_i(x(t)) \leq 1. \end{cases} \tag{3}$$

In the sequel, for brevity time t is dropped. $x(t)$, $h_i(x(t))$ and $x(t-\tau)$ are respectively denoted by x, h_i and x_τ.
Usually, the output vector can be measured by sensors in

many real systems, we assume $C_i = C \; \forall i = 1, ..., r$, so the polynomial fuzzy uncertain interval model can be rewritten as

$$\begin{cases} \dot{x} = \sum_{i=1}^{r} h_i\{A_i(x)x + A_{\tau_i}(x)x_\tau + B_i(x)u\}, \\ y = Cx, \\ x(t) = \psi(t), t \in [-\overline{\tau}, 0]. \end{cases} \tag{4}$$

Let's now use the following polynomial dynamic output-feedback controller

$$\begin{cases} \dot{\hat{x}} = \sum_{i=1}^{r} h_i\{G_i(\hat{x})\hat{x} + G_{\tau_i}(\hat{x})\hat{x}_\tau + L_i(\hat{x})y\}, \\ u = \sum_{i=1}^{r} h_i K_i(\hat{x})\hat{x}, \end{cases} \tag{5}$$

where $G_i(\hat{x}) \in \mathbb{R}^{n_x \times n_x}$, $G_{\tau_i}(\hat{x}) \in \mathbb{R}^{n_x \times n_x}$, $L_i(\hat{x}) \in \mathbb{R}^{n_x \times n_y}$ and $K_i(\hat{x}) \in \mathbb{R}^{n_u \times n_x}$ are the polynomial controller matrices to be determined.

To guarantee the positivity of the system, the key lies in the nonnegativity of the error signal, which is defined by

$$e = x - \hat{x}, \tag{6}$$

and if we choose $\tilde{x} = \begin{bmatrix} x^T & e^T \end{bmatrix}^T$ as the new augmented state vector, then the closed loop augmented polynomial fuzzy system is given as

$$\dot{\tilde{x}} = \sum_{i=1}^{r}\sum_{j=1}^{r} h_i h_j \{A_{x_{ij}}(x,\hat{x})\tilde{x} + A_{H_{ij}}(x,\hat{x})\tilde{x}_\tau\}, \tag{7}$$

with

$$A_{x_{ij}}(x,\hat{x}) = \begin{bmatrix} A^{11}_{x_{ij}}(x,\hat{x}) & -B_i(x)K_j(\hat{x}) \\ A^{21}_{x_{ij}}(x,\hat{x}) & A^{22}_{x_{ij}}(x,\hat{x}) \end{bmatrix}, \tag{8}$$

$$A^{11}_{x_{ij}}(x,\hat{x}) = A_i(x) + B_i(x)K_j(\hat{x}),$$
$$A^{21}_{x_{ij}}(x,\hat{x}) = A_i(x) - L_j(\hat{x})C + B_i(x)K_j(\hat{x}) - G_j(\hat{x}),$$
$$A^{22}_{x_{ij}}(x,\hat{x}) = G_j(\hat{x}) - B_i(x)K_j(\hat{x}),$$

$$A_{H_{ij}}(x,\hat{x}) = \begin{bmatrix} A_{\tau_i}(x) & 0 \\ A_{\tau_i}(x) - G_{\tau_j}(\hat{x}) & G_{\tau_j}(\hat{x}) \end{bmatrix}. \tag{9}$$

The problem to be solved is to find $G_j(\hat{x})$, $G_{\tau_j}(\hat{x})$, $L_j(\hat{x})$ and $K_j(\hat{x})$ such that augmented system (7) is positive and asymptotically stable.

Theorem 1: Consider polynomial fuzzy model (4) with dynamic output-feedback controller (5). If there exist Metzler matrix $G_j(\hat{x})$, polynomial matrices $G_{\tau_j}(\hat{x}) \geq 0$, $L_j(\hat{x}) \geq 0$ and $K_j(\hat{x}) \leq 0$, such that the following SOS-based conditions are satisfied, $\forall i,\ j = 1, ..., r$

$$-v_1^T(trace(\underline{A}_i(x) + G_j(\hat{x}) + (\overline{B}_i(x) - \underline{B}_i(x))K_j(\hat{x})) + \epsilon_1(x,\hat{x})I)v_1 \text{ is SOS}, \tag{10}$$

$$v_2^T([\underline{A}_i(x) + \overline{B}_i(x)K_j(\hat{x})]_{lm} - \epsilon_2(x,\hat{x})I)v_2 \text{ is SOS when } l \neq m, \tag{11}$$

$$v_3^T([G_j(\hat{x}) - \overline{B}_i(x)K_j(\hat{x})]_{lm} - \epsilon_3(x,\hat{x})I)v_3 \text{ is SOS when } l \neq m, \tag{12}$$

$$v_4^T(\underline{A}_i(x) - L_j(\hat{x})\underline{C} + \underline{B}_i(x)K_j(\hat{x}) - G_j(\hat{x}) - \epsilon_4(x,\hat{x})I)v_4 \text{ is SOS}, \tag{13}$$

$$v_5^T(\overline{A}_{\tau_i}(x) - G_{\tau_j}(\hat{x}) - \epsilon_5(x,\hat{x})I)v_5 \text{ is SOS}. \tag{14}$$

Then closed-loop polynomial fuzzy system (7) is positive and asymptotically stable

Proof: If the dynamic controller exists, then from (8), we have that $\forall i,\ j = 1, ..., r$

$$\mu\left(\begin{bmatrix} A^{11}_{x_{ij}}(x,\hat{x}) & -B_i(x)K_j(\hat{x}) \\ A^{21}_{x_{ij}}(x,\hat{x}) & A^{22}_{x_{ij}}(x,\hat{x}) \end{bmatrix}\right) < 0, \tag{15}$$

and

$$\begin{bmatrix} A^{11}_{x_{ij}}(x,\hat{x}) & -B_i(x)K_j(\hat{x}) \\ A^{21}_{x_{ij}}(x,\hat{x}) & A^{22}_{x_{ij}}(x,\hat{x}) \end{bmatrix} \text{ is Metzler.} \tag{16}$$

As we have

$$\begin{bmatrix} \underline{A}^{11}_{x_{ij}}(x,\hat{x}) & -\underline{B}_i(x)K_j(\hat{x}) \\ \underline{A}^{21}_{x_{ij}}(x,\hat{x}) & \underline{A}^{22}_{x_{ij}}(x,\hat{x}) \end{bmatrix} \leq \begin{bmatrix} A^{11}_{x_{ij}}(x,\hat{x}) & -B_i(x)K_j(\hat{x}) \\ A^{21}_{x_{ij}}(x,\hat{x}) & A^{22}_{x_{ij}}(x,\hat{x}) \end{bmatrix} \leq \begin{bmatrix} \overline{A}^{11}_{x_{ij}}(x,\hat{x}) & -\overline{B}_i(x)K_j(\hat{x}) \\ \overline{A}^{21}_{x_{ij}}(x,\hat{x}) & \overline{A}^{22}_{x_{ij}}(x,\hat{x}) \end{bmatrix},$$

where

$$\underline{A}^{11}_{x_{ij}}(x,\hat{x}) = \underline{A}_i(x) + \overline{B}_i(x)K_j(\hat{x}),$$
$$\underline{A}^{21}_{x_{ij}}(x,\hat{x}) = \underline{A}_i(x) - L_j(\hat{x})\overline{C} + \overline{B}_i(x)K_j(\hat{x}) - G_i(\hat{x}),$$
$$\underline{A}^{22}_{x_{ij}}(x,\hat{x}) = G_j(\hat{x}) - \underline{B}_i(x)K_j(\hat{x}),$$
$$\overline{A}^{11}_{x_{ij}}(x,\hat{x}) = \overline{A}_i(x) + \underline{B}_i(x)K_j(\hat{x}),$$
$$\overline{A}^{21}_{x_{ij}}(x,\hat{x}) = \overline{A}_i(x) - L_j(\hat{x})\underline{C} + \underline{B}_i(x)K_j(\hat{x}) - G_i(\hat{x}),$$
$$\overline{A}^{22}_{x_{ij}}(x,\hat{x}) = G_j(\hat{x}) - \overline{B}_i(x)K_j(\hat{x}),$$

then we deduce that

$$\mu\left(\begin{bmatrix} \underline{A}^{11}_{x_{ij}}(x,\hat{x}) & -\underline{B}_i(x)K_j(\hat{x}) \\ \underline{A}^{21}_{x_{ij}}(x,\hat{x}) & \underline{A}^{22}_{x_{ij}}(x,\hat{x}) \end{bmatrix}\right) < 0, \tag{17}$$

and

$$\begin{bmatrix} \overline{A}^{11}_{x_{ij}}(x,\hat{x}) & -\overline{B}_i(x)K_j(\hat{x}) \\ \overline{A}^{21}_{x_{ij}}(x,\hat{x}) & \overline{A}^{22}_{x_{ij}}(x,\hat{x}) \end{bmatrix} \text{ is Metzler.} \tag{18}$$

It follows from (17) that

$$trace\left(\begin{bmatrix} \underline{A}^{11}_{x_{ij}}(x,\hat{x}) & -\underline{B}_i(x)K_j(\hat{x}) \\ \underline{A}^{21}_{x_{ij}}(x,\hat{x}) & \underline{A}^{22}_{x_{ij}}(x,\hat{x}) \end{bmatrix}\right) < 0, \tag{19}$$

which is equivalent to (10). Moreover, it is obvious that (18) is equivalent to (11)-(13). The fact that $A_{H_{ij}}(x,\hat{x}) \geq 0$ implies (14), which completes the proof. □

Thus, we can now study sufficient conditions in the following Theorem to guarantee the asymptotic stability and the positivity of the augmented system (7).

