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Proportional Plus Derivative State Feedback Control of Takagi-Sugeno Fuzzy Singular Fractional Order Systems

Xuefeng Zhang* and **Kaijing Jin**

Abstract: This paper investigates the fuzzy normalization and stabilization issues of a class of singular fractional order nonlinear systems with order $0 < \alpha < 1$ based on a singular Takagi-Sugeno fuzzy model. First, we present the admissibility theorem of Takagi-Sugeno fuzzy singular fractional order systems. Next, benefited by that the fuzzy model and the state feedback controllers do not share the same membership functions, a proportional plus derivative state feedback controller is designed, which guarantees the closed-loop system normalized and admissible. Finally, a numerical simulation example is given to illustrate the effectiveness of the proposed method.

Keywords: Admissibility, fractional order systems, singular systems, Takagi-Sugeno fuzzy systems.

1. INTRODUCTION

In the past several decades, a growing number of works deal with dynamical systems described by fractional order equations. Fractional order systems (FOS) have become a popular hot research area and attract great concerns from control community [1–3]. Benefited to the unremitting efforts of researchers, a lot of research results on stability and stabilization of FOS have been reported. Matignon stability theorem [4], a well-known method, establishes a judgement condition in eigenvalue form. However, it is still difficult to analyse stability by the pole location. There are some stability results about FOS based on matrix inequalities as well. For the FOS with fractional order $0 < \alpha < 1$, the asymptotical stability conditions with complex variables are presented in [5], which are extended to the linear matrix inequality (LMI) formulation in [6,7]. Since the result in [7] contains four variables, it is inconvenient to calculate. It is worth mentioning that the result in [8] overcomes this shortcoming and reduces the complexity of the theorem since the result in [8] only contains two variables. And [9–11] extend the stability criterion in [8] to the case of the singular FOS.

As we all known, most of the physical systems in real world are nonlinear [12–15]. The robust H_∞ sliding mode controller design [16], filtering [17] of discrete-time nonlinear systems are studied. In [18], the asymptotic stability and state feedback controller design of discrete-time switched nonlinear systems are investigated under dwell-time constraints. And in [19], the quasi-synchronization of discrete-time switched systems is studied. However, it is difficult to analyze nonlinear systems directly. Fuzzy control design techniques [20,21] especially for the Takagi-Sugeno (T-S) fuzzy model [22–24], have been rapidly and successfully developed in nonlinear control frameworks such as thermal systems, batteries and neurons. This means that those nonlinear systems can be described by some linear models. In [25–27], some new criteria for the stability and stabilization of fuzzy singular systems are given since T-S fuzzy model and controller attach different membership functions. Taniguchi *et al*. [28–31] define a fuzzy singular model that expends the T-S fuzzy model in [22]. The T-S fuzzy normal system is a special case of the fuzzy singular systems where derivative matrix E in the fuzzy singular systems is nonlinear. Based on the T-S fuzzy model and suitable membership functions, nonlinear matrix E can be described by an average weighted sum of a set of constant E_l.

Singular systems, fractional order systems and T-S fuzzy systems play important roles in control theory and engineering, therefore, great efforts have been put into the research of T-S fuzzy singular FOS, such as the results shown in [32–34]. Since the admissibility issue of T-S fuzzy singular FOS is more complicated comparing with the case of T-S fuzzy systems or FOS, there are still more values to study T-S fuzzy singular FOS.

Inspired by all the above mentioned results, the main constructions are summarized as follows:

- Firstly, we extend the fuzzy singular integer order model defined in [28–31] to fuzzy singular fractional order model, which has more widespread application since integer order system is a special case of fractional order system. To the best of our knowledge, so far, there is no

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Xuefeng Zhang and Kaijing Jin are with the College of Sciences, Northeastern University, Shenyang, Liaoning 110819, China (e-mails: zhangxuefeng@mail.neu.edu.cn, jinkaijing@stumail.neu.edu.cn).
* Corresponding author.

work reported for the admissibility and stabilization issues of such a T-S fuzzy singular FOS with order $0 < \alpha < 1$.

