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Consensus for a Class of Sampled-data Heterogeneous Multi-agent Systems

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Abstract: This paper investigates the problem of consensus for one kind of sampled-data heterogeneous multi-agent system. We first discretize the continuous-time networked system by the method of sampled-data. Then we use a system transformation to transform the multi-agent system into the reduced-order error system. Two consensus algorithms with and without velocity measurement will be considered respectively. We obtain two sufficient conditions for the networked systems with and without velocity measurement to reach consensus through analyzing the stable problem of the reduced-order systems. Simulations are given to verify the effectiveness of the results.

Keywords: Consensus, heterogeneous system, multi-agent system, sampled-data system.

1. INTRODUCTION

The multi-agent system has continued to receive considerable attention over the last twenty years [1–5]. The collaboration of multiple agents can solve many problems that single agent can not solve. Therefore, we may find the wide-ranging applications of distributed control of multi-agent systems, for example, mobile robots, sensor networks, formation of aircrafts, smart tourism, and many topics about multi-agent system have been considered. Sampled-data control plays a very important role in practical systems, especially digital control systems [6–8]. It also appears in the distributed control of multi-agent systems, recently [9–13]. In [9], the protocol based on aperiodic sampled-data was used to study the containment control of continuous-time multi-agent system. Based on the sampled position data, the event-triggered containment control problem was considered in [10]. In [11], the authors also used the sampled position information to design the consensus control algorithm, where the group consensus problem for second-order multi-agent systems with switching topologies and random link failures was studied. The sampled-data method was used to studied the leaderless consensus problem of non-linear multi-agent systems with random packet losses in [12]. Most samples in existed literature about multi-agent systems are synchronous. While the asynchronous phenomena are common in the process of information transmission of multi-agent systems. Asynchronous sampled-data multi-agent systems was considered in [13], where the consensus problem was converted into the convergence problem of products of infinite general sub-stochastic matrices. For more results about distributed sampled-data control of multi-agent systems, we refer the reader to [14], where the authors summarized the results about sampled-data multi-agent systems in the past decade years.

In the past few years, the heterogeneous networked systems have a strong appeal to many academics in the field of control. The heterogeneous networked systems are used to depict the systems with different model structure or different parameter, such as the time-delay [15–26]. In [15], the heterogeneous systems consisted of different state equations, where the problem of output synchronization was investigated. In [16], the heterogeneous multi-agent systems considered were composed of first-order and second-order integrator. In [17], the authors studied the coordination problem of the continuous-time heterogeneous multi-agent system with a leader, where the interaction topologies were switching jointly connected. In [18], the authors considered the output consensus problem for a kind of linear multi-agent systems. Where the results in forms of matrix equations are obtained by proposing a new event-triggered protocol. The output regulation problem for a kind of networked systems consisted of some followers and one leader was studied in [19], where the differential graphical game was used. Based on output regulation, the containment control problem of a class of heterogeneous multi-agent system was considered in [20]. On the other hand, we can find some results on the problem of group consensus of heterogeneous net-

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worked systems in [21] and [22]. In [21], the authors considered the group consensus problem for a kind of heterogeneous multi-agent systems under the switching topology and fixed topology. In [22], a new consensus algorithm for group consensus of heterogeneous multi-agent system was proposed on the basis of the point of agents. In [23], the finite-time consensus problem of the high-order heterogeneous multi-agent system was studied. Two classes of consensus protocols on the basis of state feedback and output feedback ware provided, respectively. In [24], the authors used the sampled-data method to consider the problem of consensus for a kind of networked systems, where a sampling delay was considered. While the fixed-time convergence problem of nonlinear heterogeneous multi-agent systems was studied in [25]. To the best of author's knowledge, there are still many problems in heterogeneous multi-agent systems which are worth studying.

Inspired by the aforementioned literature, we will study the problem of consensus for sampled-data networked system with heterogeneous structure. We suppose that the system considered is made up of several first-order integrators and second-order integrators. We transform the continuous-time system to the discrete-time system by using the method of sampled-data. Moreover, we convert the multi-agent system into the reduced-order error system. The consensus algorithms with and without velocity measurement will be considered respectively. We obtain the conditions of consensus by studying the stable problem of the reduced-order models.

