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# New Explicit Criteria for Finite-time Stability of Singular Linear Systems Using Time-dependent Lyapunov Functions

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**Abstract:** In this paper, the finite time stability and $H_\infty$ finite-time stability of singular linear systems are considered. By constructing a class of time-dependent Lyapunov functions and introducing a zero term with free weighting matrices, we first establish a new explicit criterion in the form of LMIs for finite-time stability of the system. Then, an $H_\infty$ finite-time stability criterion for the system is obtained. The given results are easily verifiable and less conservative compared with some existing ones in the literature. Finally, four numerical examples are given to demonstrate the effectiveness of the proposed method.

**Keywords:** Finite-time stability, $H_\infty$ finite-time stability, singular system, time-dependent Lyapunov function.

## 1. INTRODUCTION

Singular system is the basic model in the field of system and control science, which is also called differential algebraic system, implicit system or descriptive system. In the last two decades, singular system has attracted much attention due to its extensive application in practice. Compared with the normal system, the performance analysis of singular systems becomes more complex [1–9]. For example, in the stability analysis of singular systems, it is necessary to consider not only the stability, but also the absence of pulses.

Finite-time stability is different from the usual stability concepts such as asymptotic stability and input-output stability in the sense of Lyapunov. The former mainly studies the behavior of the system in the finite time interval, while the latter focuses on the state estimation of the system in the whole time period. In the practical application, the concept of finite-time stability is more consistent with the actual situation. As a result, more and more scholars pay attention to the issue of finite time stability of singular systems. Many results about finite-time stability have been reported, such as [10–17] for linear systems, and [18–23] for singular systems. Some finite-time control problems of singular systems also have been solved in [24–27]. For more information of finite-time stability for singular systems, see references [28–33].

At present, although there are a lot of results aimed at finite-time stability of singular systems, most of them are presented by using time-independent Lyapunov functions. There are only few results that are derived from time-dependent Lyapunov functions. On the other hand, the existing results for finite-time stability of singular systems based on time-dependent Lyapunov functions are usually determined by some matrix differential inequalities that are not easy to be solved. In this paper, by constructing a class of time-dependent Lyapunov functions and introducing a zero term with free weighting matrices, new explicit conditions for finite-time stability and $H_\infty$ finite-time stability are given in terms of linear matrix inequalities, which do not contain unsolvable matrix differential inequalities. Compared with some existing results in the literature, the obtained results are easily verifiable and less conservative owing to the introduction of a zero term with free weighting matrices.

The rest of this paper is as follows: In Section 2, problem statements and preliminaries are given. Section 3 presents explicit conditions for finite-time stability and $H_\infty$ finite-time stability of singular linear systems. Section 4 gives four numerical examples to demonstrate the effectiveness of the obtained results. Section 5 summarizes this paper.

## 2. PROBLEM STATEMENTS AND PRELIMINARIES

Throughout this paper, $R^n$ stands for the vector space of all $n$-tuples of real numbers, $R^{n\times m}$ is the space of $n\times m$ ma-

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trices with real entries. For a matrix $A \in R^{n\times n}$, $A^\top$ denotes the transpose of $A$. Given a symmetric matrx $P \in R^{n\times n}$, $P > 0$ ($P < 0$) means that $P$ is positive definite (negative definite). In addition, if the dimensions of matrices and vectors are clear in context, they will not be explicitly mentioned.

Consider the following continuous-time singular linear system

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bw(t), \\ y(t) &= Cx(t) + Dw(t), \end{aligned} \tag{1}$$

where $x(t) \in R^n$ is the state vector for $t \in [0,T]$ with $T > 0$, $E \in R^{n\times n}$ is a singular matrix with $\text{rank}(E) = r < n$, $y(t) \in R^p$ is the measured output, $w(t) \in R^q$ is the disturbance input, $A, B, C, D$ are constant matrices of appropriate dimensions.

**Definition 1** [22]**:** Given positive scalars $T$, $c_1$, $c_2$ and a positive definite matrix $U$, system (1) with $w(t) \equiv 0$ is said to be finite-time stable with respect to $(T,U,c_1,c_2)$ if

$$x^\top(0)E^\top UEx(0) \le c_1 \Rightarrow x(t)^\top E^\top UEx(t) \le c_2,$$

where $t \in [0,T]$.

**Definition 2** [4]**:** Given positive scalars $T$, $c_1$, $c_2$, $\gamma$ and a positive definite matrix $U$, system (1) is said to be $H_\infty$ finite-time stable with respect to $(T, U, c_1, c_2, \gamma)$ if system (1) with $w(t) \equiv 0$ is finite-time stable with respect to ($T$, $U$, $c_1$, $c_2$), and under the zero-initial condition the output $y(t)$ satisfies $\int_0^T y^\top(s)y(s)ds \le \gamma \int_0^T w^\top(s)w(s)ds$.

**Definition 3** [27]**:** i) Singular system is regular if $\det(sE - A)$ is not identical zero. ii) Singular system is impulse-free if $\deg(\det(sE - A)) = r = \text{rank}(E)$.

We are also concerned with the following discrete-time singular linear system

$$\begin{aligned} Ex(k+1) &= Ax(k) + Bw(k), \\ y(k) &= Cx(k) + Dw(k), \end{aligned} \tag{2}$$

where $x(k) \in R^n$ is the state vector for $k = 0, 1, \cdots, T-1$, $E \in R^{n\times n}$ is a singular matrix satisfying $\text{rank}(E) = r < n$, $y(k) \in R^p$ is the measured output, $w(k) \in R^q$ is the disturbance input, $A, B, C$ and $D$ are matrices of appropriate dimensions.