• Secondly, based on this T-S fuzzy singular fractional order model, and letting that the fuzzy singular FOS and the feedback controllers share different membership functions, we propose the methods about designing a proportional plus derivative state feedback (PDSF) controller, which guarantees the asymptotical stability of closed-loop systems.

• What is more, applying a system augmentation approach, the equation constraint of admissibility condition existing most work of singular systems is eliminated, which guarantees the results are obtained in terms of strict LMIs.

The rest of paper is organized as follows: in Section 2, some preliminaries are introduced, and a more generalized T-S fuzzy fractional order model is introduced, whose derivative matrix E attaches membership function. The main results are derived in Section 3. Numerical simulation is used to demonstrate the effectiveness of our proposed results in Section 4. Finally the paper ends with the conclusion in Section 5.

Notation: Throughout this paper, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space, $\mathbb{R}^{n\times m}$ is the set of all $n\times m$ real matrices, $X > 0$ (< 0) indicates that the matrix X is positive (negative) definite, and $\mathrm{sym}\{Y\}$ denotes the expression $Y+Y^T$. We denote $a=\sin(\alpha\frac{\pi}{2})$ and $b=\cos(\alpha\frac{\pi}{2})$ in the sequel. The symbol $\star$ is used to denote the transposed element in the symmetric position of a matrix. Matrices, if not explicitly stated, are assumed to have appropriate dimensions.

2. PRELIMINARIES

To obtain the main results, we first give the following necessary definitions and lemmas, which are used in the proofs of our results.

Definition 1 [1]**:** The Caputo derivative of $f(t)$ with order α is defined by

$$D^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)}\int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}}d\tau,$$

where $m-1<\alpha\leq m$, $\Gamma(\cdot)$ is the Euler Gamma function defined by

$$\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}dt.$$

The Caputo definition for fractional order derivative is adopted throughout this paper since this definition incorporates initial values of classical integer order derivatives with clear physical interpretations.

Consider the following unforced singular FOS or the pair (E,A),

$$ED^\alpha x(t) = Ax(t),\ \ 0<\alpha<1. \tag{1}$$

Definition 2 [9]**:** (i) The pair (E,A) is said to be regular if $\det(sE-A)$ is not identically zero.

(ii) The pair (E,A) is said to be impulse free if $\deg(\det(sE-A)) = \mathrm{rank}(E)$.

(iii) The pair (E,A) is said to be stable if all the roots of $\det(sE-A)=0$ satisfy $|\arg(\mathrm{spec}(E,A))| > \alpha\frac{\pi}{2}$.

(iv) The pair (E,A) is said to be admissible if it is regular, impulse free and stable.

Lemma 1 [9]**:** The singular FOS described by (1) is admissible, iff there exist two matrices X, Y satisfying

$$\begin{bmatrix} EX & EY \\ -EY & EX\end{bmatrix} = \begin{bmatrix} X^TE^T & -Y^TE^T \\ Y^TE^T & X^TE^T\end{bmatrix} \geq 0, \tag{2}$$

and

$$\mathrm{sym}\{A(aX-bY)\}<0. \tag{3}$$

2.1. Takagi-Sugeno fuzzy model

A T-S fuzzy singular fractional order model is described by the followings.

Plant rule il: If $z_1(t)$ is H_{b1} and $\cdots$ and $z_p(t)$ is H_{bp}, then

$$E_lD^\alpha x(t) = A_ix(t)+B_iu(t),$$
$$y(t) = C_ix(t),$$

where $i=1,\ 2,\ \cdots,\ r$, $l=1,\ 2,\ \cdots,\ r_e$, H_{bg} ($b=1,\ 2,\ \cdots,$ $r\times r_e$, $g=1,\ 2,\ \cdots,\ p$) is a fuzzy set, and $r\times r_e$ is the number of IF-THEN rules. Furthermore, $z(t)=[z_1(t),\cdots,$ $z_p(t)]$ is a vector of the premise variables. All the premise variables are measurable throughout this paper. The symbol D^α denotes the fractional order derivative of function $x(t)$. $x(t)\in\mathbb{R}^n$ is the state, $u(t)\in\mathbb{R}^m$ is the control input, $y(t)\in\mathbb{R}^s$ is the measured output and $A_i\in\mathbb{R}^{n\times n}$, $B_i\in\mathbb{R}^{n\times m}$, $C_i\in\mathbb{R}^{s\times n}$ are known real constant matrices. Generally, the matrix $E_l\in\mathbb{R}^{n\times n}$ is not always nonsingular.

