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A Practical Method for Stability Analysis of Linear Fractional-order Systems with Distributed Delay

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Abstract: In this paper, an effective method using the cluster treatment of characteristic roots (CTCR) technique is investigated for the stability analysis of a general class of fractional order systems (FOSs) with distributed delay. To conclude this goal, the characteristic equation of a FOSs with distributed delay is transformed to the characteristic equation of a FOSs with multiple delays; it is shown that the stability analyses of these two systems are equivalent. The magnitude of both delays, are considered to have non-zero values so that a comprehensive analysis is performed in the parametric space of delays. For obtaining stability switch curves also the procedure advanced clustering with frequency sweeping (ACFS) method is used. The proposed method of this article determines the stability map of such systems in the parametric space of delays accurately. The significance of this proposed method is in that, a comprehensive and precise stability analysis of such systems is not presented in the literature yet and this article is the first attempt to solve this challenging problem. The practicality and effectiveness of this method is shown here with an illustrative example.

Keywords: Distributed delay, fractional order systems, stability analysis, time delay.

1. INTRODUCTION

As the transfer of mass, energy and information in the natural systems does not happen instantaneously; the existence of delay in systems is undeniable. This leads also to delay in engineering systems like hydraulic systems, production processes and other systems quite often. The existence of delay in systems can cause instability and weak performances, and hence a great attention has been paid to time-delay systems (TDSs) [1–4]. A correct understanding of delay effects on stability of a system is essential for both modeling and control of this type of systems [5, 6].

Time-delayed systems (TDSs) can be subdivided to single-delay systems, multiple time-delay systems (MTDSs) and distributed time-delay systems (DTDSs). Stability analysis of MTDSs is usually much simpler than that of DTDSs. Using a theory that is described in the next section, a given DTDSs is transformed into an equivalent MTDSs and then we will analyze its stability. Study of time-delay systems can be performed in two major branches, i.e., in time-domain-based (TDB) or in frequency domain based (FDB) approaches. Time-domain based methods use Lyapunov-Krasovskii or Lyapunov-Razumikhin functions in general, and express the stability criteria in the form of linear matrix inequalities (LMIs) [7–9]. Frequency domain methods have better performances in stability analysis of linear time-invariant (LTI) systems and can determine the stability map of this class of systems precisely [10]. Frequency based methods determine all stability regions, number of unstable roots in each delay interval and tendency of roots as well. FDB technique is for example proposed in [11] for analysis of TDSs and is developed in [12] for fractional order delay systems. In [13, 14], another FDB technique is proposed for precise stability analysis of a class of MTDSS systems that is capable of determining stable and unstable regions in parametric space of delays. In the LTI integer order systems (IOSs) with distributed delay, some useful time-domain methods as well as frequency-domain methods have been presented with the purpose of evaluating the stability of these dynamics [15, 16]. Regarding the fractional-order systems (FOSs) with distributed delay, due to the involvement of fractional mathematics in these problems, the stability analysis of these systems will be much more complicated than the integer order systems [17].

One of the most important methods for analysis in the frequency domain is the CTCR method, which is introduced for stability analysis of LTI-TDSs [18]. The CTCR method is based on two important facts about LTI systems [19]. First is that, roots of a characteristic equation changes continuously with delay and the second fact says that if a system is going toward instability, some poles of

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the system must move from the left side of the imaginary axis to the right, therefore the roots must cross the general axis inevitably [20]. This method can extract the stability regions of single delay systems in the parametric space of delays and these stability regions are called stability windows or stability pockets [21]. For multiple delay systems, stability regions can be determined and presented in 2D and 3D spaces. This method is used for determination of stability map of DTDSs in [15].