**Definition 4** [22]**:** Given positive scalars $T$, $c_1$, $c_2$ and a positive definite matrix $U$, system (2) with $w(k) \equiv 0$ is said to be finite-time stable with respect to $(T,U,c_1,c_2)$ if

$$x^\top(0)E^\top UEx(0) \le c_1 \Rightarrow x^\top(k)E^\top UEx(k) \le c_2,$$

where $k = 0, 1, \cdots, T-1$.

**Definition 5** [22]**:** Given positive scalars $T$, $c_1$, $c_2$, $\gamma$ and a positive definite matrix $U$, system (2) is said to be $H_\infty$ finite-time stable with respect to ($T$, $U$, $c_1$, $c_2$, $\gamma$) if system (2) with $w(k) \equiv 0$ is finite-time stable with respect to ($T$, $U$, $c_1$, $c_2$), and under the zero-initial condition the output $y(k)$ satisfies $\sum_{k=0}^{T} y^\top(k)y(k) \le \gamma \sum_{k=0}^{N} w^\top(k)w(k)$.

## 3. MAIN RESULTS

Since $\text{rank}(E) = r < n$, there are two nonsingular matrices $M, N$ such that $MEN = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$. Let

$$MAN = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad \bar{M} = \begin{pmatrix} 0 & 0 \\ 0 & I_{n-r} \end{pmatrix} M.$$

**Lemma 1** [7]**:** Singular system (1) is impulse-free, if and only if $A_{22}$ is nonsingular.

### 3.1. The FTS of continuous time singular systems

For system (1), we construct the following time-dependent Lyapunov function

$$V(t,x(t)) = x^\top(t)E^\top G(t)Ex(t),$$

where $G(t) = P - tQ$ for $t \in [0,T]$, $P$ and $Q$ are appropriate constant matrices. Set

$$\begin{aligned} (M^\top)^{-1}PM^{-1} &= \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}, \\ (M^\top)^{-1}QM^{-1} &= \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix}. \end{aligned}$$

It is obvious that $\bar{M}AN = \begin{pmatrix} 0 & 0 \\ A_{21} & A_{22} \end{pmatrix}$. We first establish an explicit finite-time stability criterion for system (1).

**Theorem 1:** Given positive scalars $T, c_1 < c_2$ and a positive definite matrix $U$, system (1) with $w(t) \equiv 0$ is finite-time stable with respect to $(T,U,c_1,c_2)$ if there exist a positive constant $\alpha > 1$, a positive definite matrix $P$, a symmetric matrix $Q$ and two free weighting matrices $R$ and $\tilde{R}$ such that

$$X_1 < 0, \tag{3}$$
$$X_2 < 0, \tag{4}$$
$$U < P, \;\; U < P - TQ, \tag{5}$$
$$P < \alpha U, \tag{6}$$
$$\alpha c_1 < c_2, \tag{7}$$

where

$$\begin{aligned} X_1 &= A^\top PE + E^\top PA - E^\top QE - R\bar{M}A - A^\top \bar{M}^\top R^\top, \\ X_2 &= A^\top(P - TQ)E + E^\top(P - TQ)A - E^\top QE \\ &\quad - \tilde{R}\bar{M}A - A^\top \bar{M}^\top \tilde{R}^\top. \end{aligned}$$

**Proof:** The proof will be divided into the following two parts:

**Part I:** We first show that system (1) is regular and impulse-free. Since $N$ is nonsingular, we get from (3) that

$$\begin{aligned} 0 > & N^\top(A^\top PE + E^\top PA - E^\top QE - R\bar{M}A \\ & - A^\top \bar{M}^\top R^\top)N \end{aligned}$$

$$\begin{aligned}&= N^\top A^\top M^\top (M^\top)^{-1} PM^{-1}MEN - N^\top R\bar{M}AN\\&\quad + N^\top E^\top M^\top (M^\top)^{-1} PM^{-1}MAN\\&\quad - N^\top E^\top M^\top (M^\top)^{-1} QM^{-1}MEN\\&\quad - N^\top A^\top \bar{M}^\top R^\top N\\&= \begin{pmatrix} P_{11}A_{11}+P_{12}A_{21} & P_{11}A_{12}+P_{12}A_{22}\\ 0 & 0\end{pmatrix}\\&\quad + \begin{pmatrix} A_{11}^\top P_{11}+A_{21}^\top P_{21} & 0\\ A_{12}^\top P_{11}+A_{22}^\top P_{21} & 0\end{pmatrix} - \begin{pmatrix} Q_{11} & 0\\ 0 & 0\end{pmatrix}\\&\quad - \begin{pmatrix} (N^\top R)_{12}A_{21} & (N^\top R)_{12}A_{22}\\ (N^\top R)_{22}A_{21} & (N^\top R)_{22}A_{22}\end{pmatrix}\\&\quad - \begin{pmatrix} (N^\top R)_{12}A_{21} & (N^\top R)_{12}A_{22}\\ (N^\top R)_{22}A_{21} & (N^\top R)_{22}A_{22}\end{pmatrix}^\top,\end{aligned}$$

which implies $(N^\top R)_{22}A_{22}+A_{22}^\top (N^\top R)_{22}^\top > 0$. Therefore, $A_{22}$ is nonsingular. That is, system (1) is regular, and hence it is also impulse-free by Lemma 1.

