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Output Feedback Stabilization for a Class of Cascade Nonlinear ODE-PDE Systems

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Abstract: In this paper, the output feedback stabilization problem is studied for a class of cascade nonlinear ODE-PDE systems. The nonlinear terms of ODE-subsystem are assumed to be bounded by a known constant multiplied by unmeasured states, and PDE-subsystem is a diffusion equation. Especially, the unstable diffusion equation is considered. Based on the low gain observer and a series of transformations, the output feedback stabilization problem is converted into designing proper gain parameters. Furthermore, the stability of the closed-loop system is analyzed by Lyapunov theorem. Finally, two numerical examples are given to demonstrate the effectiveness of the proposed control strategy.

Keywords: Cascade ODE-PDE systems, low gain observer, Lyapunov functional, output feedback.

1. INTRODUCTION

The actuator is an important component of a control system. In many cases, for achieving high control accuracy, the actuator dynamics cannot be neglected. There are many practical problems that actuator dynamics are dominated by partial differential equations (PDEs). For example, due to high temperature, a controller cannot be directly set into the plant. Then, a thermal conductivity body can be used to transfer heat and control the ordinary differential equations (ODEs) [1]. Furthermore, some practical problems can be modeled as a cascade ODE-PDE system, such as the stefan problem [2]. Therefore, the cascade systems with PDEs as subsystems receive much attention. The study of stabilization problem for PDEs [3–8] provided the foundation for research on stabilization of cascade systems.

In recent decades, stabilization of cascade systems consisting of ODE and PDE has been widely studied. The state feedback stabilization problem for linear ODE systems with actuator dynamics described by a heat equation and a wave equation was studied in [9, 10]. They were the early articles on cascade systems. The state feedback stabilization problem, the sliding mode control problem and the output feedback stabilization problem for linear ODE-PDE systems subject to disturbance were investigated in [11–13], respectively. Output feedback stabilization based on output signal of cascade systems is more practical and has received much attention, see [12–15].

Nonlinear ODE systems have been the main research object of stabilization problem, see [16–18]. Thus, the study of stabilization problem for nonlinear ODE-PDE system has great significance. In recent five years, preliminary research achievements have been obtained for nonlinear ODE-PDE systems. Under the assumption of input-to-state stability, the stabilization problem for ODE with actuator dynamics governed by transport and quasilinear hyperbolic was studied in [19, 20]. The problem of state-observation for cascade lower-triangular nonlinear ODE-PDE systems was studied in [21–23]. In these papers, the high gain technique, which is used to construct observer and controller of triangular nonlinear systems [24–28], played a key role in solving the problem.

In the hope of enriching the research on cascade nonlinear ODE-PDE systems, this paper studied the output feedback stabilization problem for the feedforward nonlinear system with actuator dynamics governed by the diffusion equation. A low gain observer is designed for the ODE subsystem and an output feedback controller is constructed by using the transformations step by step to make the feedforward ODE-PDE system globally exponentially stable. Compared with the existing results, the contributions of this paper are as follows:

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- There were a few achievements on stabilization problem for cascade nonlinear ODE-PDE systems [19, 20]. This paper studies the stabilization problem for cascade system consisting of feedforward nonlinear system and diffusion equation for the first time.
- First, two state transformations are applied to ODE subsystem and PDE subsystem to handle unstable subsystems, respectively. Second, a backstepping transformation with simple form is used to design the boundary controller, which makes the closed-loop system is globally exponentially stable. Compared with [1], the backstepping transformation we used and the designed controller are more simple.
- Compared with [21–23], which focused on the observer design problem for cascade lower-triangular ODE-PDE system, there are two points of particularities in this paper. On the one hand, we focus on the stabilization problem for cascade feedforward nonlinear ODE-PDE system. On the other hand, the unstable diffusion subsystem is considered in this paper, and there is no restriction on the length of the PDE domain.

The rest of the paper is organized as follows: In Section 2, the original cascade system is proposed, and some important lemmas are presented. The observer and observer-based controller are designed in the section 3. The main theorem is presented in the section 4. Finally, two numerical examples are given to demonstrate the present results.

Notations: In this paper, $\mathbb{R}^n$ is the n dimensional real space and $\mathbb{R}^{n\times m}$ is $n\times m$ real matrix space. The corresponding Euclidean norm of these two spaces is $\|\cdot\|$. Superscript T is used to represent the transposition of matrices. I_n denote n dimension identity matrix. $\mathbf{0}$ represent zero matrix with proper dimension. For a symmetric matrix $P\in\mathbb{R}^{n\times n}$, $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$ denote the largest eigenvalue and the smallest eigenvalue of P, respectively.

2. PRELIMINARIES AND PROBLEM FORMULATION

Consider the cascade system consisting of a finite-dimensional nonlinear ordinary differential system and a diffusion partial differential system. The nonlinear ODE-PDE system is described by the following equations:

$$\begin{cases}\dot{X}(t)=AX(t)+Bu(0,t)+f(X(t)),\\ u_t(x,t)=u_{xx}(x,t)+\mu u(x,t),\\ u_x(0,t)=0,\\ u_x(1,t)=U(t),\\ X(0)=X_0,\\ u(x,0)=u_0(x),\\ y(t)=(CX(t),u(1,t)),\end{cases} \tag{1}$$

where $A=\begin{pmatrix}\mathbf{0}^T & I_{n-1}\\ 0 & \mathbf{0}\end{pmatrix}\in\mathbb{R}^{n\times n}$, $\mathbf{0}=\begin{pmatrix}0 & \ldots & 0\end{pmatrix}\in\mathbb{R}^{1\times n-1}$, $B=\begin{pmatrix}0 & \ldots & 0 & 1\end{pmatrix}^T\in\mathbb{R}^{n\times 1}$, $C=\begin{pmatrix}1 & 0 & \ldots & 0\end{pmatrix}\in\mathbb{R}^{1\times n}$, μ is a known constant, the states $X_i(t), i=2,\ldots,n$ and $u(x,t), 0<x<1$ are unmeasurable, $U(t)$ is the boundary control input, the nonlinear terms f_i, $i=1,2,\ldots,n-2$ are continuous unknown functions, and $y(t)$ is the output signal. For this type of output, [12, 13, 31] investigated the stabilization for a cascade heat PDE-ODE and wave PDE-ODE subject to uncertain disturbance, respectively.