The stability analysis of FOSs is more complicated than that of integer-order systems [22], especially when they are concerned with delays. To the best of our knowledge, evaluation of CTCR method for stability analysis of fractional order distributed delay systems has not already been presented in the literature, and this study is the first effort to develop such a method for this class of systems. This work aims to present a practical method analyzing the stability of fractional LTI systems with distributed delays. The introduced method determines stable and unstable regions in a delay parametric space, and unstable roots are determined in each unstable region. Differently from present literature, both limits of the delay are taken as nonzero, which brings an interesting new perspective to the problem. Most of these earlier investigations have recognized an equivalence of distributed-delay systems to some form of discrete (lumped) delay structures. The delay distribution in these reports is almost exclusively in the interval of $[0, \tau_{max}]$. In this paper, we introduce another freedom by considering the non-zero lower bound, τ_1, of the delays. The stability question of this class of distributed-delay systems is shown to be equivalent to determining the stable regions in the domain of the two delays involved.

2. PRELIMINARIES AND DEFINITIONS

The most common definition of the fractional derivative is given in the sense of Caputo [23], and when the lower terminal is set to zero, it is defined as

$$D_t^{\alpha}x(t) = \frac{1}{\Gamma(m-\alpha)}\int_0^t \frac{x^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}}d\tau,$$
$$m-1 \le \alpha \le m, \tag{1}$$

where m is the first integer that is larger than α, and $\Gamma(\cdot)$ is the gamma function

$$\Gamma(\alpha) = \int_0^t t^{\alpha-1}\, e^{-t}dt. \tag{2}$$

The Laplace transform of the Caputo fractional derivative is

$$L\left\{D_t^{\alpha}x(t)\right\} = s^{\alpha}L\{x(t)\} - \sum_{k=0}^{m-1} s^{\alpha-k-1}x^{(k)}(0). \tag{3}$$

Consider the following LTI fractional distributed time-delay system (FDTDS) with a single delay:

$$D_t^{\alpha}x(t) = Ax(t) + B\int_{\tau_1}^{\tau_2} x(t-\eta)d\eta, \tag{4}$$

where $x(t) \in \mathbb{R}^{n\times 1}$ is the state vector, $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times n}$ are constant system matrices, Parameters τ_1 and τ_2 are non-negative, such that $(\tau_1, \tau_2) \in \mathbb{R}_+^2$. Taking Laplace transform of the system (4) yields

$$s^{\alpha}X - AX - B\int_{\tau_1}^{\tau_2} Xe^{-\eta s}d\eta = 0. \tag{5}$$

Since the integral linear operator and Laplace transform are commutative, we have

$$[s^{\alpha}I - A - B\int_{\tau_1}^{\tau_2} e^{-\eta s}d\eta]X = 0. \tag{6}$$

Thus, the system's CE is obtained as

$$\mathrm{CE}_0(s,\tau_1,\tau_2) = \det[s^{\alpha}I - A + \frac{1}{s}B(e^{-\tau_2 s} - e^{-\tau_1 s})] = 0. \tag{7}$$

Theorem 1: The stability analysis of the FDTDS (4) is analogous to that of the following FOS with two discrete independent delays:

$$D_t^{\alpha+1}x - A\dot{x} + B[x(t-\tau_2) - x(t-\tau_1)] = 0. \tag{8}$$

Proof: Taking the Laplace transform of (8) yields its CE as

$$\mathrm{CE}_1(s,\tau_1,\tau_2) = \det[s^{\alpha+1}I - As + B(e^{-\tau_2 s} - e^{-\tau_1 s})] = 0. \tag{9}$$

Using the determinant properties, one can show

$$\mathrm{CE}_0(s,\tau_1,\tau_2) = s^n\mathrm{CE}_1(s,\tau_1,\tau_2).$$

Consequently, (9) contains all the roots of (7) along with n more roots at the origin, which can be disregarded for the stability analysis. □

Although the new form (9) resembles a simpler form of (5), it is challenging to solve because of the two independent delays. The stability analysis of (4) in delay parameter space requires to detect τ_1 and τ_2 that places the eigenvalues of (4) on the imaginary axis, i.e., when $s = \pm j\omega$ satisfies the corresponding CE for some appropriate delay values $\tau^* = (\tau_1^*, \tau_2^*)$. Some roots may cross the imaginary axis and cause stability switches at those delays. Hence, the delay parameter space is decomposed into stable and unstable regions. The parametric space of (4) is defined (τ_1, τ_2) space, and the CTCR and ACSF methods are used to determine the stability regions. The advanced clustering with the frequency sweeping method [24] serves as the first step to the umbrella paradigm called