**Part II:** Taking the derivative of $V(t,x(t))$ with respect to $t$ along the trajectory of system(1), we have

$$\dot{V}(t,x(t)) = x^\top(t)(A^\top G(t)E + E^\top G(t)A - E^\top QE)x(t). \tag{8}$$

Noting that $\bar{M}E = 0$ and multiplying both sides of (1) by $-2x^\top(t)F(t)\bar{M}$ from the right, where $F(t) = R - tW$ for $t \in [0,T]$ and $W = (R-\tilde{R})/T$, we get

$$-2x^\top(t)F(t)\bar{M}Ax(t) = 0. \tag{9}$$

Combining (8) and (9) gives

$$\dot{V}(t,x(t)) = x^\top(t)\Omega(t)x(t), \tag{10}$$

where

$$\begin{aligned}\Omega(t) &= A^\top G(t)E + E^\top G(t)A - E^\top QE\\&\quad - F(t)\bar{M}A - A^\top \bar{M}^\top F^\top(t).\end{aligned}$$

Next we show that $\Omega(t) < 0$ for $t \in [0,T]$. Denote $H(t) = \xi^\top \Omega(t)\xi$, $t \in [0,T]$, $\xi \in R^n$. It is sufficient to verify that $H(t) < 0$ for $t \in [0,T]$, $\xi \in R^n$ and $\xi \neq 0$. For any $\xi \in R^n$ and $\xi \neq 0$, it is easy to see that $H(t)$ is monotone. Consequently, $H(t) < 0$ for $t \in [0,T]$ if $\Omega(0) = X_1 < 0$ and $\Omega(T) = X_2 < 0$, which is an immediate result of conditions (3) and (4). It implies that $\dot{V}(t,x(t)) \leq 0$ for $t \in [0,T]$. On the other hand, similar to the above analysis, we get from (5) that $G(t) > U$ for $t \in [0,T]$. This together with (6) and (7) yields that

$$\begin{aligned}x^\top(t)E^\top UEx(t) &\leq V(t,x(t))\\&\leq x^\top(0)E^\top G(0)Ex(0)\\&= x^\top(0)E^\top PEx(0)\\&\leq \alpha x^\top(0)E^\top UEx(0)\\&\leq \alpha c_1 \leq c_2,\ \ t \in [0,T],\end{aligned}$$

i.e., system (1) is finite-time stable with respect to $(T,U,c_1,c_2)$. This completes the proof of Theorem 1.

**Remark 1:** Although a time-dependent Lyapunov function defined by (1) has been used in the proof of Theorem 1, the given conditions are explicit linear matrix inequalities, which can be verified easily by using the LMI Toolbox in Matlab.

**Remark 2:** In the proof of Theorem 1, we introduce a zero term $F(t)\bar{M}E \equiv 0$ for $t \in [0,T]$, where the time-varying matrix $F(t) = R - t(R-\tilde{R})/T$ for certain free weighting matrices $R$ and $\tilde{R}$. Due to the introduction of the free weighting matrices $R$ and $\tilde{R}$ in (3) and (4), Theorem 1 is less conservative than some existing results in the literature. For details, please see Example 1 given in Section 4.

Next, we further present the following $H_\infty$ finite-time stability criterion for system (1).

**Theorem 2:** Given positive scalars $T$, $c_1 < c_2$, $\gamma$ and a positive definite matrix $U$, system (1) is $H_\infty$ finite-time stable with respect to $(T, U, c_1, c_2, \gamma)$ if there exist a positive constant $\alpha > 1$, a positive definite matrix $P$, a symmetric matrix $Q$ and two free weighting matrices $R$ and $\tilde{R}$ such that

$$\begin{pmatrix} X_3 & X_4\\ X_4^\top & D^\top D - \gamma I\end{pmatrix} < 0, \tag{11}$$

$$\begin{pmatrix} X_5 & X_6\\ X_6^\top & D^\top D - \gamma I\end{pmatrix} < 0, \tag{12}$$

and conditions (5)-(7) holds, where

$$\begin{aligned}X_3 &= A^\top PE + E^\top PA - E^\top QE + C^\top C\\&\quad - R\bar{M}A - A^\top \bar{M}^\top R^\top,\\X_4 &= E^\top PB + C^\top D - R\bar{M}B,\\X_5 &= A^\top (P-TQ)E + E^\top (P-TQ)A + C^\top C\\&\quad - E^\top QE - \tilde{R}\bar{M}A - A^\top \bar{M}^\top \tilde{R}^\top,\\X_6 &= E^\top (P-TQ)B + C^\top D - \tilde{R}\bar{M}B.\end{aligned}$$

**Proof:** The proof will be divided into the following two parts:

**Part I:** We first show that system (1) is regular and impulse-free. From (11), we get

$$0 > A^\top PE + E^\top PA + C^\top C - E^\top QE - R\bar{M}A - A^\top \bar{M}^\top R^\top.$$

Since $C^\top C \geq 0$ and $N$ is nonsingular, it yields that $A^\top PE + E^\top PA - E^\top QE - R\bar{M}A - A^\top \bar{M}^\top R^\top < 0$. The remaining proof is similar to that given in Theorem 1, and hence it is omitted.