Assumption 1: For all $X(t)\in\mathbb{R}^n$, there is a known constant m such that

$$|f_i(X(t))|\leq m\sum_{j=i+2}^{n}|X_j(t)|,$$

and $f_{n-1}(X(t))=f_n(X(t))=0$.

Remark 1: From Assumption 1, the system (1) can be viewed as a feedforward nonlinear system ([26, 27]) with actuator dynamics governed by diffusion equation ([9, 11–13]). Compared with the global Lipschitz condition with $f(0)=0$ in [21], the specific form of nonlinear terms in this paper is unknown. Thus, the observer and controller design can not use f_i. Furthermore, Assumption 1 contains more forms of nonlinear terms. For example, $f_i=\cos(X_{i+2})X_{i+2}$, when $X_{i+2}\in(-\infty,+\infty)$.

Lemma 1 [27]**:** There exist real numbers $a_j, b_j, j=1,2,\ldots,n$, and symmetric positive matrices P, Q satisfying the following inequalities:

$$PG+G^TP\leq -I,\quad QJ+J^TQ\leq -I,$$

where $G=A+KC$ with $K=(-a_1\ -a_2\ \ldots\ -a_n)^T$, and $J=A+B\bar{B}$ with $\bar{B}=(-b_1\ -b_2\ \ldots\ -b_n)$.

Lemma 2 (Poincaré Inequality) [32]**:** For any function $w(x,t)$, continuously differentiable for $x\in[0,1]$, the following inequalities hold:

$$\int_0^1 w^2(x,t)dx\leq 2w^2(1,t)+4\int_0^1 w_x^2(x,t)dx,$$

and

$$\int_0^1 w^2(x,t)dx\leq 2w^2(0,t)+4\int_0^1 w_x^2(x,t)dx.$$

The objective in this paper is to construct an output feedback controller to make the system (1) exponentially stable. The controller can only use the measurable signals and needs to stabilize the two subsystems.

3. OBSERVER AND OBSERVER-BASED CONTROLLER DESIGN

The following observer and controller design are carried out when $\mu>0$.

3.1. Observer design

Using the low gain observer construction technique of feedforward system ([26, 27]), the observer of the system (1) is designed as,

$$\begin{cases}\dot{\hat{X}}(t) = A\hat{X}(t) + B\hat{u}(0,t) - \Omega K(X_1(t) - \hat{X}_1(t)),\\ \hat{u}_t(x,t) = \hat{u}_{xx}(x,t) + \mu\hat{u}(x,t) + q_1(x)[u(1,t) - \hat{u}(1,t)],\\ \hat{u}_x(0,t) = 0,\\ \hat{u}_x(1,t) = U(t) + q_0[u(1,t) - \hat{u}(1,t)],\end{cases} \tag{2}$$

where K is given by Lemma 1, $\Omega = diag\left(\frac{1}{\tau}, \frac{1}{\tau^2}, \ldots, \frac{1}{\tau^n}\right)$, $\tau > 1$ and q_0 are two parameters to be designed, and $q_1(x)$ is a function to be determined. It is clear that the observer system (2) only depends on the output signal of the system (1), see [13, 26, 28].

Remark 2: The observer (2) is a kind of formal observer since it can estimate the states of system (1) only for the specific controller (controller designed in this paper). Under Assumption 1, the observer is common, by which the problem of output feedback stabilization is addressed effectively, see [26, 28].

In order to analyze the observer performance, define the state estimation errors as $\tilde{X} = \Gamma(X(t) - \hat{X}(t))$ and $\tilde{u}(x,t) = u(x,t) - \hat{u}(x,t)$, where $\Gamma = diag(\frac{1}{\tau^n}, \ldots, \frac{1}{\tau})$. Then subtracting each equation of the observer system (2) from the corresponding equation of the system (1), and using the state transformation $\tilde{X} = \Gamma(X(t) - \hat{X}(t))$, the following error system can be obtained easily:

$$\begin{cases}\dot{\tilde{X}}(t) = \frac{1}{\tau}(A+KC)\tilde{X}(t) + \frac{1}{\tau}B\tilde{u}(0,t) + F(X(t)),\\ \tilde{u}_t(x,t) = \tilde{u}_{xx}(x,t) + \mu\tilde{u}(x,t) - q_1(x)\tilde{u}(1,t),\\ \tilde{u}_x(0,t) = 0,\\ \tilde{u}_x(1,t) = -q_0\tilde{u}(1,t),\end{cases} \tag{3}$$

where $F(X(t)) = \left(-\frac{f_1(X(t))}{\tau^n}, \ldots, -\frac{f_{n-3}(X(t))}{\tau^2}, 0, 0\right)^T$.

The following Volterra integral transformation is used for the PDE subsystem:

$$\tilde{u}(x,t) = \tilde{w}(x,t) - \int_x^1 q(x,y)\tilde{w}(y,t)dy. \tag{4}$$

Select the target cascade system consisting of ODE and stable PDE as follows:

$$\begin{cases}\dot{\tilde{X}}(t) = \frac{1}{\tau}(A+KC)\tilde{X}(t) + \frac{1}{\tau}B\tilde{w}(0,t)\\ \qquad\qquad - \frac{1}{\tau}B\int_0^1 p(0,y)\tilde{w}(y,t)dy + F(X(t)),\\ \tilde{w}_t(x,t) = \tilde{w}_{xx}(x,t),\\ \tilde{w}_x(0,t) = 0,\\ \tilde{w}_x(1,t) = -c_1\tilde{w}(1,t),\end{cases} \tag{5}$$

where c_1 is a positive constant that can be selected arbitrarily. For the convenience of subsequent stability analysis, c_1 is selected to satisfy $c_1 \geq 1$ in this paper.