Theorem 2: There exists a polynomial dynamic controller (5) for polynomial fuzzy uncertain interval system (4) that provides stability and positivity to augmented system (7) if there exist a positive scalar γ, symmetric polynomial matrices $P(\hat{x}) = diag[P_1(\hat{x}), P_2] > 0$ and $S(\hat{x}) = diag[S_1(\hat{x}), S_2(\hat{x})] > 0$, a Metzler matrix $G_j(\hat{x})$ and polynomial matrices $G_{\tau_j}(\hat{x}) \geq 0$, $L_j(\hat{x}) \geq 0$ and $K_j(\hat{x}) \leq 0$ ($j = 1, ..., r$), such that the following SOS conditions are satisfied, $\forall i,\ j = 1, ..., r$

$$v_1^T(P_1(\hat{x}) - \epsilon_1(\hat{x})I)v_1 \text{ is SOS,} \tag{20}$$

$$v_2^T(P_2 - \epsilon_2 I)v_2 \text{ is SOS,} \tag{21}$$

$$v_3^T(S_1(\hat{x}) - \epsilon_3(\hat{x})I)v_3 \text{ is SOS,} \tag{22}$$

$$v_4^T(S_2(\hat{x}) - \epsilon_4(\hat{x})I)v_4 \text{ is SOS,} \tag{23}$$

$$v_5^T([\underline{A}_i(x) + \overline{B}_i(x)K_j(\hat{x})]_{lm} - \epsilon_5(x,\hat{x})I)v_5 \text{ is SOS, for } l \neq m = 1,\ ...,\ n, \tag{24}$$

$$v_6^T([G_j(\hat{x}) - \underline{B}_i(x)K_j(\hat{x})]_{lm} - \epsilon_6(x,\hat{x})I)v_6 \text{ is SOS, for } l \neq m = 1,\ ...,\ n, \tag{25}$$

$$v_7^T(\underline{A}_i(x) - L_j(\hat{x})\overline{C} + \overline{B}_i(x)K_j(\hat{x}) - G_j(\hat{x}) - \epsilon_7(x,\hat{x})I)v_7 \text{ is SOS,} \tag{26}$$

$$v_8^T(\underline{A}_{\tau_i}(x) - G_{\tau_j}(\hat{x}) - \epsilon_8(x,\hat{x})I)v_8 \text{ is SOS,} \tag{27}$$

$$-v_9^T(\Psi_{ij}(x,\hat{x}) + \Psi_{ji}(x,\hat{x}) + \epsilon_9(x,\hat{x})I)v_9 \text{ is SOS, } \forall i \leq j = 1,\ ...,\ r, \tag{28}$$

where

$$\Psi_{ij}(x,\hat{x}) = \begin{bmatrix} \Phi_{ij}^1(x,\hat{x}) & \Phi_{ij}^2(x,\hat{x}) & \Phi_{ij}^3(x,\hat{x}) & \gamma\mathcal{B}_i(x) \\ * & -S(\hat{x}) & 0 & 0 \\ * & * & -I & 0 \\ * & * & * & -I \end{bmatrix},$$

$$\begin{aligned} \Phi_{ij}^1(x,\hat{x}) &= \mathcal{A}_i^T(x)P(\hat{x}) + P(\hat{x})\mathcal{A}_i(x) \\ &\quad - \gamma\mathcal{B}_i(x)\mathcal{B}_i^T(x)P(\hat{x}) \\ &\quad - \gamma P(\hat{x})\mathcal{B}_i(x)\mathcal{B}_i^T(x) + S(\hat{x}), \end{aligned}$$

$$\Phi_{ij}^2(x,\hat{x}) = A_{H_{ij}}^T(x,\hat{x})P(\hat{x}),$$

$$\Phi_{ij}^3(x,\hat{x}) = P(\hat{x})\mathcal{B}_i(x) + \mathcal{C}^T\mathcal{K}_j^T(\hat{x}),$$

$$\mathcal{A}_i(x) = \begin{bmatrix} \overline{A}_i(x) & 0 \\ \underline{A}_i(x) & 0 \end{bmatrix},$$

$$\mathcal{B}_i(x) = \begin{bmatrix} 0 & \underline{B}_i(x) - \overline{B}_i(x) & 0 & \underline{B}_i(x) \\ 0 & \underline{B}_i(x) - \overline{B}_i(x) & -I & \underline{B}_i(x) \end{bmatrix},$$

$$\mathcal{K}_j(\hat{x}) = \begin{bmatrix} G_j(\hat{x}) & L_j(\hat{x}) & 0 & 0 \\ K_j(\hat{x}) & 0 & 0 & 0 \\ 0 & 0 & G_j(\hat{x}) & L_j(\hat{x}) \\ 0 & 0 & K_j(\hat{x}) & 0 \end{bmatrix},\ \mathcal{C} = \begin{bmatrix} 0 & I \\ 0 & 0 \\ I & -I \\ \underline{C} & 0 \end{bmatrix}.$$

v_i ($i = 1, ..., 9$) denotes vectors that are independent of x and $\hat{x}$. $\epsilon_i(x,\hat{x})$ is a slack variable to keep the positivity of the SOS condition.

Proof: It follows from (24) that $\forall i,\ j = 1, ..., r$ $\underline{A}_i(x) + \overline{B}_i(x)K_j(\hat{x})$ is Metzler.

Combining with $K_j(\hat{x}) \leq 0$ yields, for any $A_i(x) \in [\underline{A}_i(x),\ \overline{A}_i(x)]$, $B_i(x) \in [\underline{B}_i(x),\ \overline{B}_i(x)]$ and $C \in [\underline{C},\ \overline{C}]$, we get

$$A_i(x) + B_i(x)K_j(\hat{x}) \geq \underline{A}_i(x) + \overline{B}_i(x)K_j(\hat{x}) \text{ is Metzler,} \tag{29}$$

and

$$A_{x_{ij}}^{21}(x,\hat{x}) \geq \underline{A}_i(x) - L_j(\hat{x})\overline{C} + \overline{B}_i(x)K_j(\hat{x}) - G_j(\hat{x}) \geq 0. \tag{30}$$

In addition, from $G_j(\hat{x})$ being Metzler and $K_j(\hat{x}) \leq 0$, we obtain that, for any $B_i(x) \in [\underline{B}_i(x), \overline{B}_i(x)]$

$$-B_i(x)K_j(\hat{x}) \geq 0, \tag{31}$$

$$G_j(\hat{x}) - B_i(x)K_j(\hat{x}) \text{ is Metzler,} \tag{32}$$

therefore from (29)-(32) and the fact that $A_{H_{ij}}(x,\hat{x}) \geq 0$, we derive that, for any $A_i(x) \in [\underline{A}_i(x), \overline{A}_i(x)]$, $A_{\tau_i}(x) \in [\underline{A}_{\tau_i}(x), \overline{A}_{\tau_i}(x)]$, $B_i(x) \in [\underline{B}_i(x),\ \overline{B}_i(x)]$ and $C \in [\underline{C},\ \overline{C}]$, augmented system (7) is positive.

The proposed line-integral polynomial Lyapunov function has the following form

$$V(\tilde{x}) = V_1(x,\hat{x}) + V_2(e) + V_3(\tilde{x}), \tag{33}$$

where V_i ($i = 1, ..., 3$) are Lyapunov functions that are described as follows:

$$V_1(\hat{x},\hat{x}_\tau) = 2\int_\Gamma g(\psi).d\psi, \tag{34}$$

$$V_2(e_x) = e^T P_2 e, \tag{35}$$

$$V_3(\tilde{x}) = \int_{t-\tau}^{t} \tilde{x}^T(\alpha)S(\tilde{x})\tilde{x}(\alpha)d\alpha, \tag{36}$$

where Γ is a path from the origin to $\widehat{x}$, $\psi \in \mathbb{R}^n$ is the dummy vector, $d\psi \in \mathbb{R}^n$ is an infinitesimal displacement vector, $S(\widetilde{x})$ is a symmetric positive definite matrix to be determined, $g(\psi) \in \mathbb{R}^n$ is a vector function and has the following form

$$g(x,\widehat{x}) = P_1(\widehat{x})x, \tag{37}$$

with $P_1(\widehat{x}) \in \mathbb{R}^{n\times n}$ being a symmetric positive definite matrix such that

$$P_1(\widehat{x}) = \begin{bmatrix} d_{11}(\widehat{x}_1) & p_{12}(\widehat{x}) & \cdots & p_{1n}(\widehat{x}) \\ p_{12}(\widehat{x}) & d_{22}(\widehat{x}_2) & \cdots & p_{2n}(\widehat{x}) \\ \vdots & \vdots & \ddots & \vdots \\ p_{1n}(\widehat{x}) & p_{2n}(\widehat{x}) & \cdots & d_{nn}(\widehat{x}_n) \end{bmatrix},$$

where $p_{kl}(\widehat{x})$ is written as

$$p_{kl}(\widehat{x}) = \sum_{p=1}^{q} D_p \widehat{x}_k^p \widehat{x}_l^p + p_{kl}(\bar{x}), \tag{38}$$

such that $\bar{x} \subseteq \hat{x}$ and $\{\hat{x}_k, \hat{x}_l\} \cap \bar{x} = \emptyset$.
Moreover, considering Lemma 4 of [26], it can be easily proved that $V_1(\hat{x})$ is a Lyapunov function. The time derivative of Lyapunov function (33), gives

$$\begin{aligned}\dot{V}(\widetilde{x}) &= 2x^T P_1(\hat{x})\dot{x} + 2e^T P_2 \dot{e} + \widetilde{x}^T S(\hat{x})\widetilde{x} \\ &\quad - \widetilde{x}(t-\tau)^T S(\hat{x})\widetilde{x}(t-\tau) \\ &= 2\tilde{x}^T P(\hat{x})\dot{\tilde{x}} + \tilde{x}^T S(\hat{x})\widetilde{x} - \widetilde{x}(t-\tau)^T S(\hat{x})\widetilde{x}(t-\tau),\end{aligned} \tag{39}$$

where

$$P(\hat{x}) = \begin{bmatrix} P_1(\hat{x}) & 0 \\ 0 & P_2 \end{bmatrix},\ S(\hat{x}) = \begin{bmatrix} S_1(\hat{x}) & 0 \\ 0 & S_2(\hat{x}) \end{bmatrix}. \tag{40}$$

We have

$$\begin{aligned}\dot{V}(\widetilde{x}) &= 2\tilde{x}^T P(\hat{x})\dot{\tilde{x}} + \tilde{x}^T S(\hat{x})\widetilde{x} - \widetilde{x}(t-\tau)^T S(\hat{x})\widetilde{x}(t-\tau) \\ &= 2\sum_{i=1}^{r}\sum_{j=1}^{r} h_i h_j \tilde{x}^T P(\hat{x})\{A_{x_{ij}}(x,\hat{x})\tilde{x} + A_{H_{ij}}(x,\hat{x})\tilde{x}_\tau\} \\ &\quad + \tilde{x}^T S(\hat{x})\widetilde{x} - \widetilde{x}(t-\tau)^T S(\hat{x})\widetilde{x}(t-\tau).\end{aligned} \tag{41}$$

So

$$\dot{V}(\widetilde{x}) = \sum_{i=1}^{r}\sum_{i\le j}^{r} h_i h_j \tilde{\eta}^T(\Theta_{ij}(x,\hat{x}) + \Theta_{ji}(x,\hat{x}))\tilde{\eta}, \tag{42}$$

where $\widetilde{\eta}^T = [\tilde{x}^T, \tilde{x}_\tau^T]^T$.