"CTCR" to determine stability switching boundaries. To find stability switch curves we can use ACFS method by sweeping the frequency between the bounds of crossing frequency to the imaginary axis. Then, we deploy the cluster treatment of characteristic roots (CTCR) paradigm to reveal the exact and complete stability map. It is important to note that we completely adhere to the underlying guidelines of the CTCR paradigm. CTCR is constructed over two fundamental propositions. The first proposition claims the existence of a finite number of stability switching, where the only imaginary characteristic roots occur. The second proposition is on an interesting directional invariance property of the crossing tendencies of these imaginary roots [25].

The system's stability is initiated by determining the CE's pure imaginary roots and then calculating their root tendency. Let us rewrite CE (9) as

$$CE(s,\tau_1,\tau_2) = p_0(s^\alpha) + \sum_{\ell=1}^{m} p_\ell(s^\alpha)\ (e^{-\tau_2 s} - e^{-\tau_1 s})^\ell, \tag{10}$$

where p_0 and p_ℓ are real polynomials in the complex variable s^α with arbitrary order α. It is worthy to note identifying the pure imaginary roots of the dynamic system (10) is challenging, unless τ_1 and τ_2 delay terms are removed. Page length is mesured in two-coloumn format in this template. Over-length paper can be considered, but justification needs to be provided when the initial submission is made. Over-length charge will be applied for publication if accepted.

3. METHODOLOGY

As briefly explained in the previous section, one needs to determine all the imaginary axis crossings of (10) along with the time delays corresponding to these crossings. In this section, we obtain delays that cause crossing the roots from the imaginary axis. Then, the stability analysis is performed using the D-Subdivision method.

3.1. Pure imaginary roots

The following structured steps are performed to detect the imaginary axis crossing and their corresponding time delays:

1) The crossing are detected by Rekasius transformation [26]

$$e^{-\tau_\ell s} = \frac{1-\lambda_\ell s}{1+\lambda_\ell s}, \quad \lambda_\ell \in \mathbb{R}, \quad \ell = 1,2, \tag{11}$$

which is an exact expression of $e^{-\tau_l s}$ for the purely imaginary roots $s = \pm j\omega$. That is,

$$e^{-\tau_\ell s}\big|_{s=j\omega} = \frac{1-jT_\ell}{1+jT_\ell}, \quad T_\ell \in \mathbb{R}, \quad \ell = 1,2. \tag{12}$$

Furthermore, when $s = \pm j\omega$, the magnitudes of both sides of (12) is unique. Thus, it is necessitated that the phase condition be equal. In other words, the transformation is exactly held if and only if

$$\tau_\ell = \frac{2}{\omega}\left(\tan^{-1}(T_\ell) + k\pi\right), \quad k = 0,1,\ldots, \tag{13}$$

which describes an asymmetric mapping in which T_ℓ is distinct in general and is mapped into τ_ℓ sets.

2) Eliminate $e^{-\tau_1 s}$ and $e^{-\tau_2 s}$ from (10) with regard to the relation (12), forming a new CE, $CE(s,\tau_1,\tau_2)$, which is an equation only in T_1 and T_2

$$(e^{-\tau_2 s} - e^{-\tau_1 s}) = \frac{j2(T_1-T_2)}{(1+jT_1)(1+jT_2)}.$$

Then, we have

$$CE(s,T_1,T_2) = p_0(s^\alpha) + \sum_{\ell=1}^{m} p_\ell(s^\alpha)\left(\frac{j2(T_1-T_2)}{(1+jT_1)(1+jT_2)}\right)^\ell. \tag{14}$$