After calculation, the kernel function $q(x,y)$ can be designed to satisfy

$$q_{xx}(x,y) - q_{yy}(x,y) = -\mu q(x,y),$$
$$q(x,x) = -\frac{\mu}{2}x,$$
$$q_x(0,y) = 0,$$

the solution of which is $q(x,y) = -\mu y\frac{I_1(\sqrt{\mu(y^2-x^2)})}{\sqrt{\mu(y^2-x^2)}}$. $q_1(x)$ and q_0 are designed as $-q_y(x,1) - c_1 q(x,1)$ and $c_1 + \frac{\mu}{2}$.

3.2. Observer-based controller design

To construct the controller, we need to define transformations of the observer system (2) and analyze it. According to the characteristic of the observer system (2), some transformations of the ODE subsystem and the PDE subsystem do not change the structure of the observer system (2). Define transformations,

$$\hat{Z}(t) = \Gamma\hat{X}(t), \tag{6}$$

for the ODE subsystem and

$$\hat{w}(x,t) = \hat{u}(x,t) - \int_0^x s(x,y)\hat{u}(y,t)dy, \tag{7}$$

for the PDE subsystem. Let kernel function $s(x,y)$ satisfy the following partial differential equation:

$$s_{xx}(x,y) - s_{yy}(x,y) = \mu s(x,y),$$
$$s(x,x) = -\frac{\mu}{2}x,$$
$$s_y(x,0) = 0,$$

the solution of which is $s(x,y) = -\mu x\frac{I_1(\sqrt{\mu(x^2-y^2)})}{\sqrt{\mu(x^2-y^2)}}$. Then the observer system (2) can be converted into the following system:

$$\begin{cases}\dot{\hat{Z}}(t) = \frac{1}{\tau}A\hat{Z}(t) + \frac{1}{\tau}B\hat{w}(0,t) + \frac{1}{\tau}D\tilde{X}(t),\\ \hat{w}_t(x,t) = \hat{w}_{xx}(x,t) + q_1(x)\tilde{w}(1,t)\\ \qquad\qquad - \int_0^x s(x,y)q_1(y)dy\tilde{w}(1,t),\\ \hat{w}_x(0,t) = 0,\\ \hat{w}_x(1,t) = U(t) + (c_1 + \frac{\mu}{2})u(1,t) - c_1\hat{u}(1,t)\\ \qquad\qquad - \int_0^1 s_x(1,y)\hat{u}(y,t)dy,\end{cases}$$

with $D = \begin{pmatrix} -K & 0_{n\times(n-1)} \end{pmatrix}$.

Inspired by the transform idea in [21], the following transformation

$$\hat{v}(x,t) = \hat{w}(x,t) - \varphi(x)\hat{Z}(t), \tag{8}$$

is taken into account, where $\varphi(x)$ is a vector function to be derived later.

Remark 3: The backstepping transformation $w(x,t) = \hat{u}(x,t) - \int_0^x q(x,y)\hat{u}(y,t)dy + \varphi(x)\hat{Z}(t)$ is often used to cope with the stabilization problem for cascade ODE-PDE systems and boundary coupled systems ([1, 9, 33]). However, when the ODE-subsystem is a nonlinear system, the kernel function of the invertible transformation is very difficult to be found. Thus, the transformation with the simple form (8) is used in this paper, which is invertible obviously.

Our goal now is to find the function $\varphi(x)$ and controller $U(t)$ such that the observer system coincides with the following target system:

$$\begin{cases} \dot{\hat{Z}}(t) = \frac{1}{\tau}(A + B\bar{B})\hat{Z}(t) + \frac{1}{\tau}B\hat{v}(0,t) + \frac{1}{\tau}D\tilde{X}(t), \\ \hat{v}_t(x,t) = \hat{v}_{xx}(x,t) + (q_1(x) - \int_0^x s(x,y)q_1(y)dy)\tilde{w}(1,t) \\ \qquad - \frac{1}{\tau}\varphi(x)B\bar{B}\hat{Z}(t) - \frac{1}{\tau}\varphi(x)B\hat{v}(0,t) \\ \qquad - \frac{1}{\tau}\varphi(x)D\tilde{X}, \\ \hat{v}_x(0,t) = 0, \\ \hat{v}_x(1,t) = -c_2\hat{v}(1,t), \end{cases} \tag{9}$$

where $\bar{B}$ is given in Lemma 1, and c_2 is an arbitrary positive constant. The target systems (5) and (9) will be verified to be exponentially stable by choosing proper Lyapunov functional.

After calculation, $\varphi(x)$ is defined to satisfy the following equations:

$$\begin{cases} \varphi''(x) = \frac{1}{\tau}\varphi(x)A, \qquad 0 \le x \le 1, \\ \varphi(0) = \bar{B}, \\ \varphi'(0) = 0, \end{cases}$$

which expresses a second order ODE, and the explicit solution is found as

$$\varphi(x) = \begin{pmatrix} \bar{B} & \mathbf{0} \end{pmatrix} e^{\begin{pmatrix} \mathbf{0} & \frac{A}{\tau} \\ I & \mathbf{0} \end{pmatrix} x} \begin{pmatrix} I \\ \mathbf{0} \end{pmatrix}.$$

Obviously, $\varphi(x)$ is a bounded vector function, it is assumed that $\|\varphi(x)\| \le M$, with $M = \|\bar{B}\|\sqrt{n}e^{\sqrt{2n-1}}$.

To satisfy the right boundary term of the system (9), the controller is designed as,

$$\begin{aligned} U(t) = &-(c_1 + \frac{\mu}{2})u(1,t) + (c_1 - c_2)\hat{u}(1,t) \\ &+ c_2\int_0^1 s(1,y)\hat{u}(y,t)dy + c_2\varphi(1)\hat{Z}(t) \\ &+ \int_0^1 s_x(1,y)\hat{u}(y,t)dy + \varphi'(1)\hat{Z}(t). \end{aligned} \tag{10}$$

Remark 4: This type of controller structure is common in many cascade ODE-PDE systems since the solution to the differential equation that $\varphi(x)$ satisfies is an exponential function, see [9, 10]. Controller (10) contains two parts, one is about states of PDE which is used to stabilized the PDE subsystem, the other is about states of ODE which can transfer to boundary $u(0,t)$ by PDE subsystem to stabilize the ODE subsystem.