**Part II:** Taking the derivative of $V(t,x(t))$ with respect to $t$ along the trajectory of system (1), we have

$$\begin{aligned}\dot{V}(t,x(t)) =& x^\top(t)A^\top G(t)Ex(t) - x^\top(t)E^\top QEx(t)\\&+ w^\top(t)B^\top G(t)Ex(t)\end{aligned}$$

$$+x^\top(t)E^\top G(t)Ax(t)$$
$$+x^\top(t)E^\top G(t)Bw(t). \tag{13}$$

Set

$$\Phi(t) = \dot{V}(t,x(t)) + y^\top(t)y(t) - \gamma w^\top(t)w(t). \tag{14}$$

From (13) and (14), we have

$$\Phi(t) = h^\top(t)\Psi(t)h(t), \tag{15}$$

where

$$h(t) = (x^\top(t), w^\top(t))^\top,$$
$$\Psi(t) = \begin{pmatrix} \Psi_1(t) & E^\top G(t)B + C^\top D \\ B^\top G(t)E + D^\top C & D^\top D - \gamma I \end{pmatrix},$$
$$\Psi_1(t) = A^\top G(t)E + E^\top G(t)A + C^\top C - E^\top QE.$$

Noticing $\bar{M}E = 0$ and multiplying both sides of (1) by $-2x^\top(t)F(t)\bar{M}$ from the right, where $F(t) = R - tW$ for $t \in [0,T]$ and $W = (R - \tilde{R})/T$, we get

$$-2x^\top(t)[F(t)\bar{M}Ax(t) + F(t)\bar{M}Bw(t)] = 0. \tag{16}$$

Combining (15) and (16) gives $\Phi(t) = h^\top(t)\Omega(t)h(t)$, where

$$\Omega(t) = \begin{pmatrix} \Omega_1(t) & \Omega_2(t) \\ \Omega_2^\top(t) & D^\top D - \gamma I \end{pmatrix},$$

and

$$\Omega_1(t) = A^\top G(t)E + E^\top G(t)A + C^\top C - E^\top QE$$
$$- F(t)\bar{M}A - A^\top \bar{M}^\top F^\top(t),$$
$$\Omega_2(t) = E^\top G(t)B + C^\top D - F(t)\bar{M}B.$$

Next, we show that $\Omega(t) < 0$ for $t \in [0,T]$, which can be derived from conditions (11) and (12) by following the same analysis in the proof of Theorem 1. It implies that $\Phi(t) \le 0$ for $t \in [0,T]$. Since condition (5) guarantees that $V(t,x)$ is nonnegative, an integration from 0 to $T$ under the zero-initial condition yields

$$\int_0^T y^\top(s)y(s)ds \le \gamma \int_0^T w^\top(s)w(s)ds.$$

On the other hand, it is obvious that conditions (11) and (12) imply that conditions (3) and (4). By using Theorem 1, system (1) with $w(t) \equiv 0$ is finite-time stable with respect to $(T,U,c_1,c_2)$. Therefore, system (1) is $H_\infty$ finite-time stable with respect to $(T,U,c_1,c_2,\gamma)$. This completes the proof of Theorem 2.

### 3.2. The FTS of discrete time singular systems

For system (2), we construct the following time-dependent Lyapunov function

$$V(k,x(k)) = x^\top(k)E^\top G(k)Ex(k),$$

where $G(k) = P - kQ$ for $k = 0,1,\cdots,T$, $P$ and $Q$ are appropriate constant matrices.

We first establish an explicit finite-time stability criterion for system (2).

**Theorem 3:** Given positive scalars $T, c_1, c_2$ and a positive definite matrix $U$, system (1) with $w(k) \equiv 0$ is finite-time stable with respect to $(T,U,c_1,c_2)$ if there exist a positive constant $\alpha > 1$, a positive definite matrix $P$, a symmetric matrix $Q$ and two free weighting matrices $R$ and $\tilde{R}$ such that

$$A^\top(P-Q)A - E^\top PE - R\bar{M}A - A^\top\bar{M}^\top R^\top < 0, \tag{17}$$
$$X_1 < 0, \tag{18}$$
$$U < P, \quad U < P - TQ, \tag{19}$$
$$P < \alpha U, \tag{20}$$
$$\alpha c_1 < c_2, \tag{21}$$

where $X_1 = A^\top(P-TQ)A - E^\top(P-(T-1)Q)E - \tilde{R}\bar{M}A - A^\top\bar{M}^\top\tilde{R}^\top$.

**Proof:** The proof will be divided into the following three parts:

**Part I:** Condition (19) implies that $P - Q > 0$, and hence $A^\top(P-Q)A \ge 0$ (see the analysis given in the sequel Part III). Since $N$ is nonsingular, we get from (17) that

$$\begin{aligned}
0 >& N^\top(-E^\top PE - R\bar{M}A - A^\top\bar{M}^\top R^\top) \\
=& -N^\top E^\top M^\top(M^\top)^{-1}PM^{-1}MEN \\
& - N^\top R\bar{M}AN - N^\top A^\top\bar{M}^\top R^\top N \\
=& -\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \\
& - \begin{pmatrix} (N^\top R)_{11} & (N^\top R)_{12} \\ (N^\top R)_{21} & (N^\top R)_{22} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ A_{21} & A_{22} \end{pmatrix} \\
& - \begin{pmatrix} 0 & A_{21}^\top \\ 0 & A_{22}^\top \end{pmatrix}\begin{pmatrix} (N^\top R)_{11} & (N^\top R)_{12} \\ (N^\top R)_{21} & (N^\top R)_{22} \end{pmatrix}^\top \\
=& -\begin{pmatrix} (N^\top R)_{12}A_{21} & (N^\top R)_{12}A_{22} \\ (N^\top R)_{22}A_{21} & (N^\top R)_{22}A_{22} \end{pmatrix} \\
& - \begin{pmatrix} (N^\top R)_{12}A_{21} & (N^\top R)_{12}A_{22} \\ (N^\top R)_{22}A_{21} & (N^\top R)_{22}A_{22} \end{pmatrix}^\top \\
& - \begin{pmatrix} P_{11} & 0 \\ 0 & 0 \end{pmatrix},
\end{aligned}$$

which implies $(N^\top R)_{22}A_{22} + A_{22}^\top(N^\top R)_{22}^\top > 0$. Therefore, $A_{22}$ is nonsingular. That is, system (2) is regular, and hence it is also causal by Lemma 1.