The organization of this paper is as follows: In the following section, the problem formulation is given. In Section 3, the model transformation method is given to convert the multi-agent system into the reduced-order model. The main consensus results are obtained based on linear system theory. Section 4 provides simulation results, and conclusions are stated in Section 5.

Notations: We use $\mathbb{R}$ to represent the real number set. The spectral radius of matrix A is denoted as $\rho(A)$. S_n is a index set, for example, $S_r = \{1,2,\ldots,r\}$ and $S_{n-r} = \{r+1, r+2,\ldots,n\}$, respectively, in this paper. We use $|M|$ to represent the determinant of matrix M. I_n and $\mathbf{0}$ are used to represent the $n\times n$ identity matrix and zero matrix with appropriate dimension, respectively. For a complex number m, the real part and imaginary part are denoted as $\mathrm{Re}(m)$ and $\mathrm{Im}(m)$, respectively. Some necessary graph theory notions are refer to [27], here is omitted.

2. PROBLEM FORMULATION

In this paper we suppose that the multi-agent system under consideration is made up of r second-order integrators and $n-r$ first-order integrators $(r<n)$. The second-order integrator has the dynamics as follows:

$$\dot{p}_i(t) = v_i(t),\ \ \dot{v}_i(t) = u_i(t),\ \ i\in S_r, \tag{1}$$

where $u_i(t)$ is the control input, $p_i(t)$ and $v_i(t)$ are the position state and velocity state of the ith agent, respectively. The first-order integrator has the dynamics as follows:

$$\dot{p}_i(t) = u_i(t),\ \ i\in S_{n-r}. \tag{2}$$

By using the direct discretization method proposed in reference [6], we obtain the following discretized dynamics of (2) and (1), respectively,

$$p_i[k+1] = p_i[k] + Tu_i[k],\ \ i\in S_{n-r}, \tag{3}$$

$$\begin{cases} p_i[k+1] = p_i[k] + Tv_i[k] + \dfrac{T^2}{2}u_i[k], \\ v_i[k+1] = v_i[k] + Tu_i[k], \end{cases} \ i\in S_r, \tag{4}$$

where T is the sampling period.

In this paper, we select the following consensus algorithm for the first-order agent,

$$u_i[k] = \alpha\sum_{j\in N_i} a_{ij}(p_j[k]-p_i[k]),\ \ i\in S_{n-r}, \tag{5}$$

where $\alpha>0$ is the control gain. Two different kinds of the consensus protocols of the second-order integrator will be considered in this paper.

$$u_i[k] = \alpha\sum_{j\in N_i} a_{ij}(p_j[k]-p_i[k]) - \beta v_i[k],\ \ i\in S_r, \tag{6}$$

$$u_i[k] = \alpha\sum_{j\in N_i} a_{ij}(p_j[k]-p_i[k]) + \beta\sum_{j\in N_i} a_{ij}(v_j[k]-v_i[k]),\ \ i\in S_r, \tag{7}$$

where α, $\beta>0$ are the control gains. a_{ij} is the weight of the topology, if the graph contains the correspond edge, then $a_{ij}>0$, otherwise, $a_{ij}=0$.

Remark 1: The algorithms in (6) and (7) can be found in some existing literature, such as [2], where the authors have considered a lot of cooperative control problems and applied the theoretical results to practical multi-vehicle systems. In our paper, we will study sampled-data consensus problem for heterogeneous multi-agent system with (6) and (7), which is different from the results in existing literature. When the system is heterogeneous, the consensus problem will become more complicated.