3) Multiplying (14) by $(1+jT_1)^m(1+jT_2)^m$, results in a new form

$$\begin{aligned} h(s,T_1,T_2) =& ((1+jT_1)(1+jT_2))^m CE(s,T_1,T_2) \\ =& (1+jT_1)^m(1+jT_2)^m p_0(s^\alpha) \\ &+ \sum_{\ell=1}^{m} p_\ell(s^\alpha)(j2)^\ell (T_1-T_2)^\ell \\ &\times (1+jT_1)^{m-\ell}(1+jT_2)^{m-\ell}. \end{aligned} \tag{15}$$

This expression is a polynomial in s^α of which the coefficients are functions of T_1 and T_2. As a result, CE (10) with transcendental term is converted into the algebraic equation (15). To determine the crossing frequencies from the imaginary axis in (10), we substitute $s = j\omega$ into (15), and then the real and imaginary parts are separated as

$$\begin{aligned} &h(s,T_1,T_2)|_{s=j\omega} \\ &= h_\Re(\omega,T_1,T_2) + jh_\Im(\omega,T_1,T_2) = 0, \end{aligned} \tag{16}$$

where $h_\Re$ and $h_\Im$ are the real and imaginary parts of h, respectively. Equation (16) is zero for a value of $s = j\omega$ if and only if

$$\begin{aligned} h_\Re &= \sum_{i=0}^{n} a_i(\omega,T_1)T_2^i = 0, \\ h_\Im &= \sum_{i=0}^{n} b_i(\omega,T_1)T_2^i = 0. \end{aligned} \tag{17}$$

We utilize the resultant theory to eliminate T_2 from the two multivariate polynomials $h_\Re$ and $h_\Im$.

Definition 1: Consider the two multivariate polynomials in (17) in terms of ω, T_1 and T_2 with real coefficients, where $h_{\Re}$ and $h_{\Im}$ have positive degrees in terms of T_2, and $n > 0$. The resultant of $h_{\Re}$ and $h_{\Im}$ with respect to T_2 is defined by

$$R_{T_2}(h_{\Re}, h_{\Im}) = \begin{vmatrix} a_n & a_{n-1} & \dots & \dots & a_0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & \vdots & \vdots & \vdots & a_1 & a_0 \\ b_n & b_{n-1} & \dots & \dots & b_0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & \vdots & \vdots & \vdots & b_1 & b_0 \end{vmatrix}, \tag{18}$$

which is the determinant of the well-known Sylvester matrix [27,28].

Definition 2: The resultant of R_{T_2} and $\partial R_{T_2}/\partial T_1$ with respect to ω is called the *discriminant* of R_{T_2} by eliminating T_1.

Theorem 2 [24]**:** Minimum and maximum positive real roots of the discriminant of resultant of $h_{\Re}$ and $h_{\Im}$ with respect to ω, that correspond to $(T_1, T_2) \in \mathbb{R}^2$ solutions in (16) yield the exact lower and upper bounds of the crossing frequency set. This requires to study the zeros of the resultant of R_{T_2} and $\partial R_{T_2}/\partial T_1$, particularly by eliminating T_1. The resultant of R_{T_2} and $\partial R_{T_2}/\partial T_1$ becomes only a function of ω.

$$Z(\omega) = R_{T_1}(R_{T_2}, \partial R_{T_2}/\partial T_1) \tag{19}$$

which is the discriminant by Definition 2. The minimum and maximum positive real zeros of $Z(\omega)$ corresponding to the (T_1, T_2) solutions of (16) are the exact lower bound $\underline{\omega}$ and upper bound $\bar{\omega}$ of the crossing frequency set, respectively.

4) In the following, the proposed step-by-step approach is described. Notice that this method only requires frequency sweeping from the precise lower bound $\underline{\omega}$ to the precise upper bound $\overline{\omega}$ determining by the use of Theorem 1. For each $\omega \in [\underline{\omega}, \overline{\omega}]$ with an appropriately chosen step size, the following steps
   (a) Solving the equation $R_{T_2}(h_{\Re}, h_{\Im})$ for T_1.
   (b) For each T_1, if $T_2 \in \mathbb{R}$ exists and satisfies $h_{\Re} = 0$ and $h_{\Im} = 0$ then we proceed to the next step, otherwise, increase ω incrementally and the previous step is repeated.
   (c) The delay values (τ_1, τ_2) corresponding to (T_1, T_2) pairs are calculated using (13). Then, ω is incremented with the same step size used in Step (a).