$$\Theta_{ij}(x,\widehat{x}) = \begin{bmatrix} \Theta_{ij}^{11}(x,\widehat{x}) & A_{H_{ij}}^T(x,\hat{x})P(\hat{x}) \\ * & -S(\hat{x}) \end{bmatrix}, \tag{43}$$

$\Theta_{ij}^{11}(x,\widehat{x}) = P(\hat{x})A_{X_{ij}}(x,\hat{x}) + A_{X_{ij}}^T(x,\hat{x})P(\hat{x}) + S(\hat{x})$.
It follows from (28) and using the Schur complement, that:

$$\Lambda_{ij}(x,\hat{x}) + \Lambda_{ji}(x,\hat{x}) < 0, \tag{44}$$

where

$$\Lambda_{ij}(x,\widehat{x}) = \begin{bmatrix} N_{ij}(x,\hat{x}) & A_{H_{ij}}^T(x,\hat{x})P(\hat{x}) \\ * & -S(\hat{x}) \end{bmatrix},$$

and

$$\begin{aligned}N_{ij}(x,\hat{x}) &= \mathcal{A}_i^T(x)P(\hat{x}) + P(\hat{x})\mathcal{A}_i(x) \\ &\quad - \gamma\mathcal{B}_i(x)\mathcal{B}_i^T(x)P(\hat{x}) - \gamma P(\hat{x})\mathcal{B}_i(x)\mathcal{B}_i^T(x) \\ &\quad + \gamma^2\mathcal{B}_i(x)\mathcal{B}_i^T(x) \\ &\quad + (\mathcal{B}_i^T(x)P(\hat{x}) + \mathcal{K}_j(\hat{x})\mathcal{C})^T(\mathcal{B}_i^T(x)P(\hat{x}) \\ &\quad + \mathcal{K}_j(\hat{x})\mathcal{C}) + S(\hat{x}),\end{aligned} \tag{45}$$

we have then

$$\begin{aligned}&\mathcal{A}_i^T(x)P(\hat{x}) + P(\hat{x})\mathcal{A}_i(x) - \gamma\mathcal{B}_i(x)\mathcal{B}_i^T(x)P(\hat{x}) \\ &\quad - \gamma P(\hat{x})\mathcal{B}_i(x)\mathcal{B}_i^T(x) + \gamma^2\mathcal{B}_i(x)\mathcal{B}_i^T(x) \\ &\quad + (\mathcal{B}_i^T(x)P(\hat{x}) + \mathcal{K}_j(\hat{x})\mathcal{C})^T(\mathcal{B}_i^T(x)P(\hat{x}) \\ &\quad + \mathcal{K}_j(\hat{x})\mathcal{C}) + S(\hat{x}) < 0.\end{aligned} \tag{46}$$

Taking into account the following relationship

$$\begin{aligned}&P(\hat{x})\mathcal{B}_i(x)\mathcal{B}_i^T(x)P(\hat{x}) - \gamma\mathcal{B}_i(x)\mathcal{B}_i^T(x)P(\hat{x}) \\ &\quad - \gamma P(\hat{x})\mathcal{B}_i(x)\mathcal{B}_i^T(x) + \gamma^2\mathcal{B}_i(x)\mathcal{B}_i^T(x) \\ &= (P(\hat{x})\mathcal{B}_i(x) - \gamma\mathcal{B}_i(x))(\mathcal{B}_i^T(x)P(\hat{x}) - \gamma\mathcal{B}_i^T(x)) \\ &\ge 0,\end{aligned} \tag{47}$$

we obtain, from (46) the following inequality:

$$\begin{aligned}&\mathcal{A}_i^T(x)P(\hat{x}) + P(\hat{x})\mathcal{A}_i(x) - P(\hat{x})\mathcal{B}_i(x)\mathcal{B}_i^T(x)P(\hat{x}) \\ &\quad + (\mathcal{B}_i^T(x)P(\hat{x}) + \mathcal{K}_j(\hat{x})\mathcal{C})^T(\mathcal{B}_i^T(x)P(\hat{x}) + \mathcal{K}_j(\hat{x})\mathcal{C}) \\ &\quad + S(\hat{x}) < 0.\end{aligned} \tag{48}$$

Rewriting (48) yields to

$$\begin{aligned}&(\mathcal{A}_i(x) + \mathcal{B}_i(x)\mathcal{K}_j(\hat{x})\mathcal{C})^T P(\hat{x}) + P(\hat{x})(\mathcal{A}_i(x) \\ &\quad + \mathcal{B}_i(x)\mathcal{K}_j(\hat{x})\mathcal{C}) + \mathcal{C}^T\mathcal{K}_j^T(\hat{x})\mathcal{K}_j(\hat{x})\mathcal{C} + S(\hat{x}) < 0,\end{aligned} \tag{49}$$

which implies that

$$\begin{aligned}&(\mathcal{A}_i(x) + \mathcal{B}_i(x)\mathcal{K}_j(\hat{x})\mathcal{C})^T P(\hat{x}) + P(\hat{x})(\mathcal{A}_i(x) \\ &\quad + \mathcal{B}_i(x)\mathcal{K}_j(\hat{x})\mathcal{C}) + S(\hat{x}) < 0.\end{aligned} \tag{50}$$

Changing $N_{ij}(x,\hat{x})$ with (48) in (44), we get

$$\Upsilon_{ij}(x,\hat{x}) + \Upsilon_{ji}(x,\hat{x}) < 0, \tag{51}$$

where

$$\Upsilon_{ij}(x,\hat{x}) = \begin{bmatrix} \Upsilon_{ij}^{11}(x,\hat{x}) & A_{H_{ij}}^T(x,\hat{x})P(\hat{x}) \\ * & -S(\hat{x}) \end{bmatrix}. \tag{52}$$

$\Upsilon_{ij}^{11}(x,\hat{x}) = (\mathcal{A}_i(x) + \mathcal{B}_i(x)\mathcal{K}_j(\hat{x})\mathcal{C})^T P(\hat{x}) + P(\hat{x})(\mathcal{A}_i(x) + \mathcal{B}_i(x)\mathcal{K}_j(\hat{x})\mathcal{C}) + S(\hat{x})$.
Some algebraic manipulations lead to

$$\mathcal{A}_i(x) + \mathcal{B}_i(x)\mathcal{K}_j(\hat{x})\mathcal{C} = \begin{bmatrix} \overline{A}_{x_{ij}}^{11}(x,\hat{x}) & -\overline{B}_i(x)K_j(\hat{x}) \\ \overline{A}_{x_{ij}}^{21}(x,\hat{x}) & \overline{A}_{x_{ij}}^{22}(x,\hat{x}) \end{bmatrix}. \tag{53}$$

In addition, it is easy to show that

$$\begin{aligned}&\begin{bmatrix} \overline{A}_{x_{ij}}^{11}(x,\hat{x}) & -\overline{B}_i(x)K_j(\hat{x}) \\ \overline{A}_{x_{ij}}^{21}(x,\hat{x}) & \overline{A}_{x_{ij}}^{22}(x,\hat{x}) \end{bmatrix} \\ &\ge \begin{bmatrix} A_{x_{ij}}^{11}(x,\hat{x}) & -B_i(x)K_j(\hat{x}) \\ A_{x_{ij}}^{21}(x,\hat{x}) & A_{x_{ij}}^{22}(x,\hat{x}) \end{bmatrix}.\end{aligned} \tag{54}$$

Therefore, by combining (51)-(54) we obtain $\dot{V}(\tilde{x}) < 0$. Which means that the resulting closed loop polynomial augmented fuzzy system (7) is positive and asymptotically stable for any $A_i(x) \in [\underline{A}_i(x), \overline{A}_i(x)]$, $A_{\tau_i}(x) \in [\underline{A}_{\tau_i}(x), \overline{A}_{\tau_i}(x)]$, $B_i(x) \in [\underline{B}_i(x), \overline{B}_i(x)]$ and $C \in [\underline{C}, \overline{C}]$. □

Remark 1: It is clear that, under the action of the designed polynomial dynamical controller, state vectors $x(t)$ and $\hat{x}(t)$ will be nonnegative if the initial conditions satisfy $x(0) \geq \hat{x}(0) \geq 0$. One may ask why control law $u(t)$ is not positive even when $x(t)$ and $\hat{x}(t)$ are nonnegative. Define

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\hat{x}}(t) \end{bmatrix} = A_{x_{ij}}(x,\hat{x}) \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix} + A_{h_{ij}}(x,\hat{x}) \begin{bmatrix} x(t-\tau) \\ \hat{x}(t-\tau) \end{bmatrix}, \quad (55)$$

such that $A_{x_{ij}}(x,\hat{x}) = \begin{bmatrix} A_i(x) & B_i(x)K_j(\hat{x}) \\ L_j(\hat{x})C & G_j(\hat{x}) \end{bmatrix}$ and $A_{h_{ij}}(x,\hat{x}) = \begin{bmatrix} A_{\tau_i}(x) & 0 \\ 0 & G_{\tau_j}(\hat{x}) \end{bmatrix}$.

The reason for this is that the invariant set associated with (55) is not the positive orthant but the cone defined by

$$\delta = \left\{ \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix} \geq 0; \; \begin{bmatrix} I & -I \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix} \geq 0 \right\}. \quad (56)$$

If a positive system can be called positive orthant invariant, then system (55) with Metzler $G_j(\hat{x})$, $G_{\tau_j}(\hat{x}) \geq 0$, $L_j(\hat{x}) \geq 0$, and $K_j(\hat{x}) \leq 0$ can be viewed as δ invariant. We can then show that a sufficient condition to this problem is that there exist Metzler $G_j(\hat{x})$, $G_{\tau_j}(\hat{x}) \geq 0$, $L_j(\hat{x}) \geq 0$, and $K_j(\hat{x}) \leq 0$ such that (55) is asymptotically stable and δ invariant. This interpretation may be useful to seek less conservative conditions and even to establish solvable sufficient conditions for the positive stabilization problem with sign-indefinite $K_j(\hat{x})$.