Then the overall fuzzy singular FOS is inferred as follows:

$$\sum_{l=1}^{r_e}\nu_l(z(t))E_lD^\alpha x(t) = \sum_{i=1}^{r}h_i(z(t))(A_ix(t)+B_iu(t)), \tag{4}$$

$$y(t) = \sum_{i=1}^{r}h_i(z(t))C_ix(t), \tag{5}$$

where

$$h_i(z(t))\geq 0,\ \sum_{i=1}^{r}h_i(z(t))=1,\ i=1,\ 2,\ \cdots,\ r,$$
$$\nu_l(z(t))\geq 0,\ \sum_{l=1}^{r_e}\nu_l(z(t))=1,\ l=1,\ 2,\ \cdots,\ r_e.$$

$h_i(z(t))$ and $\nu_i(z(t))$ denote the normalized membership functions. To simplify notations, $\nu_i(z(t))$ and $h_i(z(t))$ are denoted as ν_i and h_i in the following analysis, respectively.

3. MAIN RESULTS

The admissibility conditions of fuzzy singular FOS are addressed in this section.

Noting $\sum_{i=1}^{r} h_i A_i = \sum_{i=1}^{r}\sum_{l=1}^{r_e} h_i \nu_l A_i$, $\sum_{l=1}^{r_e} \nu_l E_l = \sum_{i=1}^{r}\sum_{l=1}^{r_e} h_i \nu_l E_l$, fuzzy singular FOS (4) can be rewritten as

$$E^* D^\alpha x^*(t) = \sum_{i=1}^{r}\sum_{l=1}^{r_e} h_i \nu_l (A_{il}^* x^*(t) + B_i^* u(t)), \tag{6}$$

$$y(t) = \sum_{i=1}^{r} h_i C_i^* x^*(t), \tag{7}$$

where

$$x^*(t) = \begin{bmatrix} x(t) \\ D^\alpha x(t) \end{bmatrix}, \quad E^* = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix},$$

$$A_{il}^* = \begin{bmatrix} 0 & I \\ A_i & -E_l \end{bmatrix}, \quad B_i^* = \begin{bmatrix} 0 \\ B_i \end{bmatrix}, \quad C_i^* = \begin{bmatrix} C_i & 0 \end{bmatrix}.$$

Therefore, this paper discusses the admissibility for fuzzy singular FOSs (6) and (7).

Theorem 1: The fuzzy singular FOS in (6) is admissible, if there exist symmetric matrix X_1, skew-symmetric matrix Y_1, and matrices X_3, X_4, Y_3 and Y_4 such that the following LMIs hold:

$$\begin{bmatrix} X_1 & Y_1 \\ -Y_1 & X_1 \end{bmatrix} > 0, \tag{8}$$

$$\mathrm{sym}\left\{\left[\begin{array}{c|c} aX_3 - bY_3 & aX_4 - bY_4 \\ \hline A_i(aX_1 - bY_1) & \\ -E_l(aX_3 - bY_3) & -E_l(aX_4 - bY_4) \end{array}\right]\right\} < 0. \tag{9}$$

Proof: First, denoting $X = \begin{bmatrix} X_1 & 0 \\ X_3 & X_4 \end{bmatrix}$, $Y = \begin{bmatrix} Y_1 & 0 \\ Y_3 & Y_4 \end{bmatrix}$, we have

$$\begin{bmatrix} E^*X & E^*Y \\ -E^*Y & E^*X \end{bmatrix} = \begin{bmatrix} X_1 & 0 & Y_1 & 0 \\ 0 & 0 & 0 & 0 \\ -Y_1 & 0 & X_1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \geq 0,$$

$$\begin{bmatrix} E^*X & E^*Y \\ -E^*Y & E^*X \end{bmatrix}^T = \begin{bmatrix} X_1^T & 0 & -Y_1^T & 0 \\ 0 & 0 & 0 & 0 \\ Y_1^T & 0 & X_1^T & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \geq 0.$$