If there exists an imaginary root for (10) at $s = \pm j\omega_c$, wherein "c" subscription denotes crossing, for a given set of the time delays $\{\tau\} = (\tau_1, \tau_2)$, the same imaginary root also exist at all the countably infinite grid points of

$$\{\tau\} = (\tau_{1l}, \tau_{2k}) = (\tau_{10} + \frac{2\pi}{\omega_c} l, \tau_{20} + \frac{2\pi}{\omega_c} k), \quad l = 0, 1, 2, \dots, \; k = 0, 1, 2, \dots. \tag{20}$$

Definition 3 (kernel curves)**:** Assume that the set of $(\tau_{10}, \tau_{20})|_{\omega_c}$ is determined exhaustively in the (τ_1, τ_2) space for all possible ω_c values satisfying (10) and (13). This set of curves is called the kernel curve set of the system described by the CE (10), and it is denoted by $\wp_0(\tau_1, \tau_2)$.

Definition 4 (offspring curves)**:** The trajectories of (τ_1, τ_2) grid points in (14) for l = 1, 2, ..., k = 1, 2, ... corresponding to the kernel curve set are called the "offspring curves" or "offspring" in short. They are represented by $\wp_{lk}(\tau_1, \tau_2)$ where l and k identify the l^{th} and k^{th} generation offspring of the kernel τ_{10} and τ_{20}, respectively. Let denote the complete set of kernel and offspring by

$$\wp(\tau_1, \tau_2) = \wp_0(\tau_1, \tau_2) + \sum_{\substack{l=0 \\ l+k>0}}^{n} + \sum_{k=0}^{n} \wp_{lk}(\tau_1, \tau_2). \tag{21}$$

The kernel and the offspring constitute the complete (and exhaustive) distribution of (τ_1, τ_2) points for which the CE CE(s, τ_1, τ_2) has root sets with at least one imaginary pair. Outside the set of curves $\wp(\tau_1, \tau_2)$, there cannot exist a point resulting in an imaginary characteristic root for (10). These are the only locations in the (τ_1, τ_2) space where the system (10) could transit from stable to unstable posture (or vice versa). Since $\wp(\tau_1, \tau_2)$ is completely generated from the kernel using (13) and (21), they are sufficient to determine the kernel and offspring themselves exhaustively.

It should be mentioned that this method has limitations for application to time-delay systems with high order roots on the imaginary axis. To overcome this problem, it is necessary to divide the main system into several subsystems with simpler roots on the imaginary axis and then apply the CTCR method to it.

3.2. Direction of crossing

The invariance property of the root tendency, along any one of the independent time delays while the other independent time delay is fixed. The root tendency property has been used to determine the number of unstable roots in the delay parametric spaces through measuring the number of imported and exported roots from any region separated by stability switch curves [23]. The root tendency is always independent from the delays τ_1 and τ_2. It is worthy to note that the root tendency describes the root transfer orientation in $s = j\omega$ as τ increases from $\tau_\ell - \varepsilon$ to $\tau_\ell + \varepsilon$, $0 < \varepsilon \le 1$.