4. POSITIVE DYNAMIC OUTPUT-FEEDBACK CONTROLLER WITH UNMEASURABLE PREMISE VARIABLES

In this section, we consider the dynamic controller design for the following positive time delay polynomial fuzzy system with unmeasurable premise variables

$$\begin{cases} \dot{x} = \sum_{i=1}^{r} h_i(x)\{A_i(x)x + A_{\tau_i}(x)x(t-\tau) + B_i(x)u\}, \\ y = Cx, \\ x(t) = \psi(t), \; t \in [-\bar{\tau}, \; 0]. \end{cases} \quad (57)$$

The dynamic controller structure under consideration has the following form

$$\begin{cases} \dot{\hat{x}} = \sum_{j=1}^{r} h_j(\hat{x})\{G_j(\hat{x})\hat{x} + G_{\tau_j}(\hat{x})\hat{x}_\tau + L_j(\hat{x})y\}, \\ u = \sum_{k=1}^{r} h_k(\hat{x})K_k(\hat{x})\hat{x}, \end{cases} \quad (58)$$

where $G_j(\hat{x}) \in \mathbb{R}^{n_x \times n_x}$, $G_{\tau_j}(\hat{x}) \in \mathbb{R}^{n_x \times n_x}$, $L_j(\hat{x}) \in \mathbb{R}^{n_x \times n_y}$ and $K_k(\hat{x}) \in \mathbb{R}^{n_u \times n_x}$ are the polynomial controller matrices to be determined.

Then, the closed-loop polynomial system is written as follows:

$$\begin{cases} \dot{x} = \sum_{i,k=1}^{r} h_i(x)h_k(\hat{x})\{A_i(x)x + A_{\tau_i}(x)x(t-\tau) \\ \qquad + B_i(x)K_k(\hat{x})\hat{x}\}, \\ y = Cx, \\ x(t) = \psi(t), \; t \in [-\bar{\tau}, \; 0], \\ \hat{x}(t) = \phi(t), \; t \in [-\bar{\tau}, \; 0]. \end{cases} \quad (59)$$

We choose $\tilde{x} = \begin{bmatrix} x^T & e^T \end{bmatrix}^T$ as the new augmented state variable, then the augmented polynomial fuzzy system becomes

$$\dot{\tilde{x}} = \sum_{i,j,k=1}^{r} h_i(x)h_j(\hat{x})h_k(\hat{x})\{A_X^{ijk}(x,\hat{x})\tilde{x} + A_H^{ij}(x,\hat{x})\tilde{x}_\tau\}, \quad (60)$$

where

$$A_X^{ijk}(x,\hat{x}) = \begin{bmatrix} A_{x_{ik}}^{11}(x,\hat{x}) & -B_i(x)K_k(\hat{x}) \\ A_{x_{ijk}}^{21}(x,\hat{x}) & A_{x_{ijk}}^{22}(x,\hat{x}) \end{bmatrix},$$
$$A_{x_{ik}}^{11}(x,\hat{x}) = A_i(x) + B_i(x)K_k(\hat{x}),$$
$$A_{x_{ijk}}^{21}(x,\hat{x}) = A_i(x) + B_i(x)K_k(\hat{x}) - L_j(\hat{x})C - G_j(\hat{x}),$$
$$A_{x_{ijk}}^{22}(x,\hat{x}) = G_j(\hat{x}) - B_i(x)K_k(\hat{x}),$$
$$A_H^{ij}(x,\hat{x}) = \begin{bmatrix} A_{\tau_i}(x) & 0 \\ A_{\tau_i}(x) - G_{\tau_j}(\hat{x}) & G_{\tau_j}(\hat{x}) \end{bmatrix}.$$

Remark 2: Since we consider unmeasurable premise variables $\theta_j(x)$ for the polynomial fuzzy model, the membership functions of the polynomial fuzzy controller $h_j(\hat{x})$ should be allowed to depend on estimated system states $\hat{x}$ rather than on original system states x.

Theorem 3: Consider polynomial fuzzy model (57) with dynamic output-feedback controller (58). If there exist diagonal symmetric positive definite matrices $X_1(\hat{x})$ and X_2, symmetric positive definite matrices $S_1(\hat{x})$ and $S_2(\hat{x})$, polynomial matrices $Y_{1k}(\hat{x})$, $Y_{2k}(\hat{x})$, $W_{1j}(\hat{x})$, $W_{2j}(\hat{x})$, $Z_j(\hat{x})$, $W_{11_j}(\hat{x})$, $W_{12_j}(\hat{x})$ $\forall i, j, k = 1, ..., r$ and a scalar $\alpha > 0$ such that the following SOS-based conditions are satisfied

$$v_1^T(X_1(\hat{x}) - \epsilon_1(\hat{x})I)v_1 \text{ is SOS}, \quad (61)$$
$$v_2^T(X_2 - \epsilon_2 I)v_2 \text{ is SOS}, \quad (62)$$
$$v_3^T(S_1(\hat{x}) - \epsilon_3(\hat{x})I)v_3 \text{ is SOS}, \quad (63)$$
$$v_4^T(S_2(\hat{x}) - \epsilon_4(\hat{x})I)v_4 \text{ is SOS}, \quad (64)$$
$$v_5^T(\Phi_{ijk}(x,\hat{x}) - \epsilon_5(x,\hat{x})I)v_5 \text{ is SOS}, \; \forall i, \; j, \; k = 1, \; ..., \; r, \quad (65)$$
$$-v_6^T(\Omega_{ijk}(x,\hat{x}) + \Omega_{ikj}(x,\hat{x}) + \epsilon_6(x,\hat{x})I)v_6 \text{ is SOS}, \; \text{for } i \neq j, \quad (66)$$

where

$$\Phi_{ijk}(x,\hat{x}) = \begin{bmatrix} L_{ijk}(x,\hat{x}) & M_{ijk}(x,\hat{x}) \\ N_{ijk}(x,\hat{x}) & D_{ijk}(x,\hat{x}) \end{bmatrix},$$

$$\Omega_{ijk}(x,\hat{x}) = \begin{bmatrix} Q_{ijk}(x,\hat{x}) & T_{ijk}(x,\hat{x}) & A_{\tau_i}(x)X_1(\hat{x}) & 0 \\ * & U_{ijk}(x,\hat{x}) & \Omega^1_{ijk}(x,\hat{x}) & W_{12_j}(\hat{x}) \\ * & * & -S_1(\hat{x}) & 0 \\ * & * & * & -S_2(\hat{x}) \end{bmatrix},$$

with

$$\begin{aligned} Q_{ijk}(x,\hat{x}) &= A_i(x)X_1(\hat{x}) + X_1(\hat{x})(A_i(x))^T \\ &\quad + B_i(x)Y_{1k}(\hat{x}) + Y_{1k}^T(\hat{x})(B_i(x))^T + S_1(\hat{x}), \\ T_{ijk}(x,\hat{x}) &= X_1(\hat{x})(A_i(x))^T - B_i(x)Y_{2k}(\hat{x}) - C^T Z_j^T(\hat{x}) \\ &\quad + Y_{1k}^T(\hat{x})(B_i(x))^T - W_{1j}^T(\hat{x}), \\ U_{ijk}(x,\hat{x}) &= -B_i(x)Y_{2k}(\hat{x}) - Y_{2k}^T(\hat{x})(B_i(x))^T + W_{2j}^T(\hat{x}) \\ &\quad + S_2(\hat{x}), \\ \Omega^1_{ijk}(x,\hat{x}) &= A_{\tau_i}(x)X_1(\hat{x}) - W_{11_j}(\hat{x}), \\ L_{ijk}(x,\hat{x}) &= A_i(x)X_1(\hat{x}) + B_i(x)Y_{1k}(\hat{x}) + \alpha X_1(\hat{x}), \\ M_{ijk}(x,\hat{x}) &= -B_i(x)Y_{2k}(\hat{x}), \\ N_{ijk}(x,\hat{x}) &= A_i(x)X_1(\hat{x}) - Z_j(\hat{x})C + B_i(x)Y_{1k}(\hat{x}) \\ &\quad - W_{1j}^T(\hat{x}), \\ D_{ijk}(x,\hat{x}) &= W_{2j}(\hat{x}) - B_i(x)Y_{2k}(\hat{x}) + \alpha X_2. \end{aligned}$$

Then, augmented system (60) is asymptotically stable, while remaining positive.
Under these conditions, the desired polynomial controller gain matrices are obtained from

$$K_k(\hat{x}) = Y_{1k}(\hat{x})X_1^{-1}(\hat{x}), \tag{67}$$

$$L_j(\hat{x}) = Z_j(\hat{x})E_1^{-1}(\hat{x}), \tag{68}$$

$$G_j(\hat{x}) = W_{1j}(\hat{x})X_1^{-1}(\hat{x}), \tag{69}$$

$$G_{\tau_j}(\hat{x}) = W_{11_j}(\hat{x})X_1^{-1}(\hat{x}), \tag{70}$$

where $E_1(\hat{x})$ fulfils $CX_1(\hat{x}) = E_1(\hat{x})C$.

Proof: Consider the following Lyapunov-Krasovskii (L-K) functional candidate:

$$V(\tilde{x}) = V_1(x,\hat{x}) + V_2(e) + V_3(\tilde{x}), \tag{71}$$

where $V_1(x,\hat{x})$ and $V_2(e)$ are the same as (33) and

$$V_3(\tilde{x}) = \int_{t-\tau}^{t} \tilde{x}^T(\alpha)\tilde{S}(\tilde{x})\tilde{x}(\alpha)d\alpha, \tag{72}$$

with $\tilde{S}(\tilde{x}) = P(\hat{x})S(\hat{x})P(\hat{x})$,

$$P(\hat{x}) = \begin{bmatrix} P_1(\hat{x}) & 0 \\ 0 & P_2 \end{bmatrix}, \; S(\hat{x}) = \begin{bmatrix} S_1(\hat{x}) & 0 \\ 0 & S_2(\hat{x}) \end{bmatrix}. \tag{73}$$

The time derivative of L-K functional (71) gives

$$\begin{aligned} \dot{V}(\tilde{x}) &= 2x^T P_1(\hat{x})\dot{x} + 2e^T P_2\dot{e} + \tilde{x}^T\tilde{S}(\hat{x})\tilde{x} \\ &\quad - \tilde{x}(t-\tau)^T\tilde{S}(\hat{x})\tilde{x}(t-\tau) \\ &= 2\tilde{x}^T P(\hat{x})\dot{\tilde{x}} + \tilde{x}^T\tilde{S}(\hat{x})\tilde{x} - \tilde{x}(t-\tau)^T\tilde{S}(\hat{x})\tilde{x}(t-\tau), \end{aligned} \tag{74}$$

then

$$\begin{aligned} \dot{V}(\tilde{x}) &= 2\sum_{i,j,k=1}^{r} h_i(x)h_j(\hat{x})h_k(\hat{x})\tilde{x}^T P(\hat{x})\{A_X^{ijk}(x,\hat{x})\tilde{x} \\ &\quad + A_H^{ij}(x,\hat{x})\tilde{x}_\tau\} + \tilde{x}^T\tilde{S}(\hat{x})\tilde{x} \\ &\quad - \tilde{x}(t-\tau)^T\tilde{S}(\hat{x})\tilde{x}(t-\tau) \\ &= \sum_{i,j,k=1}^{r} h_i(x)h_j(\hat{x})h_k(\hat{x})\tilde{\sigma}^T \\ &\quad \times (\Pi_{ijk}(x,\hat{x}) + \Pi_{ikj}(x,\hat{x}))\tilde{\sigma}, \end{aligned} \tag{75}$$

where $\tilde{\sigma}^T = [\tilde{x}^T, \tilde{x}_\tau^T]^T$,

$$\Pi_{ijk}(x,\hat{x}) = \begin{bmatrix} \Pi^{11}_{ijk}(x,\hat{x}) & (A_H^{ij})^T(x,\hat{x})P(\hat{x}) \\ * & -\tilde{S}(\hat{x}) \end{bmatrix}. \tag{76}$$