**Part II:** Let $\Delta V$ be the difference of $V(\cdot)$. We have

$$\begin{aligned}
\Delta V =& x^\top(k+1)E^\top G(k+1)Ex(k+1) \\
& - x^\top(k)E^\top G(k)Ex(k) \\
=& x^\top(k)(A^\top G(k+1)A - E^\top G(k)E)x(k).
\end{aligned} \tag{22}$$

Noting that $\bar{M}E=0$ and multiplying both sides of (1) by $-2x^\top(k)F(k)\bar{M}$ from the right, where $F(k)=R-kW$ for $k=0,1,\cdots,T-1$ and $W=(R-\tilde{R})/(T-1)$, we get

$$-2x^\top(k)F(k)\bar{M}Ax(k)=0. \tag{23}$$

Combining this and conditions (22) and (23) gives

$$V(x(k+1))-V(x(k))=x^\top(k)\Omega(k)x(k),$$

where

$$\begin{aligned}\Omega(k)=&A^\top G(k+1)A-E^\top G(k)E\\ &-F(k)\bar{M}A-A^\top\bar{M}^\top F^\top(k).\end{aligned}$$

Next we show that $\Omega(k)<0$ for $k=0,1,\cdots,T-1$. Denote $H(k)=\xi^\top\Omega(k)\xi$, $k=0,1,\cdots,T-1$, $\xi\in R^n$. It is sufficient to verify that $H(k)<0$ for $k=0,1,\cdots,T-1$, $\xi\in R^n$ and $\xi\neq 0$. For any $\xi\in R^n$ and $\xi\neq 0$, it is easy to see that $H(k)$ is monotone. Consequently, $H(k)<0$ for $k=0,1,\cdots,T-1$ if

$$\begin{aligned}\Omega(0)=&A^\top(P-Q)A-E^\top PE-R\bar{M}A-A^\top\bar{M}^\top R^\top\\ &<0,\end{aligned}$$

and $\Omega(T-1)=X_1<0$, which is an immediate result of conditions (17) and (18).

**Part III:** Similar to the same discussion in Part II, we can obtain from condition (19) that $G(k)>U$ for $k=0,1,\cdots,T$. Therefore, conditions (19)-(21) imply that

$$\begin{aligned}x^\top(k)E^\top UEx(k)\le& V(x(k))\\ \le& x^\top(0)E^\top PEx(0)\\ \le& \alpha x^\top(0)E^\top UEx(0)\\ \le& \alpha c_1\\ \le& c_2,\quad k=0,1,\cdots,T,\end{aligned}$$

i.e., system (2) is finite-time stable with respect to $(T,U,c_1,c_2)$. This completes the proof of Theorem 3.

Next, we further present the following $H_\infty$ finite-time stability criterion for system (2).

**Theorem 4:** Given positive scalars $T,c_1,c_2,\gamma$ and a positive definite matrix $U$, system (2) is $H_\infty$ finite-time stable with respect to $(T,U,c_1,c_2,\gamma)$ if there exist a positive constant $\alpha>1$, a positive definite matrix $P$, a symmetric matrix $Q$ and two free weighting matrices $R$ and $\tilde{R}$ such that

$$\begin{pmatrix} X_2 & X_3^\top\\ X_3 & B^\top(P-Q)B+D^\top D-\gamma I\end{pmatrix}<0, \tag{24}$$

$$\begin{pmatrix} X_4 & X_5^\top\\ X_5 & B^\top(P-TQ)B+D^\top D-\gamma I\end{pmatrix}<0, \tag{25}$$

and conditions (19)-(21) holds, where

$$\begin{aligned}X_2=&A^\top(P-Q)A+C^\top C-E^\top PE-R\bar{M}A\\ &-A^\top\bar{M}^\top R^\top,\\ X_3=&B^\top(P-Q)A+D^\top C-B^\top\bar{M}^\top R^\top,\\ X_4=&A^\top(P-TQ)A+C^\top C-\tilde{R}\bar{M}A\\ &-E^\top(P-(T-1)Q)E-A^\top\bar{M}^\top\tilde{R}^\top,\\ X_5=&B^\top(P-TQ)A+D^\top C-B^\top\bar{M}^\top\tilde{R}^\top.\end{aligned}$$

**Proof:** The proof will be divided into the following two parts:

**Part I:** We first show that system (2) is regular and causal. From (24), we get

$$A^\top PA+C^\top C-E^\top PE-R\bar{M}A-A^\top\bar{M}^\top R^\top<0.$$

Since $A^\top PA\ge 0$, $C^\top C\ge 0$ and $N$ is nonsingular, it yields that $-E^\top PE-R\bar{M}A-A^\top\bar{M}^\top R^\top<0$. The remain proof is similar to that given in Theorem 1, and hence it is omitted.