Let $P[k] = [p_1[k],\ldots,p_r[k],\ p_{r+1}[k],\ldots,p_n[k]]^T$, $V[k] = [v_1[k],\ldots,v_r[k]]^T$. The discrete-time multi-agent systems (3) and (4) with algorithms (5) and (6) can be expressed in the following compact form respectively:

$$\begin{bmatrix} P[k+1] \\ V[k+1] \end{bmatrix} = \Xi_1 \begin{bmatrix} P[k] \\ V[k] \end{bmatrix}, \tag{8}$$

where

$$\Xi_1 = \begin{bmatrix} I_r - \frac{\alpha T^2}{2}\mathcal{L}_{11} & -\frac{\alpha T^2}{2}\mathcal{L}_{12} & (T - \frac{\beta T^2}{2})I_r \\ -\alpha T\mathcal{L}_{21} & I_{(n-r)} - \alpha T\mathcal{L}_{22} & \mathbf{0} \\ -\alpha T\mathcal{L}_{11} & -\alpha T\mathcal{L}_{12} & (1-\beta T)I_r \end{bmatrix}.$$

$\mathcal{L}_{11}$, $\mathcal{L}_{12}$, $\mathcal{L}_{21}$, $\mathcal{L}_{22}$ have the following definitions:

$$\mathcal{L} = \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} \\ \mathcal{L}_{21} & \mathcal{L}_{22} \end{bmatrix},$$

and

$$\mathcal{L}_{11} = \begin{bmatrix} l_{11} & \dots & l_{1r} \\ \vdots & \dots & \vdots \\ l_{r1} & \cdots & l_{rr} \end{bmatrix},$$

$$\mathcal{L}_{12} = \begin{bmatrix} l_{1(r+1)} & \cdots & l_{1n} \\ \vdots & \dots & \vdots \\ l_{r(r+1)} & \cdots & l_{rn} \end{bmatrix},$$

$$\mathcal{L}_{21} = \begin{bmatrix} l_{(r+1)1} & \cdots & l_{(r+1)r} \\ \vdots & \dots & \vdots \\ l_{n1} & \cdots & l_{nr} \end{bmatrix},$$

$$\mathcal{L}_{22} = \begin{bmatrix} l_{(r+1)(r+1)} & \cdots & l_{(r+1)n} \\ \vdots & \dots & \vdots \\ l_{n(r+1)} & \cdots & l_{nn} \end{bmatrix}.$$

Here $\mathcal{L}$ is the Laplacian matrix associated with topology $\mathcal{G}$ which is defined as $\mathcal{L} = [l_{ij}] \in \mathbb{R}^{n\times n}$, where $l_{ii} = \sum_{j=1,j\neq i}^{n} a_{ij}$ and $l_{ij} = -a_{ij}, \forall i \neq j$.

By using the same method to the above, the discretized multi-agent systems (3) and (4) with algorithms (5) and (7) respectively can be expressed in the following compact form:

$$\begin{bmatrix} P[k+1] \\ V[k+1] \end{bmatrix} = \Psi_1 \begin{bmatrix} P[k] \\ V[k] \end{bmatrix}, \tag{9}$$

where

$$\Psi_1 = \begin{bmatrix} I_r - \frac{\alpha T^2}{2}\mathcal{L}_{11} & -\frac{\alpha T^2}{2}\mathcal{L}_{12} & TI_r - \frac{\beta T^2}{2}\mathcal{L}_r \\ -\alpha T\mathcal{L}_{21} & I_{(n-r)} - \alpha T\mathcal{L}_{22} & \mathbf{0} \\ -\alpha T\mathcal{L}_{11} & -\alpha T\mathcal{L}_{12} & I_r - \beta T\mathcal{L}_r \end{bmatrix}.$$

$\mathcal{L}_{11}$, $\mathcal{L}_{12}$, $\mathcal{L}_{21}$, $\mathcal{L}_{22}$ are the same as the aforementioned. α and T are the control gain and sampling period respectively. $\mathcal{L}_r$ is the Laplacian matrix of a subgraph of $\mathcal{G}$ with node set S_r.