$\Pi^{11}_{ijk}(x,\hat{x}) = P(\hat{x})A_X^{ijk}(x,\hat{x}) + (A_X^{ijk})^T(x,\hat{x})P(\hat{x}) + \tilde{S}(\hat{x})$.
If the following condition holds, then $\dot{V}(\tilde{x}) < 0$ at $\tilde{x} \neq 0$

$$\Pi_{ijk}(x,\hat{x}) + \Pi_{ikj}(x,\hat{x}) < 0. \tag{77}$$

Define $X(\hat{x}) = P^{-1}(\hat{x})$ and $S(\hat{x}) = P^{-1}(\hat{x})\tilde{S}(\hat{x})P^{-1}(\hat{x})$. Pre and post-multiplying (77) by $\mathbb{X}(\hat{x}) = diag[X(\hat{x}), X(\hat{x})]$, we get

$$\Sigma_{ijk}(x,\hat{x}) + \Sigma_{ikj}(x,\hat{x}) < 0, \tag{78}$$

where

$$\Sigma_{ijk}(x,\hat{x}) = \begin{bmatrix} \Sigma^{11}_{ijk}(x,\hat{x}) & X(\hat{x})(A_H^{ij})^T(x,\hat{x}) \\ * & -S(\hat{x}) \end{bmatrix}. \tag{79}$$

$\Sigma^{11}_{ijk}(x,\hat{x}) = A_X^{ijK}(x,\hat{x})X(\hat{x}) + X(\hat{x})(A_X^{ijk})^T(x,\hat{x}) + S(\hat{x})$.
Define $X(\hat{x}) = \begin{bmatrix} X_1(\hat{x}) & 0 \\ 0 & X_2 \end{bmatrix}$, $S(\hat{x}) = \begin{bmatrix} S_1(\hat{x}) & 0 \\ 0 & S_2(\hat{x}) \end{bmatrix}$, $Y_{1k}(\hat{x}) = K_k(\hat{x})X_1(\hat{x})$, $Y_{2k}(\hat{x}) = K_k(\hat{x})X_2$, and $E_1(\hat{x})C = CX_1(\hat{x})$ leads to the following inequality:

$$\delta_{ijk}(x,\hat{x}) + \delta_{ikj}(x,\hat{x}) < 0, \tag{80}$$

where

$$\delta_{ijk}(x,\hat{x}) = \begin{bmatrix} a^{ik}(x,\hat{x}) & b^{ijk}(x,\hat{x}) & A_{\tau_i}(x)X_1(\hat{x}) & 0 \\ * & c^{ijk}(x,\hat{x}) & d^{ij}(x,\hat{x}) & G_{\tau_j}(\hat{x})X_2 \\ * & * & -S_1(\hat{x}) & 0 \\ * & * & * & -S_2(\hat{x}) \end{bmatrix},$$

with

$$\begin{aligned} a^{ik}(x,\hat{x}) &= A_i(x)X_1(\hat{x}) + X_1(\hat{x})(A_i(x))^T + B_i(x)Y_{1k}(\hat{x}) \\ &\quad + Y_{1k}^T(\hat{x})(B_i(x))^T + S_1(\hat{x}), \\ b^{ijk}(x,\hat{x}) &= X_1(\hat{x})(A_i(x))^T - B_i(x)Y_{2k}(\hat{x}) \end{aligned}$$

$$-C^T E_1^T(\hat{x})L_j^T(\hat{x}) + Y_{1k}^T(\hat{x})(B_i(x))^T - X_1(\hat{x})G_j^T(\hat{x}),$$

$$c^{ijk}(x,\hat{x}) = -B_i(x)Y_{2k}(\hat{x}) - Y_{2k}^T(\hat{x})(B_i(x))^T + X_2 G_j^T(\hat{x}) + S_2(\hat{x}),$$

$$d^{ij}(x,\hat{x}) = A_{\tau_i}(x)X_1(\hat{x}) - G_{\tau_j}(\hat{x})X_1(\hat{x}).$$

Considering that $Z_j(\hat{x}) = L_j(\hat{x})E_1(\hat{x})$, $W_{1j}(\hat{x}) = G_j(\hat{x})X_1(\hat{x})$, $W_{2j}(\hat{x}) = G_j(\hat{x})X_2$, $W_{11_j}(\hat{x}) = G_{\tau_j}(\hat{x})X_1(\hat{x})$, $W_{12_j}(\hat{x}) = G_{\tau_j}(\hat{x})X_2$, we get

$$\Omega_{ijk}(x,\hat{x}) + \Omega_{ikj}(x,\hat{x}) < 0. \tag{81}$$

SOS condition (66) implies (81), then we have $\dot{V}(\tilde{x}) < 0$. Therefore, augmented polynomial fuzzy system (60) is asymptotically stable.
To guarantee the positivity of augmented system polynomial fuzzy system (60), we have to prove that all $A_X^{ijk}(x,\hat{x})$are Metzler, or equivalent, that there exists scalar α such that $A_X^{ijk}(x,\hat{x}) + \alpha I \geq 0$.
Multiplying on the right by $diag[X_1(\hat{x}), X_2]$ we get (65).
Once the SOS conditions are solved and programmed, we can obtain the polynomial controller gain matrices

$$K_k(\hat{x}) = Y_{1k}(\hat{x})X_1^{-1}(\hat{x}), \tag{82}$$

$$L_j(\hat{x}) = Z_j(\hat{x})E_1^{-1}(\hat{x}), \tag{83}$$

$$G_j(\hat{x}) = W_{1j}(\hat{x})X_1^{-1}(\hat{x}), \tag{84}$$

$$G_{\tau_j}(\hat{x}) = W_{11_j}(\hat{x})X_1^{-1}(\hat{x}). \tag{85}$$

Therefore, if there exist symmetric polynomial diagonal matrices $X(\hat{x})$ and $S(\hat{x})$, polynomial matrices $G_j(\hat{x})$, $G_{\tau_j}(\hat{x})$, $L_j(\hat{x})$ and $K_k(\hat{x})$ such that the SOS conditions in Theorem (3) are satisfied, then augmented system (60) is positive and asymptotically stable. □

Theorem 4: For a positive scalar γ, there exists a solution to the problem of the general polynomial dynamic controller for closed-loop system (59) if there exist symmetric positive definite matrices $P(\hat{x}) = diag[P_1(\hat{x}), P_2]$ and $Q(\hat{x}) = diag[Q_1(\hat{x}), Q_2(\hat{x})]$, polynomial matrices $G_{\tau_j}(\hat{x}) \geq 0$, $L_j(\hat{x}) \geq 0$, $K_k(\hat{x}) \leq 0$ and Metzler matrices $G_j(\hat{x})$ such that the following SOS-based conditions are satisfied $\forall i,\ j,\ k = 1, ..., r$

$$v_1^T(P_1(\hat{x}) - \epsilon_1(\hat{x})I)v_1 \text{ is SOS}, \tag{86}$$

$$v_2^T(P_2 - \epsilon_2 I)v_2 \text{ is SOS}, \tag{87}$$

$$v_3^T(S_1(\hat{x}) - \epsilon_3(\hat{x})I)v_3 \text{ is SOS}, \tag{88}$$

$$v_4^T(S_2(\hat{x}) - \epsilon_4(\hat{x})I)v_4 \text{ is SOS}, \tag{89}$$

$$v_5^T([\underline{A}_i(x) + \overline{B}_i(x)K_k(\hat{x})]_{lm} - \epsilon_5(x,\hat{x})I)v_5 \text{ is SOS, for } l \neq m, \tag{90}$$

$$v_6^T([G_j(\hat{x}) - \underline{B}_i(x)K_k(\hat{x})]_{lm} - \epsilon_6(x,\hat{x})I)v_6 \text{ is SOS, for } l \neq m, \tag{91}$$

$$v_7^T(\underline{A}_i(x) - L_j(\hat{x})\overline{C} + \overline{B}_i(x)K_k(\hat{x}) - G_j(\hat{x}) - \epsilon_7(x,\hat{x})I)v_7 \text{ is SOS}, \tag{92}$$

$$v_8^T(\underline{A}_{\tau_i}(x) - G_{\tau_j}(\hat{x}) - \epsilon_8(x,\hat{x})I)v_8 \text{ is SOS}, \tag{93}$$

$$-v_9^T(\Xi_{ijk}(x,\hat{x}) + \Xi_{ikj}(x,\hat{x}) + \epsilon_5(x,\hat{x})I)v_9 \text{ is SOS}, \ \forall i,\ j \leq k, \tag{94}$$

where

$$\Xi_{ijk}(x,\hat{x}) = \begin{bmatrix} \Xi_i^1(x,\hat{x}) & \Xi_{ijk}^2(x,\hat{x}) & \Xi_{ijk}^3(x,\hat{x}) & \gamma\mathcal{B}_i(x) \\ * & -S(\hat{x}) & 0 & 0 \\ * & * & -I & 0 \\ * & * & * & -I \end{bmatrix},$$

$$\Xi_i^1(x,\hat{x}) = \mathcal{A}_i^T(x)P(\hat{x}) + P(\hat{x})\mathcal{A}_i(x) - \gamma\mathcal{B}_i(x)\mathcal{B}_i^T(x)P(\hat{x}) - \gamma P(\hat{x})\mathcal{B}_i(x)\mathcal{B}_i^T(x) + S(\hat{x}),$$

$$\Xi_{ijk}^2(x,\hat{x}) = (A_H^{ij}(x,\hat{x}))^T P(\hat{x}),$$

$$\Xi_{ijk}^3(x,\hat{x}) = P(\hat{x})\mathcal{B}_i(x) + \mathcal{C}^T\mathcal{K}_{jk}^T(\hat{x}),$$

$$\mathcal{A}_i(x) = \begin{bmatrix} \overline{A}_i(x) & 0 \\ \underline{A}_i(x) & 0 \end{bmatrix},$$

$$\mathcal{B}_i(x) = \begin{bmatrix} 0 & \underline{B}_i(x) - \overline{B}_i(x) & 0 & \underline{B}_i(x) \\ 0 & \underline{B}_i(x) - \overline{B}_i(x) & -I & \underline{B}_i(x) \end{bmatrix},$$

$$\mathcal{K}_{jk}(\hat{x}) = \begin{bmatrix} G_j(\hat{x}) & L_j(\hat{x}) & 0 & 0 \\ K_k(\hat{x}) & 0 & 0 & 0 \\ 0 & 0 & G_j(\hat{x}) & L_j(\hat{x}) \\ 0 & 0 & K_k(\hat{x}) & 0 \end{bmatrix},$$

$$\mathcal{C} = \begin{bmatrix} 0 & I \\ 0 & 0 \\ I & -I \\ \underline{C} & 0 \end{bmatrix}.$$