4. MAIN RESULT

Based on the observer (2) and the controller (10), the globally exponential stability analysis of the closed-loop system is presented in this section.

Theorem 1: Under Assumption 1, there exists an output feedback controller with the form (10), such that the closed-loop system consisting of (1), (2) and (10) is globally exponentially stable.

Proof: Choose Lyapunov functional candidates for the error system (5) and the observer system (9) as $V_1 = \tilde{X}(t)^T P\tilde{X}(t) + \frac{a}{2}\int_0^1 \tilde{w}^2(x,t)dx$ and $V_2 = \hat{Z}(t)^T Q\hat{Z}(t) + \frac{1}{2}\int_0^1 \hat{v}^2(x,t)dx$.

For analyzing the stability of the closed-loop system consisting of (1), (2) and (10), the Lyapunov functional candidate $V = V_1 + \frac{1}{\tau}V_2$ is considered. Then, the derivative of V along (5) and (9) is obtained,

$$\begin{aligned} &\dot{V}|_{(5)(9)} \\ &\le -\frac{1}{\tau}||\tilde{X}(t)||^2 - \frac{1}{\tau^2}||\hat{Z}(t)||^2 + \frac{2}{\tau^2}\hat{Z}(t)^T QD\tilde{X}(t) \\ &\quad + \frac{2}{\tau}\tilde{X}(t)^T PB\tilde{w}(0,t) - ac_1\tilde{w}^2(1,t) - a\int_0^1 \tilde{w}_x^2(x,t)dx \\ &\quad + \frac{1}{\tau}\int_0^1 \hat{v}(x,t)(q_1(x) - \int_0^x s(x,y)q_1(y)dy)dx\tilde{w}(1,t) \\ &\quad - \frac{1}{\tau}\int_0^1 \hat{v}_x^2(x,t)dx - \frac{1}{\tau^2}\int_0^1 \hat{v}(x,t)\varphi(x)D\tilde{X}(t)dx \\ &\quad - \frac{c_2}{\tau}\hat{v}^2(1,t) - \frac{2}{\tau}\tilde{X}(t)^T PB\int_0^1 q(0,y)\tilde{w}(y,t)dy \\ &\quad + \frac{2}{\tau^2}\hat{Z}(t)^T QB\hat{v}(0,t) + 2\tilde{X}(t)^T PF(X(t)) \\ &\quad - \frac{1}{\tau^2}\int_0^1 \hat{v}(x,t)\varphi(x)B\hat{v}(0,t)dx \\ &\quad - \frac{1}{\tau^2}\int_0^1 \hat{v}(x,t)\varphi(x)B\bar{B}\hat{Z}(t)dx. \end{aligned}$$

The nonlinear term is analyzed as follows:

$$||F(X(t))|| \le \frac{k}{\tau^2}(||\hat{Z}(t)|| + ||\tilde{X}(t)||),$$

where $k = m(n-1)$ and the above transformations of ODE subsystem are applied to cope with $X(t)$.

By Young's inequality, one can have

$$|2\tilde{X}(t)^T PF(X(t))|$$

$$\leq (\frac{2k}{\tau^2}||P|| + \frac{4k^2}{\tau^2}||P||^2)||\tilde{X}(t)||^2 + \frac{1}{4\tau^2}||\hat{Z}(t)||^2.$$

Applying the Young's inequality to cope with the cross terms, the following estimations are obtained:

$$\left|\frac{2}{\tau^2}\hat{Z}(t)^T QD\tilde{X}(t)\right| \leq \frac{1}{4\tau^2}||\hat{Z}(t)||^2 + \frac{4||QD||^2}{\tau^2}||\tilde{X}(t)||^2,$$
$$\left|\frac{2}{\tau^2}\hat{Z}(t)^T QB\hat{v}(0,t)\right| \leq \frac{4||QB||^2}{\tau^2}\hat{v}^2(0,t) + \frac{1}{4\tau^2}||\hat{Z}(t)||^2,$$
$$\left|\frac{2}{\tau}\tilde{X}(t)^T PB\tilde{w}(0,t)\right| \leq \frac{||PB||^2}{\tau^2}||\tilde{X}(t)||^2 + \tilde{w}^2(0,t),$$

$$\left|-\frac{1}{\tau^2}\int_0^1 \hat{v}(x,t)\varphi(x)D\tilde{X}(t)dx\right|$$
$$\leq \frac{M||D||}{2\tau^2}\int_0^1 \hat{v}^2(x,t)dx + \frac{M||D||}{2\tau^2}||\tilde{X}(t)||^2,$$
$$\left|-\frac{1}{\tau^2}\int_0^1 \hat{v}(x,t)\varphi(x)B\hat{v}(0,t)dx\right|$$
$$\leq \frac{M}{2\tau^2}\int_0^1 \hat{v}^2(x,t)dx + \frac{M}{2\tau^2}\hat{v}^2(0,t),$$
$$\left|-\frac{1}{\tau^2}\int_0^1 \hat{v}(x,t)\varphi(x)B\bar{B}\hat{Z}(t)dx\right|$$
$$\leq \frac{2M^2||B\bar{B}||^2}{\tau^2}\int_0^1 \hat{v}^2(x,t)dx + \frac{1}{8\tau^2}||\hat{Z}(t)||^2,$$
$$\left|\frac{2}{\tau}\tilde{X}^T PB\int_0^1 q(0,y)\tilde{w}(y,t)dy\right|$$
$$\leq \frac{||PB||^2}{\tau^2}||\tilde{X}||^2 + k_1^2\int_0^1 \tilde{w}^2(x,t)dx,$$

and

$$\left|\frac{1}{\tau}\int_0^1 \hat{v}(x,t)(q_1(x) - \int_0^x s(x,y)q_1(y)dy)dx\tilde{w}(1,t)\right|$$
$$\leq \frac{1}{4\tau^2}\int_0^1 \hat{v}^2(x,t)dx + k_2^2\tilde{w}^2(1,t),$$

where $k_1 = \max_{y\in[0,1]}|q(0,y)|$ and $k_2 = \max_{x\in[0,1]}|q_1(x) - \int_0^x s(x,y) q_1(y)dy|$.