And from inequality (8), one has

$$\begin{bmatrix} E^*X & E^*Y \\ -E^*Y & E^*X \end{bmatrix} = \begin{bmatrix} E^*X & E^*Y \\ -E^*Y & E^*X \end{bmatrix}^T \geq 0.$$

From inequality (9), it can be obtained

$$\begin{aligned} &\mathrm{sym}\{A_{il}^*(aX - bY)\} \\ &= \mathrm{sym}\left\{\begin{bmatrix} 0 & I \\ A_i & -E_l \end{bmatrix}\begin{bmatrix} aX_1 - bY_1 & 0 \\ aX_3 - bY_3 & aX_4 - bY_4 \end{bmatrix}\right\} \\ &= \mathrm{sym}\left\{\left[\begin{array}{c|c} aX_3 - bY_3 & aX_4 - bY_4 \\ \hline A_i(aX_1 - bY_1) & \\ -E_l(aX_3 - bY_3) & -E_l(aX_4 - bY_4) \end{array}\right]\right\} < 0. \end{aligned}$$

Applying Lemma 1, we can obtain that system (6) with $u(t) = 0$ is admissible. This completes the proof. □

Remark 1: The admissibility conditions in Theorem 1 are conservative because they require every subsystems are admissible. And from LMIs presented in Theorems 1, we can obtain that E_l, $l = 1, 2, \cdots, r_e$, are required to be nonsingular. And it is infeasible when matrices E_l is singular.

Next, we propose a PDSF controller as follows to normalize and stabilize fuzzy singular FOS (6).

Controller rule jl: If q_1 is H_{b1} and $\cdots$ and q_p is H_{bp}, then $u(t) = K_{jl}^* x^*(t)$,

where $j = 1, 2, \cdots, r$, $l = 1, 2, \cdots, r_e$. Then the overall controller is obtained as

$$u(t) = \sum_{j=1}^{r}\sum_{l=1}^{r_e} \mu_j \nu_l K_{jl}^* x^*(t), \tag{10}$$

where $K_{jl}^* = \begin{bmatrix} K_{jl} & F_{jl} \end{bmatrix}$. $\mu_j \nu_l$ is the membership function of the controller. Inspired by [25–27], the membership functions of controller are different from those of the fuzzy plant. It can be seen that when $h_j = \mu_j$, the designed controller (10) is the classical parallel distributed compensation controller. By substituting (10) into (6), fuzzy singular FOS (6) is represented as

$$E^* D^\alpha x^*(t) = \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{l=1}^{r_e} h_i \mu_j \nu_l (A_{il}^* + B_i^* K_{jl}^*) x^*(t), \tag{11}$$

where $A_{il}^* + B_i^* K_{jl}^* = \begin{bmatrix} 0 & I \\ A_i + B_i K_{jl} & -E_l + B_i F_{jl} \end{bmatrix}$. The admissibility conditions for closed-loop system (11) are presented as follows.

Theorem 2: The fuzzy singular FOS in (11) is admissible, if there exist symmetric matrix X_1, skew-symmetric matrix Y_1, and matrices X_3, X_4, Y_3, Y_4 and Z_{il}, $i = 1, 2, \cdots, r$, $l = 1, 2, \cdots, r_e$, such that (8) and the following LMIs hold:

$$\mathrm{sym}\{A_{il}^*(aX - bY) + B_i^* Z_{jl}^*\} < 0, \tag{12}$$

where $X = \begin{bmatrix} X_1 & 0 \\ X_3 & X_4 \end{bmatrix}$, $Y = \begin{bmatrix} Y_1 & 0 \\ Y_3 & Y_4 \end{bmatrix}$. The matrix gains are given as

$$K_{il}^* = Z_{il}^*(aX - bY)^{-1}, \ i = 1, 2, \cdots, r, \ l = 1, 2, \cdots, r_e.$$

Proof: From Theorem 1, fuzzy singular FOS (11) is admissible if (8) and the following inequalities hold

$$\mathrm{sym}\{(A_{il}^* + B_i^* K_{jl}^*)(aX - bY)\}$$

$$= \mathrm{sym}\{A_{il}^*(aX-bY)+B_i^*K_{jl}^*(aX-bY)\}$$
$$< 0.$$

Then denoting $Z_{jl}^* = K_{jl}^*(aX-bY)$, inequality (12) is obtained. This completes the proof. □

Noting that above Theorem 2 requires every subsystem admissible, it is conservative and harsh. In order to reduce the conservatism, the next theorem is addressed.