Proof: By following the same procedure as the proof of Theorem 2, we can conclude that (60) is asymptotic stable for any $A_i(x) \in [\underline{A}_i(x), \overline{A}_i(x)]$, $A_{\tau_i}(x) \in [\underline{A}_{\tau_i}(x), \overline{A}_{\tau_i}(x)]$, $B_i(x) \in [\underline{B}_i(x), \overline{B}_i(x)]$ and $C \in [\underline{C}, \overline{C}]$. □

Remark 3: In this paper, the SOS conditions are solved via SeDuMi in addition to SOSTools. For more details of how to solve the SDPs using SeDuMi, see [21].

Remark 4: Our method presents a new way that can reduces the conservatism from several points of view such as

- The SOS approach allows to study the fuzzy polynomial models which are more general than the well-known T-S fuzzy models [41,42]. These models include polynomial matrices in their consequent parts of each rule.
- Polynomial fuzzy model is used. However, with LMI approaches, only T-S fuzzy model can be investigated.
- The use of fuzzy polynomial models allows to reduce the number of rules if then and consequently, the complexity and the computation time [41,42].

- The stability conditions are obtained based on polynomial Lyapunov functions that include common quadratic Lyapunov functions as special cases, which are extensively used for LMI approaches.
- The control design is developed by considering unmeasurable premise variables which are more general than when the premise variables are assumed to be measurable.
- A general model describing positive polynomial fuzzy uncertain interval system is studied, which is more applicable in practice, for example electronic circuits with nonlinear elements.

Hence, with a more general framework for both modeling and control, our SOS-based approach indeed provides more relaxed analysis and design conditions than the existing LMI approach using T-S fuzzy systems.

5. ILLUSTRATIVE EXAMPLES

In this section, simulation examples are used to demonstrate the effectiveness of the proposed approaches.

5.1. Example 1

Consider the following continuous-time polynomial uncertain interval fuzzy system with time-delay

$$\begin{cases} \dot{x} = \sum_{i=1}^{2} h_i\{A_i x + A_{\tau_i} x_\tau + B_i(x)u\}, \\ y = Cx, \end{cases} \tag{95}$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \; A_1 = \begin{bmatrix} -a-1\pm 0.5 & 0.6643 \\ 1.961 & -1.74 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} -a+0.2172\pm 0.5 & 0.6643 \\ 1.961 & -1.74 \end{bmatrix},$$
$$A_{\tau_1} = A_{\tau_2} = \begin{bmatrix} 0.04\pm 0.01 & 0.01 \\ 0 & 0.12 \end{bmatrix},$$
$$B_1(x_2) = B_2(x_2) \begin{bmatrix} b+0.1x_2^2 \pm 0.3 \\ 0 \end{bmatrix},$$
$$C = \begin{bmatrix} 1\pm 0.1 & 0 \end{bmatrix}.$$

where parameters a and b are constant scalars to be determined.
The membership functions are defined as

$$h_1 = \frac{sin(x_1)+0.2172x_1}{1.2172x_1}, \; h_2 = 1-h_1.$$

As opposed to the T-S fuzzy model, the polynomial fuzzy model considers to have polynomial matrices $A_i(x)$ and $B_i(x)$ in consequent parts of each fuzzy rule. It is stated in [21] that SOS problems with zero-order polynomial matrices reduce to LMI problems. In other words, we cannot use an LMI design framework for one or higher degree polynomial matrices. Hence, we compare our approach with different orders of Lyapunov polynomial matrix $P_1(\hat{x})$. Theorem 2 is applied to all the combinations with $0 \le a \le 5$ and $15 \le b \le 18$ for zero, second- and fourth-order positive polynomial Lyapunov matrix $P_1(\hat{x})$.

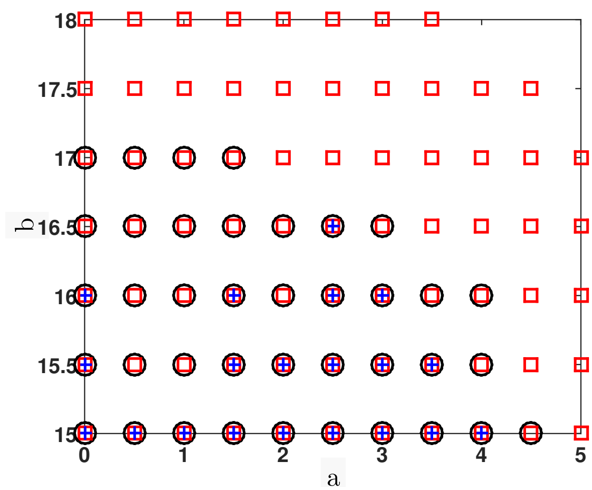

Fig. 1. Stability regions given by Theorem 2 with $P_1(\hat{x})$ of degree 4 indicated by "□", of degree 2 indicated by "○" and of degree 0 indicated by "+".

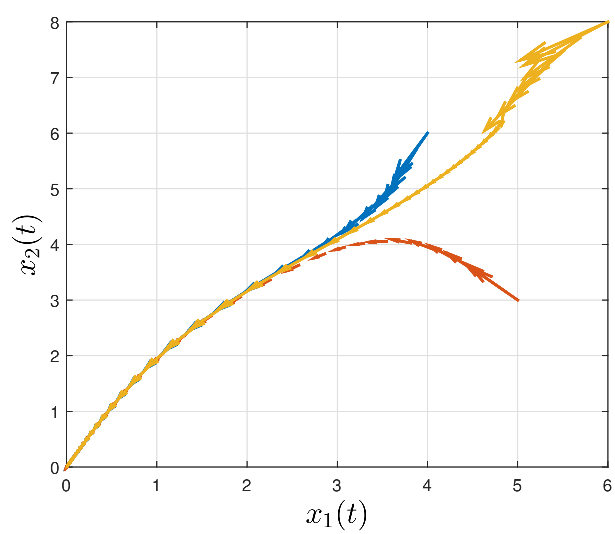

Fig. 2. Behaviors in the $x_1(t)$-$x_2(t)$ plane.

It can be seen from Fig. 1 that Theorem 2 with polynomial matrix $P_1(\hat{x})$ of degrees 4 and 2 are able to offer a larger stability range than with constant matrix P_1. We can remark that higher order polynomial Lyapunov functions rather than the common quadratic Lyapunov function get more relaxed results and provide less conservative results.

Now, we consider $a = 1.5$ and $b = 17$. Fig. 2 shows control results, for several initial states, by the developed polynomial controller given in Theorem 2. In fact, the controller guarantees the asymptotic stability of the controlled

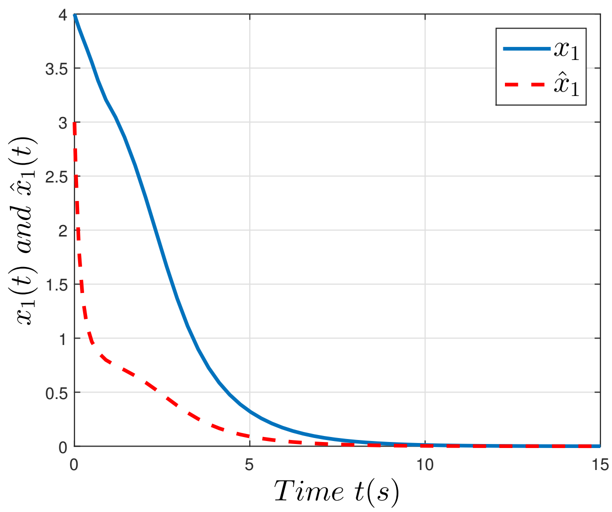

Fig. 3. Response of state $x_1(t)$ and its estimation $\hat{x}_1(t)$ for Example 1.

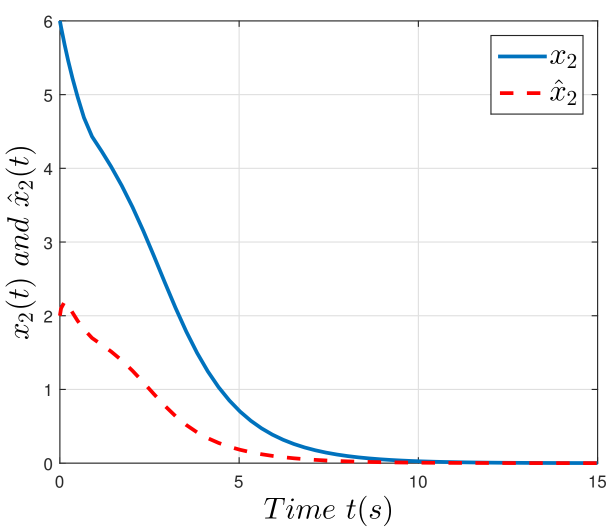

Fig. 4. Response of state $x_2(t)$ and its estimation $\hat{x}_2(t)$ for Example 1.

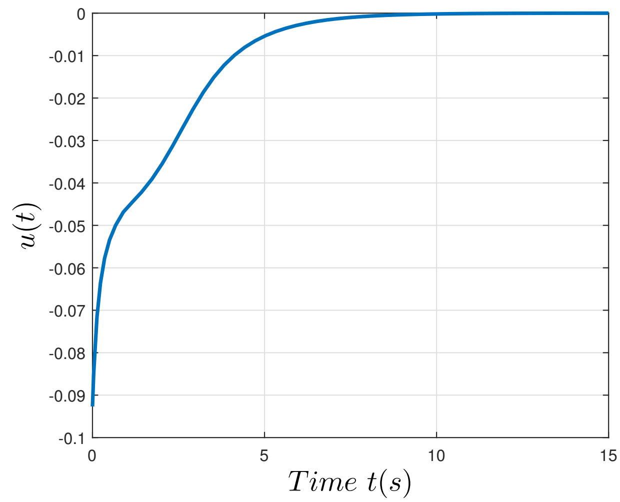

Fig. 5. Control $u(t)$.