Definition 1: The heterogeneous multi-agent system (8) is said to achieve consensus if for any initial conditions, one has

$$\lim_{k\to\infty} |p_i[k] - p_j[k]| = 0, \ \forall i, j \in S_n,$$
$$\lim_{k\to\infty} |v_i[k] - v_j[k]| = 0, \ \forall i, j \in S_r.$$

3. CONSENSUS ANALYSIS

In this section, we will apply linear system theory to analyze the consensus problem of the aforementioned multi-agent systems. First we convert the multi-agent system under consideration into the reduced-order system by a model transformation. Let $z_i = p_i - p_1$, $i = 2, \dots, r$, $z_j = p_j - p_{r+1}$, $j = r+2, \dots, n$, $w_i = v_i - v_1$, $i = 2, \dots, r$. Denote

$$Z[k] = [z_2, \ \dots, \ z_r, \ z_{r+2}, \ \dots, \ z_n, \ w_2, \ \dots, \ w_r]^T.$$

It follows from (8) that

$$Z[k+1] = \Xi_2 Z[k], \tag{10}$$

where

$$\Xi_2 = \begin{bmatrix} I_{r-1} - \frac{\alpha T^2}{2}\overline{\mathcal{L}}_{11} & -\frac{\alpha T^2}{2}\overline{\mathcal{L}}_{12} & (T - \frac{\beta T^2}{2})I_{r-1} \\ -\alpha T\overline{\mathcal{L}}_{21} & I_{(n-r-1)} - \alpha T\overline{\mathcal{L}}_{22} & \mathbf{0} \\ -\alpha T\overline{\mathcal{L}}_{11} & -\alpha T\overline{\mathcal{L}}_{12} & (1-\beta T)I_{r-1} \end{bmatrix},$$

and $\overline{\mathcal{L}}_{11}$, $\overline{\mathcal{L}}_{12}$, $\overline{\mathcal{L}}_{21}$, $\overline{\mathcal{L}}_{22}$ have the following definitions:

$$\overline{\mathcal{L}} = \begin{bmatrix} \overline{\mathcal{L}}_{11} & \overline{\mathcal{L}}_{12} \\ \overline{\mathcal{L}}_{21} & \overline{\mathcal{L}}_{22} \end{bmatrix},$$

with

$$\overline{\mathcal{L}}_{11} = \begin{bmatrix} l_{22} - l_{12} & \dots & l_{2r} - l_{1r} \\ \vdots & \dots & \vdots \\ l_{r2} - l_{12} & \cdots & l_{rr} - l_{1r} \end{bmatrix},$$

$$\overline{\mathcal{L}}_{12} = \begin{bmatrix} l_{2(r+2)} - l_{1(r+2)} & \cdots & l_{2n} - l_{1n} \\ \vdots & \dots & \vdots \\ l_{r(r+2)} - l_{1(r+2)} & \cdots & l_{rn} - l_{1n} \end{bmatrix},$$

$$\overline{\mathcal{L}}_{21} = \begin{bmatrix} l_{(r+2)2} - l_{(r+1)2} & \cdots & l_{(r+2)r} - l_{(r+1)r} \\ \vdots & \dots & \vdots \\ l_{n2} - l_{(r+1)2} & \cdots & l_{nr} - l_{(r+1)r} \end{bmatrix},$$

$$\overline{\mathcal{L}}_{22} = \begin{bmatrix} l_{(r+2)(r+2)} - l_{(r+1)(r+2)} & \cdots & l_{(r+2)n} - l_{(r+1)n} \\ \vdots & \dots & \vdots \\ l_{n(r+2)} - l_{(r+1)(r+2)} & \cdots & l_{nn} - l_{(r+1)n} \end{bmatrix}.$$

Similarly, we can deduce the reduced model of (9) as follows:

$$Z[k+1] = \Psi_2 Z[k], \tag{11}$$

where

$$\Psi_2 = \begin{bmatrix} I_{r-1} - \frac{\alpha T^2}{2}\overline{\mathcal{L}}_{11} & -\frac{\alpha T^2}{2}\overline{\mathcal{L}}_{12} & \psi_{13} \\ -\alpha T\overline{\mathcal{L}}_{21} & \psi_{22} & \mathbf{0} \\ -\alpha T\overline{\mathcal{L}}_{11} & -\alpha T\overline{\mathcal{L}}_{12} & \psi_{33} \end{bmatrix}, \tag{12}$$