Theorem 3: The fuzzy singular FOS in (11) is admissible, if the membership functions satisfy $\mu_j - \rho_j h_j \geq 0$ for all j with $0 < \rho_j < 1$, and there exist matrices $Q < 0$, $X_1, X_3, X_4, Y_1, Y_3, Y_4, \Delta_i = \Delta_i^T, Z_{il}$, $i = 1, 2, \cdots, r$, $l = 1, 2, \cdots, r_e$, such that (8) and the following LMIs hold:

$$\mathrm{sym}\{A_{il}^*(aX-bY)+B_i^*Z_{jl}^*\}-\Delta_i<0, \tag{13}$$

$$\rho_i\mathrm{sym}\{A_{il}^*(aX-bY)+B_i^*Z_{il}^*\}+(1-\rho_i)\Delta_i-Q_{ii}<0, \tag{14}$$

$$\begin{aligned}&\rho_j\mathrm{sym}\{A_{il}^*(aX-bY)+B_i^*Z_{jl}^*\}+(1-\rho_j)\Delta_i\\&\quad+\rho_i\mathrm{sym}\{A_{jl}^*(aX-bY)+B_j^*Z_{il}^*\}+(1-\rho_i)\Delta_j\\&\quad-Q_{ij}-Q_{ij}^T<0,\ \ i<j,\end{aligned} \tag{15}$$

where X and Y are defined in Theorem 2, and

$$Q=\begin{bmatrix}Q_{11} & Q_{12} & \cdots & Q_{1r}\\ \star & Q_{22} & \cdots & Q_{2r}\\ \vdots & \vdots & \ddots & \vdots\\ \star & \star & \cdots & Q_{rr}\end{bmatrix}.$$

Then, the gain matrices are solved as

$$K_{il}^*=Z_{il}^*(aX_1-bY_1)^{-1},\ i=1,2,\cdots,r,\ l=1,2,\cdots,r_e.$$

Proof: Applying Lemma 1, we just need to prove

$$\mathrm{sym}\{\sum_{i=1}^r\sum_{j=1}^r\sum_{l=1}^{r_e}h_i\mu_j\nu_l(A_{il}^*+B_i^*K_{jl}^*)(aX-bY)\}<0.$$

Denoting

$$Z_{jl}^*=K_{jl}^*(aX-bY),\ j=1,2,\cdots,r,\ l=1,2,\cdots,r_e,$$

we have

$$\sum_{i=1}^r\sum_{j=1}^r\sum_{l=1}^{r_e}h_i\mu_j\nu_l\mathrm{sym}\{A_{il}^*(aX-bY)+B_i^*Z_{jl}^*\}<0. \tag{16}$$

To reduce the complexity, we consider the following inequalities,

$$\sum_{i=1}^r\sum_{j=1}^r h_i\mu_j\mathrm{sym}\{A_{il}^*(aX-bY)+B_i^*Z_{jl}^*\}<0, \tag{17}$$

for $l = 1, 2, \cdots, r_e$. For the purpose of reducing the conservatism further, the following equalities are introduced:

$$\sum_{i=1}^r\sum_{j=1}^r h_i(h_j-\mu_j)\Delta_i=0, \tag{18}$$

where $\Delta_i = \Delta_i^T$ is an arbitrary matrix. From (17) and (18), we obtain