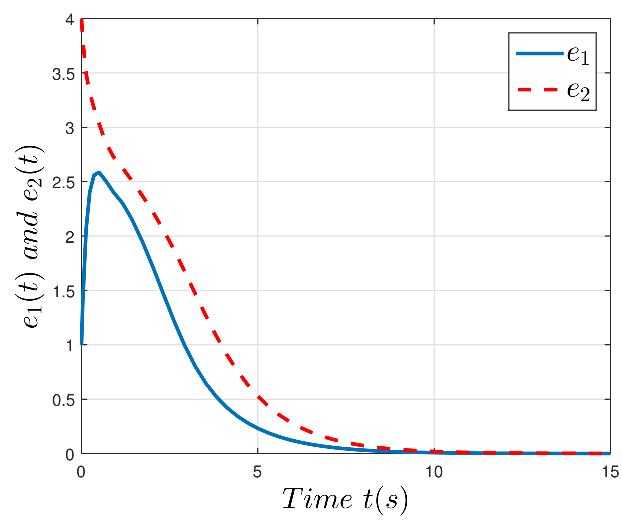

Fig. 6. Evolution of estimation errors $e_1(t)$ and $e_2(t)$.

system. Figs. 3 and 4 show some simulation results using polynomial dynamic controller (5) derived from Theorem 2, starting from initial conditions $x_0 = \begin{bmatrix} 4 & 6 \end{bmatrix}^T$ and $\hat{x}_0 = \begin{bmatrix} 3 & 2 \end{bmatrix}^T$, for $\tau = 0.8$. We can observe that the evolutions of state vector $x(t)$ and of the estimation $\hat{x}(t)$ are always in the positive orthant and converge to zero. These properties can be seen in Figs. 3 and 4 that plot the state evolutions from the given initial conditions. Fig. 5 shows the control input. We can note, from Fig. 6, the convergence of the estimation errors. In addition, the estimation errors always remain nonnegative. These facts show the effectiveness of the proposed approach.
The following polynomial matrices are obtained by solving the SOS conditions in Theorem 2:

$$L_1(\hat{x}_2) = \begin{bmatrix} 1.6231e^{-4}\hat{x}_2^2 + 0.6375 \\ 4.7027e^{-4}\hat{x}_2^2 + 2.8865 \end{bmatrix},$$

$$L_2(\hat{x}_2) = \begin{bmatrix} 1.6231e^{-4}\hat{x}_2^2 + 0.615 \\ 4.7027e^{-4}\hat{x}_2^2 + 2.9106 \end{bmatrix},$$

$$K_1(\hat{x}_2) = \begin{bmatrix} 1.409e^{-3}\hat{x}_2^2 - 0.0231 & 6.325e^{-5}\hat{x}_2^2 - 0.0180 \end{bmatrix},$$

$$K_2(\hat{x}_2) = \begin{bmatrix} 1.409e^{-3}\hat{x}_2^2 - 0.0244 & 6.326e^{-5}\hat{x}_2^2 - 0.0181 \end{bmatrix},$$

where $1e^{-s} = 10^{-s}$, $s \geq 0$.

5.2. Example 2

Consider the following continuous-time polynomial system with time-delay:

$$\begin{cases} \dot{x} = \sum_{i=1}^{2} h_i\{A_i(x)x + A_{\tau_i}x_\tau + B_i(x)u\}, \\ y = Cx, \end{cases} \tag{96}$$

where

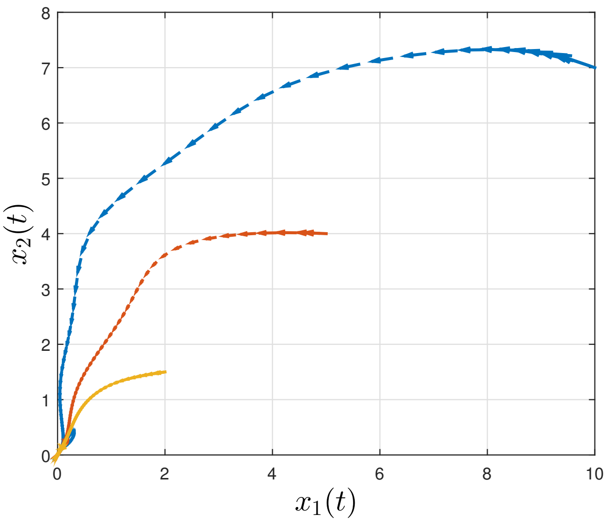

Fig. 7. Behaviors in the $x_1(t)$-$x_2(t)$ plane (with feedback).

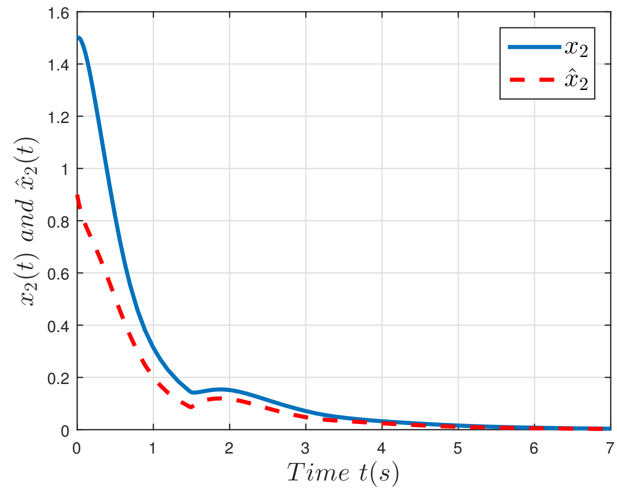

Fig. 9. Response of state $x_2(t)$ and its estimations.

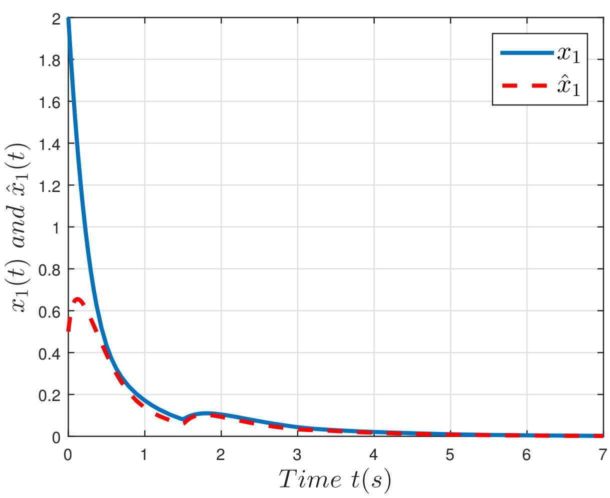

Fig. 8. Response of state $x_1(t)$ and its estimations.

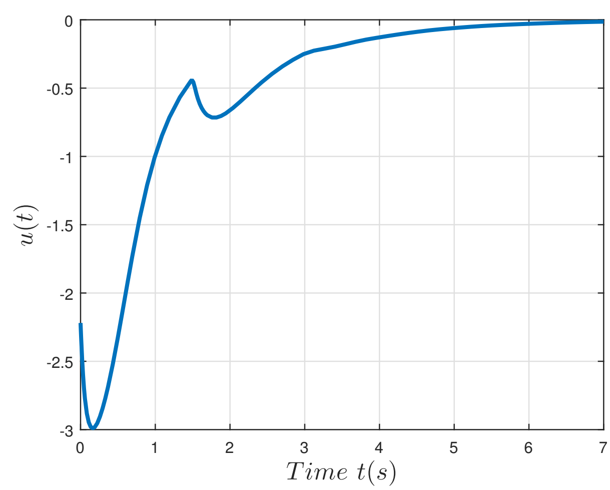

Fig. 10. Control $u(t)$.

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},\ A_1(x_2) = \begin{bmatrix} -4 & x_2^2 \\ 2.5 & -3 \end{bmatrix},$$
$$A_2(x_2) = \begin{bmatrix} -4 & x_2^2+1 \\ 7 & -5 \end{bmatrix},\ A_{\tau_1} = \begin{bmatrix} 0.14 & 0 \\ 0.1 & 0.12 \end{bmatrix},$$
$$A_{\tau_2} = \begin{bmatrix} 0.1 & 0.1 \\ 0 & 0.12 \end{bmatrix},\ \tau = 1.5,$$
$$B_1(x_2) = \begin{bmatrix} 0.3x_2^2 \\ 0.14 \end{bmatrix},\ B_2(x_2) = \begin{bmatrix} 0.3x_2^2 - 0.15 \\ 0.16 \end{bmatrix},$$
$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

The membership functions are defined as

$$h_1 = \sin^2(x_2),\ h_2 = 1 - h_1.$$

Fig. 7 shows control results, for several initial states, by using the polynomial controller proposed in Theorem 3. In fact, the controller guarantees the asymptotic stability of the controlled system. Figs. 8 and 9 show the simulation results of the numerical example from initial conditions $x_0 = \begin{bmatrix} 2 & 1.5 \end{bmatrix}^T$ and $\hat{x}_0 = \begin{bmatrix} 0.5 & 0.9 \end{bmatrix}^T$. It can be seen from these figures that state vector $x(t)$, as well as estimated state vector $\hat{x}(t)$, are nonnegative and converge. Fig. 10 shows the control input. Fig. 11 also shows the convergence and the nonnegativity of the estimation errors. By using the MATLAB SOSTools, it can be seen that the conditions in Theorem 3 are feasible, for $\alpha = 10$, with the following solution

$$Z_1(\hat{x}_2) = Z_2(\hat{x}_2) = \begin{bmatrix} 0.7031\hat{x}_2^2 + 0.6578 \\ 7.6881e^{-11}\hat{x}_2^2 + 0.3096 \end{bmatrix},$$
$$Y_{11} = Y_{12} = \begin{bmatrix} -0.0296 & -5.697 \end{bmatrix}.$$

5.3. Example 3

Consider the following continuous-time polynomial fuzzy uncertain interval system

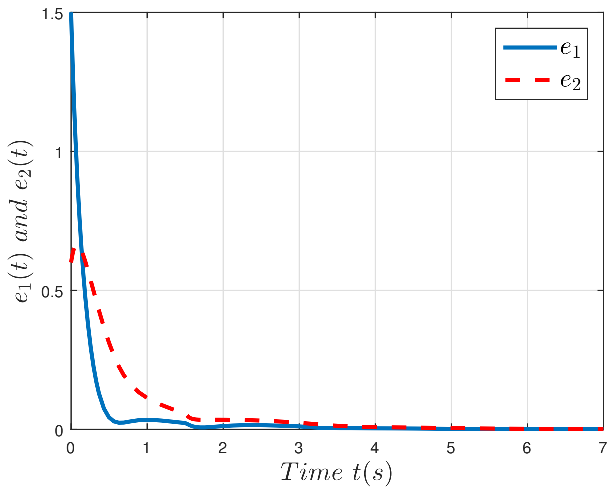

Fig. 11. Evolution of estimation errors $e_1(t)$ and $e_2(t)$.