The root sensitivities associated with each purely imaginary characteristic root crossing $j\omega$ with respect to one of the time delay τ_ℓ is defined as

$$S^s_{\tau_\ell}\Big|_{s=j\omega_c} = \frac{ds}{d\tau_\ell}\Big|_{s=j\omega_c} = -\frac{\partial \mathrm{CE}/\partial \tau_\ell}{\partial \mathrm{CE}/\partial s}\Big|_{s=j\omega_c}, \quad \ell = 1,2, \tag{22}$$

the corresponding root tendency with respect to one of the delays is given by

$$\begin{aligned}\text{Root Tendency} &= RT\big|^{\tau_\ell}_{s=j\omega_c} = \mathrm{sign}\Big(\Re\Big(S^s_{\tau_\ell}\big|_{s=j\omega_c}\Big)\Big)\\ &= \mathrm{sign}\Big(\Re\Big(-\frac{\partial \mathrm{CE}/\partial \tau_\ell}{\partial \mathrm{CE}/\partial s}\Big)\Big)\Big|_{\substack{s=j\omega_c\\ \tau=\tau_\ell}}.\end{aligned} \tag{23}$$

Thus, it can be used as a criterion in assessing the stability outlook of the characteristic equation (10).

4. ILLUSTRATIVE EXAMPLE

In this section, the proposed method's effectiveness is studied by determining the stability of a FDTDS.

4.1. Example

Consider a FDTDS

$$\begin{aligned}D_t^{0.5}x(t) &= \frac{d^{0.5}x(t)}{dt^{0.5}} = \begin{bmatrix}0 & 1\\ -0.4 & -1\end{bmatrix}x(t)\\ &+ \begin{bmatrix}-0.3 & -0.1\\ -0.2 & -0.4\end{bmatrix}\int_{\tau_1}^{\tau_2} x(t-\eta)d\eta + \begin{bmatrix}0\\1\end{bmatrix}u(t),\\ y(t) &= \begin{bmatrix}1 & 0\end{bmatrix}x(t),\end{aligned} \tag{24}$$

whose characteristic equation is

$$\begin{aligned}\mathrm{CE}(s,\tau_1,\tau_2) &= s^2(s+s^{0.5}+0.4)\\ &+s(-0.7s^{0.5}-0.46)\,(e^{-\tau_2 s}-e^{-\tau_1 s})+0.1(e^{-\tau_2 s}-e^{-\tau_1 s})^2,\end{aligned} \tag{25}$$

and also its closed loop transfer function is

$$H(s,\tau_1,\tau_2) = \frac{s(s+0.1(e^{-\tau_2 s}-e^{-\tau_1 s}))}{\mathrm{CE}(s,\tau_1,\tau_2)}. \tag{26}$$

It is note that the delay free system ($\tau_1 = \tau_2 = 0$) is asymptotically stable. Applying the criterion expressed the exponential term in (25) is eliminated, so we have

$$\begin{aligned}h(s,T_1,T_2) &= s^2(s+s^{0.5}+0.4)(1+jT_1)^2(1+jT_2)^2\\ &+ j2s(-0.7s^{0.5}-0.46)\\ &\times (T_1-T_2)(1+jT_1)(1+jT_2)\\ &-0.4(T_1-T_2)^2.\end{aligned} \tag{27}$$

Substituting $s = \omega e^{\frac{j\pi}{2}} = j\omega$, $s^2 = -\omega^2$ and $s^{0.5} = \omega^{0.5}e^{\frac{j\pi}{4}} = \omega^{0.5}(\frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2})$ into (27) and equating the real and imaginary parts of the obtained relation to zero, we obtain

$$\begin{aligned}h_{\Re} =&\big(\omega^2(0.4+0.5\sqrt{2}\omega^{0.5})T_1^2 + \omega(0.7\sqrt{2}\omega^{0.5}\\ &-\sqrt{2}\omega^{1.5} - 2\omega^2 + 0.92)T_1 + 0.7\sqrt{2}\omega^{1.5}\\ &+0.4\omega^2 + 0.5\sqrt{2}\omega^{2.5} - 0.4\big)T_2^2\\ &+\big(-\omega(\sqrt{2}\omega^{1.5} + 0.7\sqrt{2}\omega^{0.5} + 2\omega^2 + 0.92)T_1^2\\ &+(1.6\omega^2 + 0.8 + 2\sqrt{2}\omega^{2.5})T_1\\ &+\omega(2\omega^2 + \sqrt{2}\omega^{1.5} - 0.7\sqrt{2}\omega^{0.5} - 0.92)\big)T_2\\ &+(-0.7\sqrt{2}\omega^{1.5} + 0.5\sqrt{2}\omega^{2.5} + 0.4\omega^2 - 0.4)T_1^2\\ &+\omega(0.92 + 2\omega^2 + 0.7\sqrt{2}\omega^{0.5} + \sqrt{2}\omega^{1.5})T_1\\ &-\omega^2(0.5\sqrt{2}\omega^{0.5} + 0.4)\\ =&0,\end{aligned} \tag{28}$$