It is clear that $\tilde{w}(0,t) = \tilde{w}(1,t) - \int_0^1 \tilde{w}_x(x,t)dx$ and $\hat{v}(0,t) = \hat{v}(1,t) - \int_0^1 \hat{v}_x(x,t)dx$, then the following useful inequalities can be obtained by Schwarz inequality:

$$|\tilde{w}(0,t)|^2 \leq 2\tilde{w}^2(1,t) + 2\int_0^1 \tilde{w}_x^2(x,t)dx,$$
$$|\hat{v}(0,t)|^2 \leq 2\hat{v}^2(1,t) + 2\int_0^1 \hat{v}_x^2(x,t)dx,$$

and from Lemma 2, one gets the following two inequalities:

$$-\int_0^1 \tilde{w}_x^2(x,t)dx \leq \frac{1}{2}\tilde{w}^2(1,t) - \frac{1}{4}\int_0^1 \tilde{w}^2(x,t)dx,$$
$$-\int_0^1 \hat{v}_x^2(x,t)dx \leq \frac{1}{2}\hat{v}^2(1,t) - \frac{1}{4}\int_0^1 \hat{v}^2(x,t)dx.$$

Then the the derivative of V along (5) and (9) can be rewritten as follows:

$$\begin{aligned}\dot{V}|_{(5)(9)} \leq & -\nu_1||\tilde{X}(t)||^2 - \frac{1}{8\tau^2}||Z(t)||^2 - \nu_3\int_0^1 \hat{v}^2(x,t)dx \\ & -\nu_4\int_0^1 \hat{v}_x^2(x,t)dx - \nu_2\hat{v}^2(1,t) - \nu_5\tilde{w}^2(1,t) \\ & -\nu_6\int_0^1 \tilde{w}^2(x,t)dx - \nu_7\int_0^1 \tilde{w}_x^2(x,t)dx,\end{aligned} \tag{11}$$

where the coefficients $\nu_i, i = 1,\ldots,7$, have the following forms:

$$\nu_1 = \frac{1}{\tau} - \frac{2k||P|| + 4k^2||P||^2 + 4||QD||^2 + 2||PB||^2 + \frac{M||D||}{2}}{\tau^2}$$
$$\nu_2 = \frac{c_2}{\tau} - \frac{8||QB||^2 + M}{\tau^2} - \frac{1}{4\tau},$$
$$\nu_3 = \frac{1}{8\tau} - \frac{1}{4\tau^2} - \frac{2M^2||B\bar{B}||^2}{\tau^2} - \frac{M}{2\tau^2} - \frac{M||D||}{2\tau^2},$$
$$\nu_4 = \frac{1}{2\tau} - \frac{8||QB||^2 + M}{\tau^2},$$
$$\nu_5 = \frac{5a}{8} - 2 - k_2^2,\ \ \nu_6 = \frac{3a}{16} - k_1^2,\ \ \nu_7 = \frac{a}{4} - 2.$$

From expressions of ν_1, ..., ν_4, we can choose

$$\begin{aligned}\tau = \max\Big\{ & \frac{16k||P|| + 32k^2||P||^2 + 32||QD||^2 + 16||PB||^2}{7} \\ & + \frac{4M||D||}{7},\ \frac{32||QB||^2 + 4M}{4c_2 - 1},\ 16||QB||^2 + 2M, \\ & 4 + 32M^2||B\bar{B}||^2 + 8M(1 + ||D||)\Big\},\end{aligned} \tag{12}$$

and from coefficients $\nu_5,\ldots,\nu_7$, one can choose

$$a = \max\left\{\frac{16}{5} + \frac{8k_2^2}{5}, 16k_1^2, 8\right\}. \tag{13}$$

(12) and (13) make the seven coefficients non-negative. It follows from (11) and (12) that

$$\begin{aligned}\dot{V}|_{(5)(9)} \leq & -\frac{1}{8\tau}||\tilde{X}(t)||^2 - \frac{a}{16}\int_0^1 \tilde{w}^2(x,t)dx \\ & -\frac{1}{16\tau}\int_0^1 \hat{v}^2(x,t)dx - \frac{1}{8\tau^2}||\hat{Z}(t)||^2 \\ \leq & -\beta V,\end{aligned}$$

where $\beta = \min\left\{\frac{1}{8\tau\lambda_{\max}(P)}, \frac{1}{8\tau\lambda_{\max}(Q)}, \frac{1}{8}\right\}$, then we can obtain

$$\begin{aligned}& ||\tilde{X}(t)|| + ||\tilde{w}(x,t)|| + ||\hat{Z}(t)|| + ||\hat{v}(x,t)|| \\ & \leq 2\sqrt{\alpha_1}e^{-\frac{\beta}{2}t}(||\tilde{X}(0)|| + ||\tilde{w}(x,0)|| + ||\hat{Z}(0)|| \\ & \quad + ||\hat{v}(x,0)||),\end{aligned} \tag{14}$$

where $\alpha_1 = \frac{\max\left\{\lambda_{\max}(P), \frac{\lambda_{\max}(Q)}{\tau}, \frac{a}{2}, \frac{1}{2\tau}\right\}}{min\left\{\lambda_{\min}(P), \frac{\lambda_{\min}(Q)}{\tau}, \frac{a}{2}, \frac{1}{2\tau}\right\}}$. (14) indicates that systems (5) and (9) are exponentially stable at $\tilde{X}(t) = 0$, $\tilde{w}(x,t) = 0$, $\hat{Z}(t) = 0$, and $\hat{v}(x,t) = 0$.