$$\begin{aligned}&\sum_{i=1}^r\sum_{j=1}^r h_i\mu_j\mathrm{sym}\{A_{il}^*(aX-bY)+B_i^*Z_{jl}^*\}\\&=\sum_{i=1}^r\sum_{j=1}^r h_i(\mu_j+\rho_jh_j-\rho_jh_j)\\&\quad\times\mathrm{sym}\{A_{il}^*(aX-bY)+B_i^*Z_{jl}^*\}\\&\quad+\sum_{i=1}^r\sum_{j=1}^r h_i(h_j-\rho_jh_j)\Delta_i-\sum_{i=1}^r\sum_{j=1}^r h_i(\mu_j-\rho_jh_j)\Delta_i\\&=\sum_{i=1}^r\sum_{j=1}^r h_ih_j\\&\quad\times(\mathrm{sym}\{\rho_j(A_{il}^*(aX-bY)+B_i^*Z_{jl}^*)\}+(1-\rho_j)\Delta_i)\\&\quad+\sum_{i=1}^r\sum_{j=1}^r h_i(\mu_j-\rho_jh_j)\\&\quad\times(\mathrm{sym}\{A_{il}^*(aX-bY)+B_i^*Z_{jl}^*\}-\Delta_i).\end{aligned} \tag{19}$$

Considering $\mu_j - \rho_j h_j > 0$, and $\mathrm{sym}\{(A_{il}^*(aX-bY)+B_i^*Z_{jl}^*\}-\Delta_i<0$, we obtain

$$\begin{aligned}&\sum_{i=1}^r\sum_{j=1}^r h_i\mu_j\mathrm{sym}\{A_{il}^*(aX-bY)+B_i^*Z_{jl}^*\}\\&\leq\sum_{i=1}^r\sum_{j=1}^r h_ih_j\\&\quad\times(\mathrm{sym}\{\rho_j(A_{il}^*(aX-bY)+B_i^*Z_{jl}^*)\}+(1-\rho_j)\Delta_i)\\&=\sum_{i=1}^r h_i^2(\mathrm{sym}\{\rho_i(A_{il}^*(aX-bY)+B_i^*Z_{il}^*)\}+(1-\rho_i)\Delta_i)\\&\quad+\sum_{j=1}^r\sum_{i<j}h_ih_j\\&\quad\times(\mathrm{sym}\{\rho_j(A_{il}^*(aX-bY)+B_i^*Z_{jl}^*)\}+(1-\rho_j)\Delta_i\\&\quad+\mathrm{sym}\{\rho_i(A_{il}^*(aX-bY)+B_i^*Z_{il}^*)\}+(1-\rho_i)\Delta_j).\end{aligned} \tag{20}$$

Due to

$$\mathrm{sym}\{\rho_i(A_{il}^*(aX-bY)+B_i^*Z_{jl}^*)\}+(1-\rho_i)\Delta_i<Q_{ii},$$

and

$$\begin{aligned}&\mathrm{sym}\{\rho_j(A_{il}^*(aX-bY)+B_i^*Z_{jl}^*)\}+(1-\rho_j)\Delta_i\\&\quad+\mathrm{sym}\{\rho_i(A_{jl}^*(aX-bY)+B_j^*Z_{ijl}^*)\}+(1-\rho_i)\Delta_j\\&<Q_{ij}+Q_{ij}^T,\end{aligned}$$

it yields from (20) that

$$\begin{aligned}&\sum_{i=1}^r\sum_{j=1}^r h_i\mu_jx^T(t)\mathrm{sym}\{(A_{il}^*(aX-bY)+B_i^*Z_{jl}^*)\}x(t)\\&\leq x^T(t)Qx(t)<0.\end{aligned} \tag{21}$$

Therefore, $\sum_{i=1}^r\sum_{j=1}^r h_i\mu_j\mathrm{sym}\{((A_{il}^*(aX-bY)+B_i^*Z_{jl}^*)\}<0$, which implies the admissibility of system (11). This completes the proof. □

Remark 2: The admissibility conditions in Theorem 2 are a special case of Theorem 3. If there exist solutions for the admissibility conditions in Theorem 2, choosing $\Delta_i = 0$, $Q_{ij} = 0$ for $i < j$, the LMIs (13)-(15) are satisfied. Thus, the admissibility conditions in Theorem 3 are more relax than that in Theorem 2.