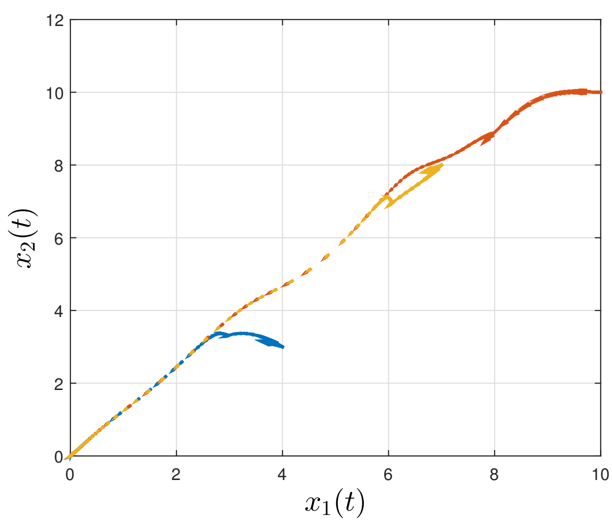

Fig. 12. Behaviors in the $x_1(t)$-$x_2(t)$ plane.

$$\begin{cases} \dot{x} = \sum_{i=1}^{2} h_i\{A_i x + A_{\tau_i} x_\tau + B_i(x)u\}, \\ y = Cx, \end{cases} \tag{97}$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},\ A_1 = \begin{bmatrix} -0.7732 \pm 0.002 & 0.6643 \pm 0.01 \\ 1.6 \pm 0.02 & -1.713 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -0.7732 \pm 0.002 & 0.6643 \pm 0.01 \\ 1.54 \pm 0.02 & -1.5 \end{bmatrix},$$

$$A_{\tau_1} = A_{\tau_2} = \begin{bmatrix} 0.1 \pm 0.020 & 0 \\ 0 & 0.2 \end{bmatrix},$$

$$B_1(x_1) = B_2(x_1) \begin{bmatrix} 15 + 0.01x_1^2 \pm 0.3 \\ 0.04 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 1 \pm 0.5 \end{bmatrix}.$$

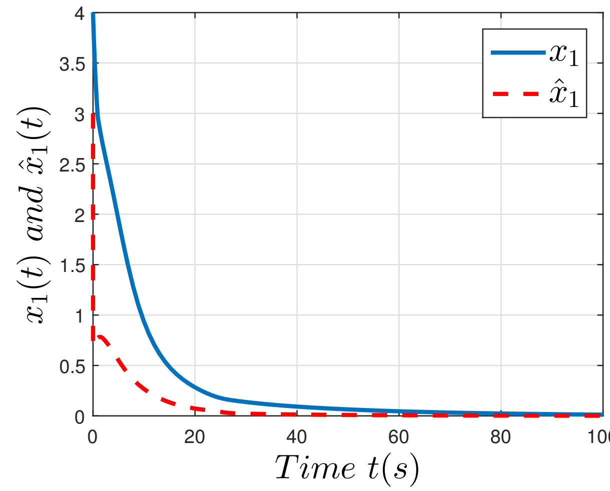

Fig. 13. Response of state $x_1(t)$ and its estimations.

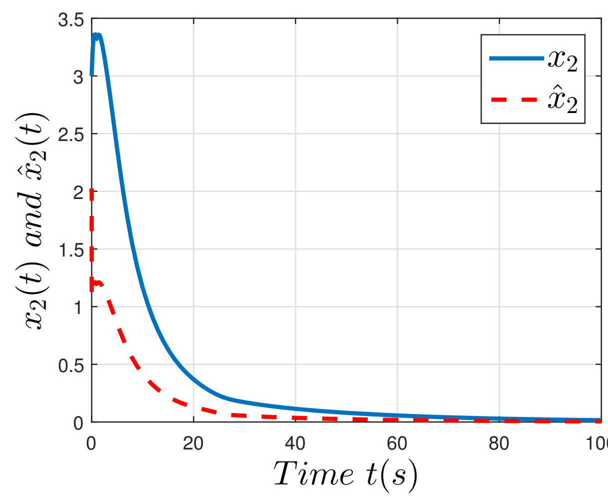

Fig. 14. Response of state $x_2(t)$ and its estimations.

The membership functions are defined as

$$h_1 = \sin^2(x_1),\ h_2 = 1 - h_1.$$

Fig. 12 shows control results, for several initial states, by considering the polynomial controller proposed in Theorem 4. Fig. 12 shows control results, for several initial states, by considering the polynomial controller proposed in Theorem 4.

The simulations presented in Figs. 13 and 14 show that state vector $x(t)$, as well as the estimated state vector $\hat{x}(t)$, are nonnegative and converge. These properties can be seen in Figs. 13 and 14, which represent the state evolutions from given initial conditions $x_0 = \begin{bmatrix} 4 & 3 \end{bmatrix}^T$ and $\hat{x}_0 = \begin{bmatrix} 3 & 2 \end{bmatrix}^T$, for $\tau = 0.9$, when $A_i = \overline{A}_i$, $B_i(x) = \overline{B}_i(x)$ and $C = \overline{C}$. In fact, the controller guarantees the asymptotic stability of the controlled system. Fig. 15 shows the control input. It is possible to see, from Fig. 16, that the

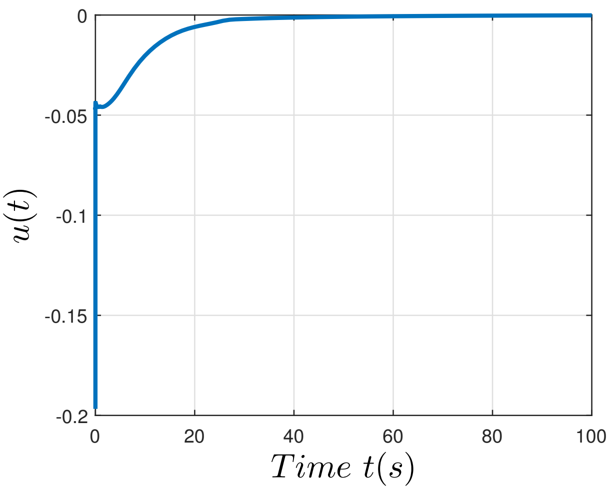

Fig. 15. Control $u(t)$.

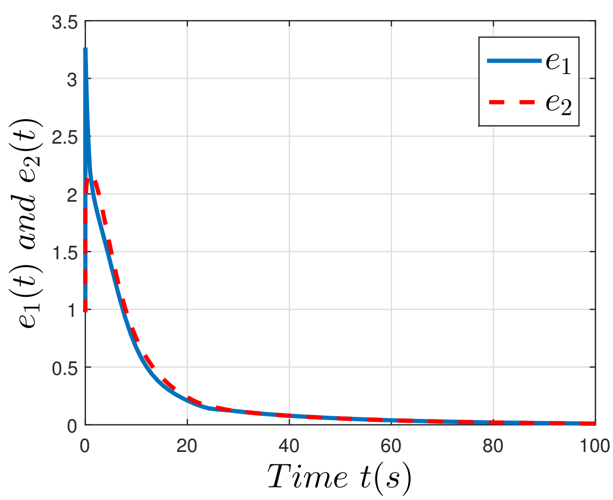

Fig. 16. Evolution of estimation errors $e_1(t)$ and $e_2(t)$.

estimation errors are nonnegative. These facts show the effectiveness of the proposed approach

By using the MATLAB SOSTools, it can be seen that the conditions in Theorem 4 are feasible, for $\gamma = 5$, with the following solution

$$L_1(\hat{x}_1) = \begin{bmatrix} 61.5201\hat{x}_1^2 + 61.5545 \\ 245.195\hat{x}_1^2 + 245.0933 \end{bmatrix},$$

$$L_2(\hat{x}_1) = \begin{bmatrix} -3.2468e^{-5}\hat{x}_1^2 + 0.1469 \\ -5.3541e^{-5}\hat{x}_1^2 + 0.9647 \end{bmatrix},$$

$$K_1(\hat{x}_1) = \begin{bmatrix} 1.210e^{-3}\hat{x}_1^2 - 1.893e^{-2} & 4.108e^{-5}\hat{x}_1^2 - 1.321e^{-2} \end{bmatrix},$$

$$K_2(\hat{x}_1) = \begin{bmatrix} 9.420e^{-5}\hat{x}_1^2 - 5.929e^{-2} & 2.912e^{-5}\hat{x}_1^2 - 1.244e^{-2} \end{bmatrix}.$$

6. CONCLUSION

In this paper, we have presented an SOS based design approach of the dynamic output-feedback controller for positive polynomial fuzzy uncertain interval systems with time delay and unmeasurable state variables. The design problem has been firstly analysed by assuming measurable premise variables and then extended for unmeasurable ones. The positive stabilization problem under polynomial fuzzy dynamic output feedback control has been solved. It has been shown that all the proposed conditions are solvable in terms of SOS and can symbolically and numerically be solved via the recently developed SOSTools and an SDP solver. The designed polynomial fuzzy controller not only trails the estimated signals but also guarantees the positivity of the estimations. To illustrate the validity of the design approach, illustrative examples have been provided. These examples have shown the utility of our SOS approach for the positive polynomial fuzzy dynamic control design. Our next subjects are to apply the advanced SOS robust stabilization conditions to more complex positive polynomial systems, e.g., uncertain positive polynomial fuzzy systems with delay in the presence of external disturbances, uncertain positive polynomial fuzzy systems in the presence of sensor and/or actuator faults, etc. We focus on the area of faults detection and isolation, where polynomial interval observers and polynomial interval observer-based controllers using decomposed control laws can have a great importance and open the doors for multiple lines of research.

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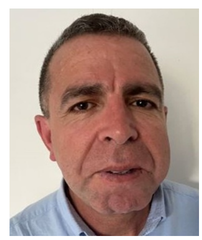

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