From the above transformations, the following inequality can be obtained:

$$\begin{aligned}&||X(t)|| + ||u(x,t)|| + ||\hat{X}(t)|| + ||\hat{u}(x,t)|| \\ &\leq 6\sqrt{\alpha_2} e^{-\frac{\beta}{2}t}(||X(0)|| + ||u(x,0)|| + ||\hat{X}(0)|| \\ &\quad + ||\hat{u}(x,0)||),\end{aligned} \tag{15}$$

and

$$\begin{aligned}\alpha_2 =& \alpha_1 \max\left\{3||\Gamma^{-1}||^2 + p_2 M^2, p_1, p_2\right\} \\ &\times \max\left\{3||\Gamma||^2 + 2M^2||\Gamma||^2, p_3, p_4\right\},\end{aligned}$$

where $p_1 = 4 + 4\max_{y\in[0,1]} \int_0^y q^2(x,y)dx$, $p_2 = 12 + 12\max_{y\in[0,1]} \int_y^1 l^2(x,y)dx$, $p_3 = 2 + 2\max_{y\in[0,1]} \int_0^y p^2(x,y)dx$, and $p_4 = 4 + 4\max_{y\in[0,1]} \int_y^1 s^2(x,y)dx$, with $l(x,y)$ and $p(x,y)$ being the kernels of inverse transformation $\hat{u} = \hat{w} + \int_0^x l(x,y)\hat{w}(y,t)dy$ and $\tilde{w} = \tilde{u} + \int_x^1 p(x,y)\tilde{u}(y,t)dy$.

It is clear that the closed-loop system consisting of (1), (2) and controller (10) is globally exponentially stable. □

Remark 5: From Theorem 1, the closed-loop system is globally exponentially stable, thus it has a unique solution.

Remark 6: As shown in the proof of Theorem 1, the parameter τ and boundary damping are used to dominate the nonlinearities and coupling terms. For any $c_1 \geq 1$ and $c_2 > 0$, if parameter τ is picked sufficiently large, the observer, error system and controller converge.

Corollary 1: When $\mu \leq 0$, we can design the observer and controller of the system (1) as follows:

$$\begin{cases}\dot{\hat{X}}(t) = A\hat{X}(t) + B\hat{u}(0,t) - \Omega K(X_1(t) - \hat{X}_1(t)), \\ \hat{u}_t(x,t) = \hat{u}_{xx}(x,t) + \mu\hat{u}(x,t), \\ \hat{u}_x(0,t) = 0, \\ \hat{u}_x(1,t) = U(t) + c_1[u(1,t) - \hat{u}(1,t)],\end{cases} \tag{16}$$

and

$$U(t) = -c_1 u(1,t) + c_1\varphi(1)\hat{Z}(t) + \varphi'(1)\hat{Z}(t), \tag{17}$$

where

$$\varphi(x) = \begin{pmatrix} \bar{B} & \mathbf{0} \end{pmatrix} e^{\begin{pmatrix} \mathbf{0} & -\mu I + \frac{A}{\tau} \\ I & \mathbf{0} \end{pmatrix} x} \begin{pmatrix} I \\ \mathbf{0} \end{pmatrix}.$$

The closed-loop system consisting of (1), (16) and controller (17) is exponentially stable.

5. NUMERICAL EXAMPLES

In this section, we present two numerical examples to demonstrate the effectiveness of the proposed output feedback controllers. Choose the trapezoidal method to discretize ODE-subsystem and the finite difference scheme to discretize PDE-subsystem. Time and space steps are taken as 0.001 and 0.05, respectively. A three-dimensional nonlinear ODE-PDE system is

$$\begin{cases}\dot{X}_1(t) = X_2(t) + \omega(t, X(t))X_3(t), \\ \dot{X}_2(t) = X_3(t), \\ \dot{X}_3(t) = u(0,t), \\ u_t(x,t) = u_{xx}(x,t) + \mu u(x,t), \\ u_x(0,t) = 0, \\ u_x(1,t) = U(t),\end{cases} \tag{18}$$

where $|\omega(t,X(t))|$ is bounded by a known constant ε, and it is not difficult to verify that Assumption 1 holds.

Let $\mu = 5$ in (18). By (2), the observer of (18) can be designed as,

$$\begin{cases}\dot{\hat{X}}_1(t) = \hat{X}_2(t) + \frac{0.6}{\tau}(X_1(t) - \hat{X}_1(t)), \\ \dot{\hat{X}}_2(t) = \hat{X}_3(t) + \frac{0.7}{\tau^2}(X_1(t) - \hat{X}_1(t)), \\ \dot{\hat{X}}_3(t) = \hat{u}(0,t) + \frac{0.2}{\tau^2}(X_1(t) - \hat{X}_1(t)), \\ \hat{u}_t(x,t) = \hat{u}_{xx}(x,t) + 5\hat{u}(x,t) + \Big(5(c_1+1)\frac{I_1(\sqrt{2(1-x^2)})}{\sqrt{2(1-x^2)}} \\ \qquad + 5\frac{I_2(\sqrt{2(1-x^2)})}{1-x^2}\Big)(u(1,t) - \hat{u}(1,t)), \\ \hat{u}_x(0,t) = 0, \\ \hat{u}_x(1,t) = U(t) + q_0(u(1,t) - \hat{u}(1,t)).\end{cases} \tag{19}$$