and

$$\begin{aligned}h_{\Im} =&\big(-\omega^2(0.5\sqrt{2}\omega^{0.5} + \omega)T_1^2 + 2\omega(0.35\sqrt{2}\omega^{0.5}\\ &+0.5\sqrt{2}\omega^{1.5} + 0.4\omega)T_1 + \omega(\omega^2 - 0.92\\ &-0.7\sqrt{2}\omega^{0.5} + 0.5\sqrt{2}\omega^{1.5})\big)T_2^2\\ &+\big(2\omega(-0.35\sqrt{2}\omega^{0.5} + 0.5\sqrt{2}\omega^{1.5} + 0.4\omega)T_1^2\\ &+4\omega^2(\omega + 0.5\sqrt{2}\omega^{0.5})T_1 - 2\omega(0.4\omega\\ &+0.5\sqrt{2}\omega^{0.5}(0.7+\omega))\big)T_2 + \omega(0.7\sqrt{2}\omega^{0.5}\\ &+0.5\sqrt{2}\omega^{1.5} + \omega^2 + 0.92)T_1^2 + 2\omega(-0.4\omega\\ &+0.35\sqrt{2}\omega^{0.5} - 0.5\sqrt{2}\omega^{1.5})T_1\\ &-\omega^2(0.5\sqrt{2}\omega^{0.5} + \omega)\\ =&0.\end{aligned} \tag{29}$$

Using homomorphism resultant algorithm (18) eliminate T_2 from $h_{\Re}$ and $h_{\Im}$,

$$R_{T_2}(h_{\Re}, h_{\Im}) = 0. \tag{30}$$

Discriminant of the resultant of $h_{\Re}$ and $h_{\Im}$ in Theorem 1 is the resultant of R_{T_2} and $\partial R_{T_2}/\partial T_1$ with eliminating T_1,

$$Z(\omega) = R_{T_1}(R_{T_2}, \frac{\partial R_{T_2}}{\partial T_1}) = 0. \tag{31}$$

The minimum and maximum positive real amount of ω are estimated as $[\underline{\omega}, \overline{\omega}]$ is $[0.1,\ 0.9]$. One can now use this ω range and the frequency sweeping method in previous section, in order to extract the stability maps on $\tau_1 - \tau_2$ domain by sweeping the frequency from 0.1 to 0.9. Fig. 1 shows stability switching curves in the 2D space of τ_1 and τ_2. The red and blue stability switching curves are representing the kernel and offspring sets, respectively. The number of unstable roots in the both unstable and stable regions are determined and shown by NU in Fig. 1. The

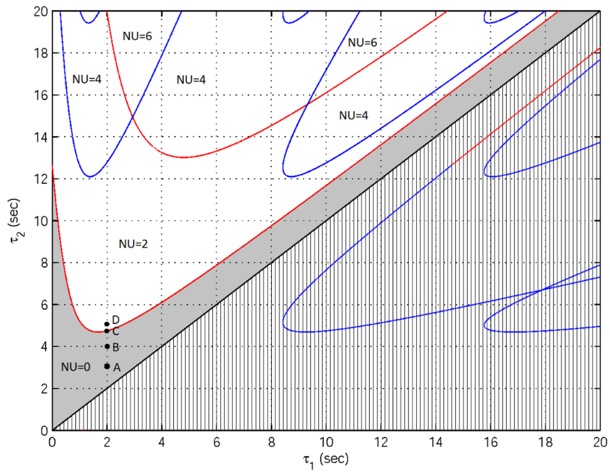

Fig. 1. Stability map in the domain of the time delays τ_1 and τ_2. The shaded regions represent are stable.