**Part II:** Let $\Delta V$ be the difference variation of $V(\cdot)$. Then, we have

$$\begin{aligned}\Delta V=&x^\top(k+1)E^\top G(k+1)Ex(k+1)\\ &-x^\top(k)E^\top G(k)Ex(k)\\ =&x^\top(k)A^\top G(k+1)Ax(k)\\ &+2x^\top(k)A^\top G(k+1)Bw(k)\\ &+w^\top(k)B^\top G(k+1)Bw(k)\\ &-x^\top(k)E^\top G(k)Ex(k).\end{aligned} \tag{26}$$

Set

$$\begin{aligned}\Phi(k)=&V(x(k+1))-V(x(k))+y^\top(k)y(k)\\ &-\gamma w^\top(k)w(k),\quad k=0,1,\cdots,T-1.\end{aligned} \tag{27}$$

From (26) and (27), we have

$$\Phi(k)=\begin{pmatrix}x^\top(k) & w^\top(k)\end{pmatrix}\Psi(k)\begin{pmatrix}x(k)\\ w(k)\end{pmatrix}, \tag{28}$$

where

$$\begin{aligned}\Psi(k)=&\begin{pmatrix}\Psi_1(k) & \Psi_2^\top(k)\\ \Psi_2(k) & B^\top G(k+1)B+D^\top D-\gamma I\end{pmatrix},\\ \Psi_1(k)=&A^\top G(k+1)A-E^\top G(k)E+C^\top C,\\ \Psi_2(k)=&B^\top G(k+1)A+D^\top C.\end{aligned}$$

Since $\bar{M}E=0$, multiplying both sides of (2) by $-2x^\top(k)F(k)\bar{M}$ from the right, where $F(k)=R-kW$ is defined as in the proof of Theorem 1, we get

$$-2x^\top(k)[F(k)\bar{M}Ax(k)+F(k)\bar{M}Bw(k)]=0. \tag{29}$$

Combining this and conditions (28) and (29) gives

$$\Phi(k)=\begin{pmatrix}x^\top(k) & w^\top(k)\end{pmatrix}\Omega(k)\begin{pmatrix}x(k)\\ w(k)\end{pmatrix}, \tag{30}$$

where

$$\Omega(k) = \begin{pmatrix} \Omega_1(k) & \Omega_2^\top(k) \\ \Omega_2(k) & B^\top G(k+1)B + D^\top D - \gamma I \end{pmatrix},$$

$$\Omega_1(k) = A^\top G(k+1)A + C^\top C - E^\top G(k)E$$

$$- F(k)\bar{M}A - A^\top \bar{M}^\top F^\top(k),$$

$$\Omega_2(k) = B^\top G(k+1)A + D^\top C - B^\top \bar{M}^\top F^\top(k).$$

We can show that $\Omega(k) < 0$ for $k = 0, 1, \ldots, T-1$, which can be derived from conditions (24) and (25) by following the proof of Theorem 3. Consequently, summing $\Phi(k)$ from $k = 0$ to $T$, under the zero-initial condition, it holds $\sum_{k=0}^{N} y^\top(k)y(k) \le \gamma \sum_{k=0}^{N} w^\top(k)w(k)$. On the other hand, it is obvious that conditions (24) and (25) imply that conditions (17) and (18). By using Theorem 3, system (2) with $w(k) \equiv 0$ is finite-time stable with respect to $(T, U, c_1, c_2)$. Therefore, system (2) is $H_\infty$ finite-time stable with respect to $(T, U, c_1, c_2, \gamma)$. This completes the proof of Theorem 4.

## 4. EXAMPLES

**Example 1:** Consider the singular system (1) with $w(t) \equiv 0$, $E = \begin{pmatrix} 1 & -0.9 \\ 0 & 0 \end{pmatrix}$ and $A = \begin{pmatrix} 0.2 & 0.6 \\ 0.4 & 1 \end{pmatrix}$. Let $U$ be an identity matrix, $T = 50$. Choose $\alpha = 1.01, c_1 = 0.1, c_2 = 0.102$. By a simple algebraic computation, we can find

$$M = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}, \quad \bar{M} = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}.$$

Using the LMI toolbox of Matlab to solve inequalities (3)-(7), we get that they are feasible with appropriate matrices $P$, $Q$, $R$ and $\tilde{R}$. According to Theorem 1, system (1) is finite-time stable with respect to $(T, U, c_1, c_2)$. The state trajectory of system (1) under the initial condition $x(0) = (0.51, -0.2)^\top$ and the system response from [0,50] are shown in Figs. 1 and 2. However, when applying Theorem 1 in [19] and [31] to the above example, we find it is infeasible. Although Theorem 1 in [33] can be applied to this example, it yields that $c_2 = 0.7$ which is conservative than the estimation given by our result. The system response from 0 to 50s are shown in Fig. 1.

**Example 2:** Consider the singular system (1), where

$$E = \begin{pmatrix} -1 & 1.5 \\ 0 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 0.1 & 0.01 \\ 0.02 & 0.41 \end{pmatrix},$$

$$B = \begin{pmatrix} 0.12 & 0.05 \\ 0.04 & 0.02 \end{pmatrix}, \quad C = \begin{pmatrix} 0.1 & 0.02 \\ 0.03 & 0.12 \end{pmatrix},$$

and $D = \begin{pmatrix} 0.02 & 0.2 \\ 0.03 & 0.02 \end{pmatrix}$. Let $U$ be an identity matrix, $T = 50$. Choose $\alpha = 1.71$, $c_1 = 0.5$, $c_2 = 0.86, \gamma = 0.1$.

Using the LMI toolbox of Matlab to solve inequalities (5)-(7), (11) and (12), we obtain that they are feasible with

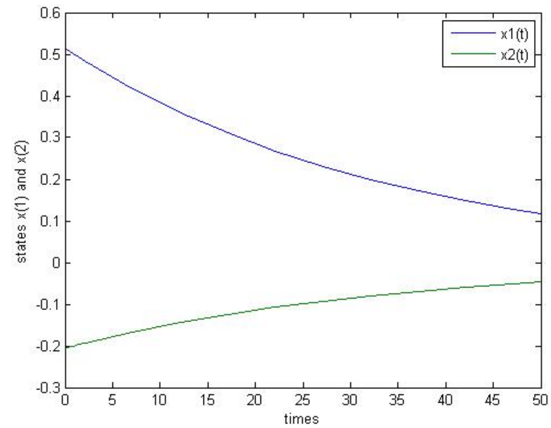


Fig. 1. The state trajectory of system (1).