Assume that the initial conditions of the systems (18) and (19) are $X(0) = (-30\ 5\ 0.2)^T$, $\hat{X}(0) = (0\ 0\ 0)^T$, $u(x,0) = x + \frac{1}{10}$ and $\hat{u}(x,0) = 0.1$, $x \in (0,1)$. Choosing $\varepsilon = 0.1$ and Hurwitz coefficients $b_1 = 0.2$, $b_2 = 1.2$, and $b_3 = 0.6$, one can calculate that the value of τ is 276.0172. We set $c_1 = 30$ and $c_2 = 30$ here, then $q_0 = 32.5$ and the following controller can be obtained:

$$\begin{aligned}U(t) =& -32.5u(1,t) + 30\varphi(1)\begin{pmatrix}\frac{\hat{X}_1(t)}{\tau^3} & \frac{\hat{X}_2(t)}{\tau^2} & \frac{\hat{X}_3(t)}{\tau}\end{pmatrix}^T \\ &- \int_0^1\Big(155\frac{I_1(\sqrt{6(1-y^2)})}{\sqrt{6(1-y^2)}} + 5\frac{I_2(\sqrt{6(1-y^2)})}{1-y^2}\Big) \\ &\times \hat{u}(y,t)dy + \varphi'(1)\begin{pmatrix}\frac{\hat{X}_1(t)}{\tau^3} & \frac{\hat{X}_2(t)}{\tau^2} & \frac{\hat{X}_3(t)}{\tau}\end{pmatrix}^T.\end{aligned} \tag{20}$$

The numerical results are depicted in the following five pictures. Figs. 1, 2 and 3 show states and corresponding

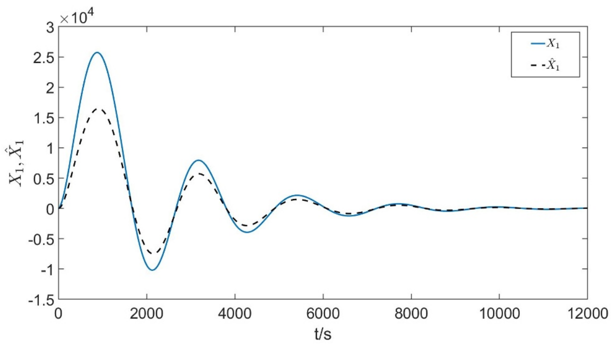

Fig. 1. The states X_1 and $\hat{X}_1$ of the closed-loop system consisting of (18), (19) and (20) with $\mu = 5$.

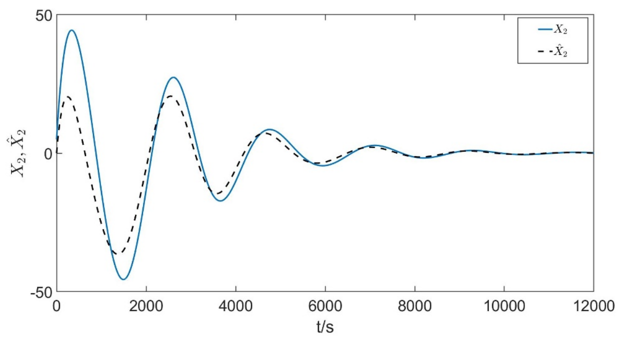

Fig. 2. The states X_2 and $\hat{X}_2$ of the closed-loop system consisting of (18), (19) and (20) with $\mu = 5$.

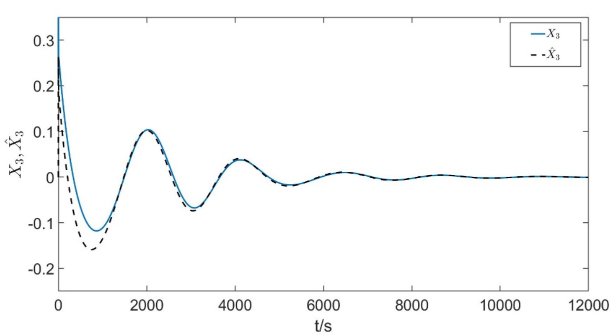

Fig. 3. The states X_3 and $\hat{X}_3$ of the closed-loop system consisting of (18), (19) and (20) with $\mu = 5$.

observer states of the ODE-subsystem, and Figs. 4 and 5 present the state and observer state of the PDE-subsystem. Figs. 1-5 validate our results clearly.

Let $\mu = -5$ in (18), which is considered as a numerical experiment of corollary. From (16), the observer of the system (18) with $\mu = -5$ is as follows:

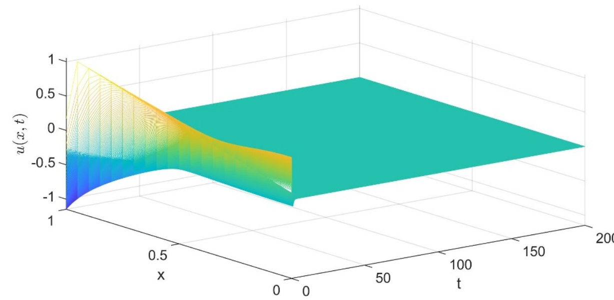

Fig. 4. The state $u(x,t)$ of the closed-loop system consisting of (18), (19) and (20) with $\mu = 5$.

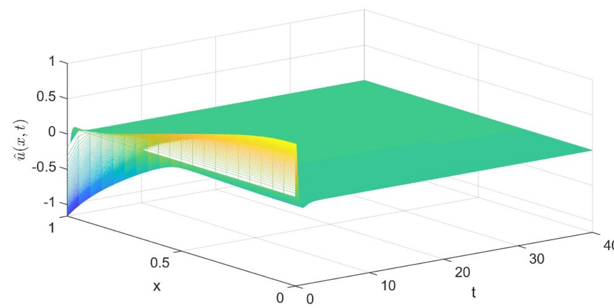

Fig. 5. The state $\hat{u}(x,t)$ of the closed-loop system consisting of (18), (19) and (20) with $\mu = 5$.