Theorem 4: Fuzzy singular FOS (11) with $h_i = v_i$ and $r = r_e$ is admissible, if the membership functions satisfy $\mu_j - \rho_j h_j \geq 0$ for all j with $0 < \rho_j < 1$, and there exist matrices $Q < 0$, X_1, X_3, X_4, Y_1, Y_3, Y_4, $\Delta_i = \Delta_i^T$, Z_{ii}, $i = 1, 2, \cdots, r$, such that (8) and the following LMIs hold:

$$\mathrm{sym}\{A_{ii}^*(aX - bY) + B_i^* Z_{jj}\} - \Delta_i < 0, \tag{22}$$

$$\rho_i \mathrm{sym}\{A_{ii}^*(aX - bY) + B_i^* Z_{ii}\} + (1-\rho_i)\Delta_i - Q_{ii} < 0, \tag{23}$$

$$\begin{aligned} &\rho_j \mathrm{sym}\{A_{ii}^*(aX - bY) + B_i^* Z_{jj}\} + (1-\rho_j)\Delta_i \\ &\quad + \rho_i \mathrm{sym}\{A_{jj}^*(aX - bY) + B_j^* Z_{ii}\} + (1-\rho_i)\Delta_j \\ &\quad - Q_{ij} - Q_{ij}^T < 0, \quad i < j, \end{aligned} \tag{24}$$

where X, Y are defined in Theorem 2 and Q is defined in Theorem 3. Then, the gain matrices are obtained as

$$K_{ii}^* = Z_{ii}(aX - bY)^{-1}, \; i = 1, 2, \; \cdots, \; r.$$

4. AN EXAMPLE

In this section, a simulation example is provided to validate the effectiveness of the proposed controller design schemes.

Example 1: Consider a fractional order electrical circuit system shown in Fig. 1 with inductors L_1, L_2, resistances R_1, R_2, and a source current i_z, where $L_1 = \frac{5-3\sin(i_1)}{2}$ and $R_2 = \sin(i_2) + 2$ are nonlinear terms. i_1 and i_2 represent current passing through the inductor L_1 and L_2, respectively.

As the statements in [35], let the voltage $u_L(t)$ on the inductor L with its current $i_L(t)$ be related by the following formula

$$u_L(t) = L\frac{d^\alpha i_L(t)}{dt^\alpha}, \; 0 < \alpha < 1.$$

Using the Kirchhoffs laws, we can write the electrical circuit equations

$$\begin{bmatrix} L_1 & -L_2 \\ 0 & 0 \end{bmatrix} \frac{d^\alpha}{dt^\alpha} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$$

$$= \begin{bmatrix} -R_1 & R_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} \begin{bmatrix} i_z \end{bmatrix}.$$

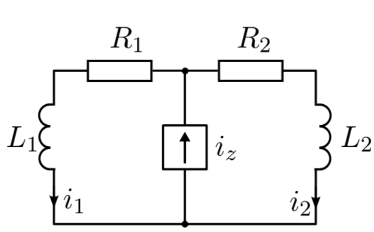

Fig. 1. Electrical circuit illustration.

Denote the state of system as $x(t) = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T = \begin{bmatrix} i_1 & i_2 \end{bmatrix}^T$, and $x_1 \in [0,\ 2]$, $x_2 \in [0,\ 4]$. Choosing $\alpha = 0.8$, $L_2 = 1$, $R_1 = 1$, and considering there exist two nonlinear terms $\frac{5-3\sin(x_1)}{2}$ and $\sin(x_2) + 2$, the following T-S fuzzy model is obtained:

IF $\frac{5-3\sin(x_1)}{2}$ is 'small' and $\sin(x_2)+2$ is 'small', THEN

$$E_1 D^\alpha x(t) = A_1 x(t) + Bu(t);$$

IF $\frac{5-3\sin(x_1)}{2}$ is 'small' and $\sin(x_2)+2$ is 'big', THEN

$$E_1 D^\alpha x(t) = A_2 x(t) + Bu(t);$$

IF $\frac{5-3\sin(x_1)}{2}$ is 'big' and $\sin(x_2)+2$ is 'small', THEN

$$E_2 D^\alpha x(t) = A_1 x(t) + Bu(t);$$

IF $\frac{5-3\sin(x_1)}{2}$ is 'big' and $\sin(x_2)+2$ is 'big', THEN

$$E_2 D^\alpha x(t) = A_2 x(t) + Bu(t),$$

where

$$E_1 = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \; E_2 = \begin{bmatrix} 4 & -1 \\ 0 & 0 \end{bmatrix},$$
$$A_1 = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}, \; A_2 = \begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix},$$
$$B = \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