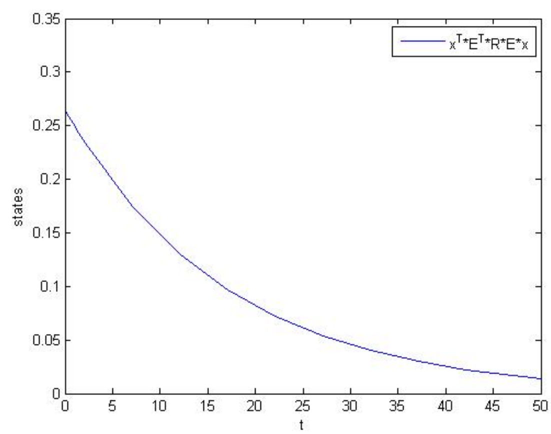


Fig. 2. The system response from 0 to 50 s.

appropriate matrices $P$, $Q$, $R$ and $\tilde{R}$. According to Theorem 2, system (1) is $H_\infty$ finite-time stable with respect to $(T, U, c_1, c_2, \gamma)$.

**Example 3:** Consider the discrete-time singular system (2) with $w(k) \equiv 0$ $E = \begin{pmatrix} 1 & -1.1 \\ 0 & 0 \end{pmatrix}$ and $A = \begin{pmatrix} 0.1 & 0.05 \\ 0.2 & 1 \end{pmatrix}$. Let $U$ be an identity matrix and $T = 100$.

Choose $\alpha = 2$, $c_1 = 0.1$, $c_2 = 0.21$. Using the LMI toolbox of Matlab to solve inequalities (17)-(21), we get that they are feasible with appropriate matrices $P$, $Q$, $R$ and $\tilde{R}$. Then Theorem 3 implies that system (2) with $w(k) \equiv 0$ is finite-time stable with respect to $(T, U, c_1, c_2)$. However, when applying Theorem 7 in [28] to the above example, we find it is infeasible.

**Example 4:** Consider the discrete-time singular system (2) with

$$E = \begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 0.3 & 0.1 \\ 1 & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} 0.3 & 0.04 \\ 0.3 & 0.2 \end{pmatrix}, \quad C = \begin{pmatrix} 0.02 & 0.01 \\ 0.33 & 0.01 \end{pmatrix},$$

and $D = \begin{pmatrix} 0.04 & 0.2 \\ 0.14 & 0.03 \end{pmatrix}$. Let $U$ be an identity matrix and $T = 100$.

Choose $\alpha = 1.5$, $c_1 = 0.2$, $c_2 = 0.31$, and $\gamma = 3$. Using the LMI toolbox of Matlab, we find that inequalities (19)-(21), (24) and (25), are feasible with appropriate matrices $P$, $Q$, $R$, and $\tilde{R}$. According to Theorem 4, the system is $H_\infty$ finite-time stable with respect to $(T, U, c_1, c_2, \gamma)$.

## 5. CONCLUSION

In this paper, the finite time stability and $H_\infty$ finite-time stability of singular linear systems are studied. By introducing a class of time-dependent Lyapunov functions and a zero term with free weighting matrices, sufficient conditions for finite-time stability and $H_\infty$ finite-time stability of the system are given in terms linear matrix inequalities. Finally, four numerical examples are given to verify the validity of our results. For finite-time stability of singular linear time-delay systems, it remains for further study.

## REFERENCES

[1] Z. Wu, H. Su, and J. Chu, "Robust stabilization for uncertain discrete singular systems with state delay," *International Journal of Robust and Nonlinear Control*, vol. 18, no. 16, pp. 1532-1550, 2008.

[2] B. Zhang, S. Xu, Q. Ma, and Z. Zhang, "Output-feedback stabilization of singular LPV systems subject to inexact scheduling parameters," *Automatica*, vol. 104, pp. 1-7, 2019.

[3] J. E. Feng, J. Lam, and S. Xu, "Finite-time functional observers for descriptor systems," *International Journal of Control, Automation, and Systems*, vol. 7, no. 3, pp. 341-347, 2009.

[4] L. Hao and X. Zhao, "Finite-time $H_\infty$ control of switched systems with mode-dependent average dwell time," *Journal of the Franklin Institute*, vol. 351, no. 3, pp. 1301-1315, 2014.

[5] G. Zhuang, S. Xu, B. Zhang, J. Xia, Y. Chu, and Y. Zou, "Unified filters design for singular Markovian jump systems with time-varying delays," *Journal of the Franklin Institute*, vol. 353, no. 15, pp. 3739-3768, 2016.

[6] N. Chaibi and E. H. Tissir, "Delay dependent robust stability of singular systems with time-varying delay," *International Journal of Control, Automation, and Systems*, vol. 10, no. 3, pp. 632-638, 2012.

[7] S. Xu and J. Lam, *Robust Control and Filtering of Singular Systems*, Springer, Berlin, 2006.

[8] J. Zhao, Z. Hu, and L. Zhang, "Stability and stabilization for discrete-time singular systems with infinite distributed delays and actuator failures," *International Journal of Control, Automation, and Systems*, vol. 10, no. 4, pp. 721-726, 2012.

[9] G. Zhuang, J. Xia, J. E. Feng, B. Zhang, J. Lu, and Z. Wang, "Admissibility analysis and stabilization for neutral descriptor hybrid systems with time-varying delays," *Nonlinear Analysis Hybrid Systems*, vol. 33, pp. 311-321, 2019.