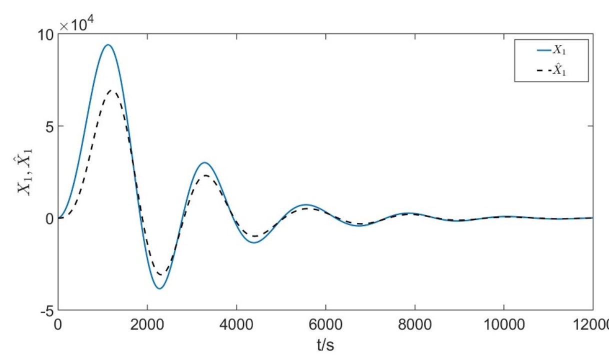

Fig. 6. The states X_1 and $\hat{X}_1$ of the closed-loop system consisting of (18), (21) and (22) with $\mu = -5$.

$$\begin{cases} \dot{\hat{X}}_1(t) = \hat{X}_2(t) + \dfrac{0.6}{\tau}(X_1(t) - \hat{X}_1(t)), \\ \dot{\hat{X}}_2(t) = \hat{X}_3(t) + \dfrac{0.7}{\tau^2}(X_1(t) - \hat{X}_1(t)), \\ \dot{\hat{X}}_3(t) = \hat{u}(0,t) + \dfrac{0.2}{\tau^2}(X_1(t) - \hat{X}_1(t)), \\ \hat{u}_t(x,t) = \hat{u}_{xx}(x,t) - 5\hat{u}(x,t), \\ \hat{u}_x(0,t) = 0, \\ \hat{u}_x(1,t) = U(t) + c_1(u(1,t) - \hat{u}(1,t)). \end{cases} \tag{21}$$

Choosing $c_1 = 1$, the controller (17) is as follows:

$$U(t) = -u(1,t) + \varphi(1)\begin{pmatrix} \frac{\hat{X}_1(t)}{\tau^3} & \frac{\hat{X}_2(t)}{\tau^2} & \frac{\hat{X}_3(t)}{\tau} \end{pmatrix}^T + \varphi'(1)\begin{pmatrix} \frac{\hat{X}_1(t)}{\tau^3} & \frac{\hat{X}_2(t)}{\tau^2} & \frac{\hat{X}_3(t)}{\tau} \end{pmatrix}^T. \tag{22}$$

Under the same initial condition and τ, the numerical results are shown in Figs. 6-10, which validate the effective-

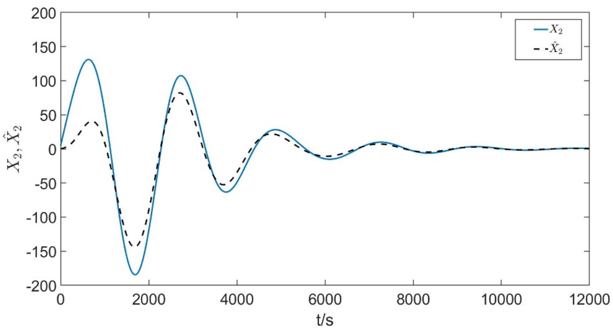

Fig. 7. The states X_2 and $\hat{X}_2$ of the closed-loop system consisting of (18), (21) and (22) with $\mu = -5$.

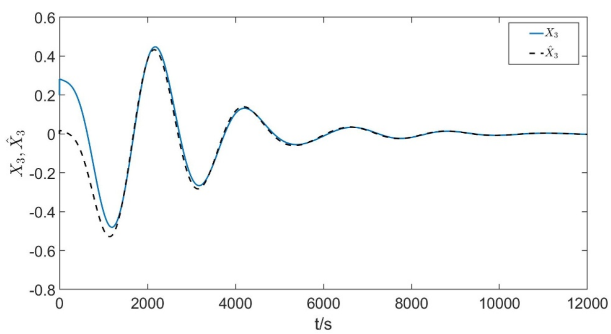

Fig. 8. The states X_3 and $\hat{X}_3$ of the closed-loop system consisting of (18), (21) and (22) with $\mu = -5$.

ness of the corollary.

Remark 7: Parameters $c_1 \geq 1$ and $c_2 > 0$ can be chosen arbitrarily. Then parameter q_0 and function $q_1(x)$ can be determined. τ is given in (12). In numerical examples, we find two points. One is that the greater the parameter τ is, the smaller the controller is and the longer the convergence time is. The other is that, under $\mu \leq 0$, the control performance is better when c_1 is small.

6. CONCLUSIONS

This paper has studied output feedback stabilization problem for the cascade feedforward nonlinear ODE-PDE system, where the unstable diffusion PDE subsystem has been considered. First, under a linear growth condition (Assumption 1), low gain observers have been designed. Two output feedback controllers have been constructed by the transformations step by step. Second, the globally exponential stability of the closed-loop system has been obtained by constructing proper Lyapunov functional.

Our work will be further studied from the following three aspects: (i) Study the stabilization problem for triangular nonlinear system with actuator described by other types of PDE, such as second-order wave equation, Euler-Bernoulli beam equation; (ii) Stabilization problem for cascade triangular nonlinear ODE-PDE systems with disturbance is still an open issue. (iii) Tracking problem for cascade nonlinear ODE-PDE system or boundary coupled system may be solved.

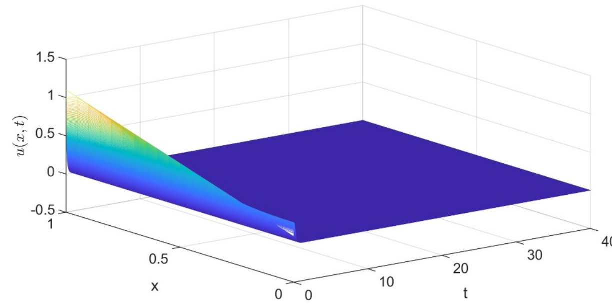

Fig. 9. The state $u(x,t)$ of the closed-loop system consisting of (18), (21) and (22) with $\mu = -5$.

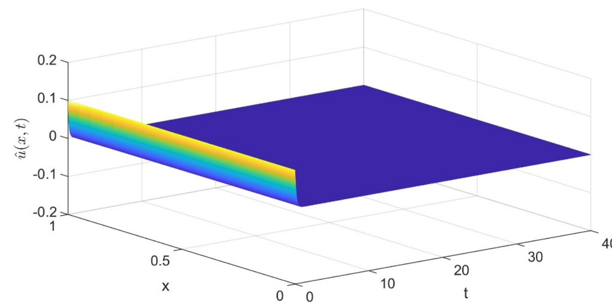

Fig. 10. The state $\hat{u}(x,t)$ of the closed-loop system consisting of (18), (21) and (22) with $\mu = -5$.

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