We choose the following membership functions:

$$v_1 = \frac{4 - \frac{5-3\sin(x_1)}{2}}{4-1} = \frac{1+\sin(x_1)}{2}, \; v_2 = 1 - v_1,$$
$$h_1 = \frac{3-(\sin(x_2)+2)}{3-1} = \frac{1-\sin(x_2)}{2}, \; h_2 = 1 - h_1,$$
$$\mu_1 = \frac{x_2^2}{10}, \; \mu_2 = 1 - \mu_1.$$

Based on Theorem 3, by setting $\rho_1 = 0.1$, $\rho_2 = 0.4$, and solving LMIs (8) and (13)-(15), the PDSF controller gain matrices are obtained as follows:

$$\begin{aligned} K_{11}^* &= K_{12}^* \\ &= \begin{bmatrix} -11.5142 & -11.6811 & -28.7513 & -0.4896 \end{bmatrix}, \\ K_{21}^* &= K_{22}^* \\ &= \begin{bmatrix} -11.5343 & -11.7011 & -28.7980 & -0.4933 \end{bmatrix}. \end{aligned}$$

Designing the PDSF controller with above parameters, then, the state trajectories of the closed-loop system (11) are plotted in Fig. 2. It implies that the closed-loop system (11) is admissible and the PDSF controller design method in Theorem 3 is effective.

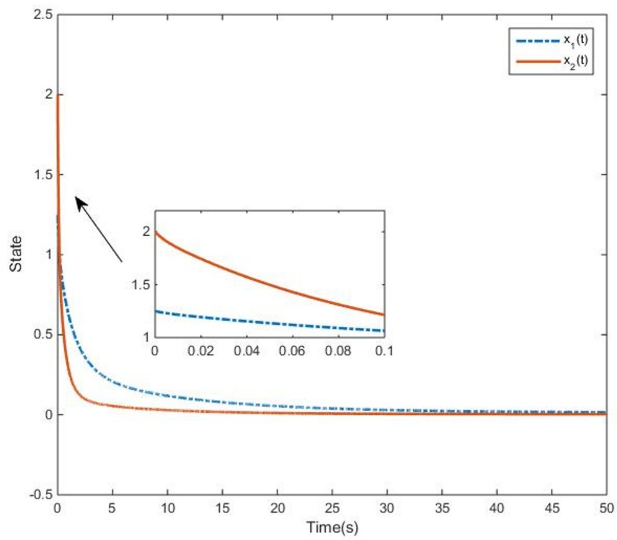

Fig. 2. State trajectories of the closed-loop system based on PDSF controller.

5. CONCLUSION

The issues of normalization and stabilization of T-S fuzzy singular FOS with fractional order $0 < \alpha < 1$ are investigated in this paper. In order to facilitate the admissibility analysis, a equivalent system in admissibility is introduced. And the system matrices and controllers share different membership function, resulting in a more relaxed admissibility analysis result by PDSF controller. The effectiveness of the proposed method has been illustrated by the provided example. In the future, we will consider the problems of output feedback control for the nonlinear systems subject to parameter uncertainties in the frame structure of this paper.

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Xuefeng Zhang received his B.Sc. degree in applied mathematics, an M.S. degree in control theory and control engineering, and a Ph.D. degree in control theory and control engineering from Northeastern University, Shenyang, China, in 1989, 2004, and 2008, respectively, where he is currently with the college of Sciences. He has published more than 100 journal and conference papers and 3 books. He is the Associate Editor of IEEE Access and the Committee member of Technical Committee on Fractional and Control of Chinese Association of Automation. His research interests include fractional order control systems and singular systems.

Kaijing Jin received her B.Sc. degree in information and computing science from Bohai University in 2018. She is currently pursuing an M.S. degree in operations research and cybernetics with Northeastern University, Shenyang, China, where she is also with the School of Sciences. Her research interests include fractional order systems, Takagi-Sugeno fuzzy systems and singular systems.