[10] S. B. Stojanovic, "New results for finite-time stability of discrete-time linear systems with interval time-varying delay," *Discrete Dynamics in Nature and Society*, vol. 2015, pp. 1-15, 2015.

[11] S. Xu, J. Lam, B. Zhang, and Y. Zou, "A new result on the delay-dependent stability of discrete systems with time-varying delays," *International Journal of Robust and Nonlinear Control*, vol. 24, no. 16, pp. 2512-2521, 2014.

[12] T. Yu, J. Cao, and C. Huang, "Finite-time cluster synchronization of coupled dynamical systems with impulsive effects," *Discrete and Continuous Dynamical Systems - B*, vol. 22, no. 11, 2017.

[13] S. B. Stojanovic, D. L. Debeljkovic, and D. S. Antic, "Robust finite-time stability and stabilization of linear uncertain time-delay systems," *Asian Journal of Control*, vol. 15, no. 5, pp. 1548-1554, 2013.

[14] T. Huang and Y. Sun, "New criteria for finite-time stability of linear time-delay systems," *Proc. of International Conference on Control, Automation and Systems*, pp. 1137-1140, 2019.

[15] F. Amato, G. Carannante, G. D. Tommasi, and A. Pironti, "Input-output finite-time stability of linear systems: Necessary and sufficient conditions," *IEEE Transactions on Automatic Control*, vol. 57, no. 12, pp. 3051-3063, 2012.

[16] Z. Zhang, Z. Zhang, and H. Zhang, "Finite-time stability analysis and stabilization for uncertain continuous-time system with time-varying delay," *Journal of The Franklin Institute-engineering and Applied Mathematics*, vol. 352, no. 3, pp. 1296-1317, 2015.

[17] S. B. Stojanovic, D. L. Debeljkovic, and M. A. Misic, "Finite-time stability for a linear discrete-time delay systems by using discrete convolution: An LMI approach," *International Journal of Control, Automation, and Systems*, vol. 14, no. 4, pp. 1144-1151, 2016.

[18] N. H. Muoi, G. Rajchakit, and V. N. Phat, "LMI Approach to finite-time stability and stabilization of singular linear discrete delay systems," *Acta Applicandae Mathematicae*, vol. 146, no. 1, pp. 81-93, 2016.

[19] S. B. Stojanovic, D. L. Debeljkovic, and D. Antic, "Finite-time stability and stabilization of singular state-delay systems using improved estimation of a lower bound on a Lyapunov-like functional," *Bulletin of the Polish Academy of Sciences-technical Sciences*, vol. 63, no. 2, pp. 479-487, 2015.

[20] W. Xue and W. Mao, "Admissible finite-time stability and stabilization of uncertain discrete singular systems," *Journal of Dynamic Systems Measurement and Control-Transactions of the ASME*, vol. 135, no. 3, pp. 18-31, 2013.

[21] J. Yao, J. E. Feng, L. Sun, and Y. Zheng, "Input-output finite-time stability of time-varying linear singular systems," *Journal of Control Theory and Technology*, vol. 10, no. 3, pp. 287-291, 2012.

[22] Q. Ma and Y. Sun, "$H_\infty$ finite time stability for discrete-time singular linear time-varying systems," *Proc. of International Conference on Control, Automation and Systems*, pp. 885-888, 2019.

[23] R. Yang, L. Sun, G. Zhang, and Q. Zhang, "Finite-time stability and stabilization of nonlinear singular time-delay systems via Hamiltonian method," *Journal of the Franklin Institute*, vol. 356, no. 12, pp. 5961-5992, 2019.

[24] F. Amato, M. Ariola, and P. Dorato, "Finite-time control of linear systems subject to parametric uncertainties and disturbances," *Automatica*, vol. 37, no. 9, pp. 1459-1463, 2001.

[25] F. Amato and M. Ariola, "Finite-time control of discrete-time linear systems," *IEEE Transactions on Automatic Control*, vol. 50, no. 5, pp. 724-729, 2005.

[26] Z. Lin, Y. Shen, and C. Li, "Finite-time control of discrete-time systems with time-varying exogenous disturbance," *Communications in Nonlinear Science and Numerical Simulation*, vol. 14, no. 2, pp. 361-370, 2009.

[27] P. Niamsup and V. N. Phat, "A new result on finite-time control of singular linear time-delay systems," *Applied Mathematics Letters*, vol. 60, pp. 1-7, 2016.

[28] S. Wo and X. Han, "Finite-time stability analysis of discrete-time linear singular systems," *Mathematical Problems in Engineering*, vol. 2014, pp. 1-6, 2014.

[29] N. T. Thanh, P. Niamsup, and V. N. Phat, "Finite-time stability of singular nonlinear switched time-delay systems: A singular value decomposition approach," *Journal of the Franklin Institute*, vol. 354, no. 8, pp. 3502-3518, 2017.

[30] F. Amato, C. Cosentino, and A. Merola, "Sufficient conditions for finite-time stability and stabilization of nonlinear quadratic systems," *IEEE Transactions on Automatic Control*, vol. 55, no. 2, pp. 430-434, 2010.

[31] S. Zhao, J. Sun, and L. Liu, "Finite-time stability of linear time-varying singular systems with impulsive effects," *International Journal of Control*, vol. 81, no. 11, pp. 1824-1829, 2008.

[32] X. Yang, X. Li, and J. Cao, "Robust finite-time stability of singular nonlinear systems with interval time-varying delay," *Journal of the Franklin Institute*, vol. 355, no. 3, pp. 1241-1258, 2018.

[33] S. Wo and X. Han, "Finite-time stability for continuous-time linear singular systems," *Advanced Materials Research*, pp. 383-387, 2013.

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