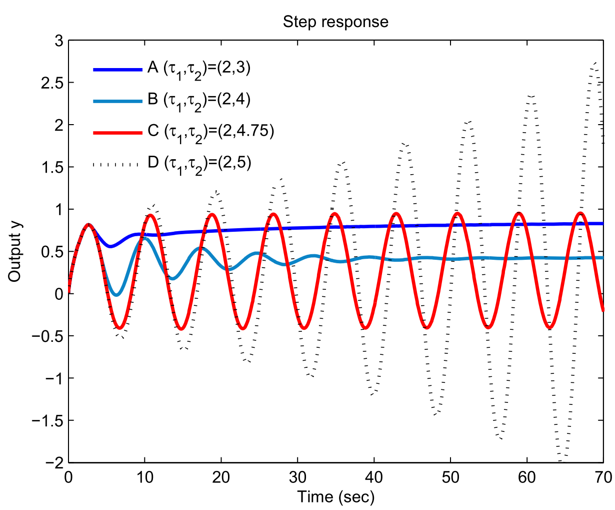

Fig. 2. Step responses of (24) for A, B, C and D points shown in Fig. 1.

gray region shows an stable region wherein the number of unstable roots is zero. The straight line $\tau_1 = \tau_2$ is when the feedback control is removed and the system is stable. For small values of $\tau_1 = \tau_2$, which is basically the regions right above the line $\tau_1 = \tau_2$, the system is still stable due to the lack of the distributed integral's destabilizing power. This is also true for the narrow gray strip next to the straight line $\tau_1 = \tau_2$ and right above that. Since the assumption is $\tau_2 \geq \tau_1$, the regions under the line $\tau_1 = \tau_2$ is unacceptable. Arbitrary points A to D are selected in different stability regions to study the stability of the system by integrating its response to the unit step input. Fig. 2, response of the system results for various τ_1 and τ_2 to verify the accuracy of stability map given in Fig. 1. In addition Fig. 3 shows, system's root locus at the points A, B, C and D using a root approximation algorithm called Quasi Polynomial mapping-based Root-finder (QPmR) [29]. There is a good agreement between the obtained stability regions in Fig. 1 and the root locus shown in Fig. 3.

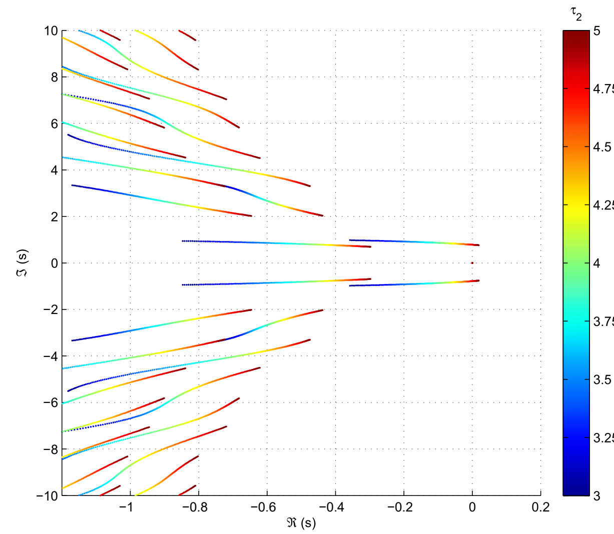

Fig. 3. Root Locus of (25) in $\tau_1 = 2$ and $\tau_2 = [3,5]$ for A, B, C and D points shown in Fig. 1.

5. CONCLUSION

This paper proposes a method to study the stability of fractional-order systems with distributed delays. This work is one of the few first attempts for the stability analysis of linear fractional-order dynamic systems with distributed delays to the best authors' knowledge. For this purpose, an integration of two effective the CTCR and ACSF methods was used, which effectively identifies stable and unstable areas in the delay space with a comprehensive image of the system stability. The proposed method efficiency was verified through an example by depicting its complete stability map.

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