$$\psi_{13} = TI_{r-1} - \frac{\beta T^2}{2}\overline{\mathcal{L}}_r,$$
$$\psi_{22} = I_{(n-r-1)} - \alpha T\overline{\mathcal{L}}_{22},$$
$$\psi_{33} = I_{m-1} - \beta T\overline{\mathcal{L}}_r,$$

$$\overline{\mathcal{L}}_r = \begin{bmatrix} \bar{l}_{22}-l_{12} & \dots & l_{2r}-l_{1r} \\ \vdots & \dots & \vdots \\ l_{r2}-l_{12} & \cdots & \bar{l}_{rr}-l_{1r} \end{bmatrix}.$$

In order to deduce the main results of this paper, we first give some necessary assumptions and lemma as follows. It is worth pointing out that the graph considered in this paper is directed.

Assumption 1: The interaction topology $\mathcal{G}$ has a directed spanning tree.

Assumption 2: If the $(r+1)$th agent can obtain the information of the kth agent, then the jth agent can obtain the information of the kth agent $k \in \{2, \dots, r\}$, $j \in \{r+2, \dots, n\}$, i.e., $\overline{\mathcal{L}}_{21} = 0$.

Assumption 3: $\overline{\mathcal{L}}_r = \overline{\mathcal{L}}_{11}$, $\overline{\mathcal{L}}_r$ and $\overline{\mathcal{L}}_{11}$ are as defined in (12).

Lemma 1 [27]**:** If the graph $\mathcal{G}$ has a directed spanning tree, then all the eigenvalues of $\overline{\mathcal{L}}_{11}$ and $\overline{\mathcal{L}}_{22}$ associated with the graph $\mathcal{G}$ have positive real parts.

At this point, we have obtained the reduced order model by transformation, and given relevant assumptions and lemma. The similar transformation used in this paper can be found in [28]. In the following section, we will analyze the stability of the reduced-order model, and then deduce the conditions for the system to achieve consensus.

Theorem 1: With Assumptions 1 and 2, the heterogeneous multi-agent system (8) can achieve consensus asymptotically if α, β and T satisfy the following conditions:

$$\begin{aligned} &\text{(i)}\ \ \alpha < \min_{i=1,\dots,n-r-1}\left\{\frac{2Re(\eta_i)}{T|\eta_i|^2}\right\}, \\ &\text{(ii)}\ \ f_j(\alpha,\beta,\mu_j,T) < 256, \quad j=1,\dots,r-1, \end{aligned} \tag{13}$$

where

$$\begin{aligned} &f_j(\alpha,\beta,\mu_j,T) \\ &= (\alpha T^2 Re(\mu_j)+2\beta T-4)^2 \\ &\pm \alpha T^2 Im(\mu_j)(2\sqrt{m_{2j}^2+4m_{1j}}+2m_{2j})^{\frac{1}{2}} \\ &\mp(\alpha T^2 Re(\mu_j)+2\beta T-4)(2\sqrt{m_{2j}^2+4m_{1j}}-2m_{2j})^{\frac{1}{2}} \\ &+\sqrt{m_{2j}^2+4m_{1j}}+\alpha^2T^4Im^2(\mu_j), \\ &m_{1j} = \alpha^2T^4(2\beta T+\alpha T^2Re(\mu_j)-8)^2Im^2(\mu_j), \\ &m_{2j} = 4\beta^2T^2+4\alpha T^2(\beta T-4)Re(\mu_j)+\alpha^2T^4Im^2(\mu_j), \end{aligned}$$

and μ_j is the jth eigenvalue of $\overline{\mathcal{L}}_{11}$, η_i is the ith eigenvalue of $\overline{\mathcal{L}}_{22}$.

Proof: It followers from the above analysis we know that the reduced model (10) is asymptotically stable implies the heterogeneous multi-agent system (8) can reach consensus. Therefore, we should verify that $\rho(\Xi_2) < 1$, Ξ_2 is in formula (10). Obviously, we can decompose Ξ_2 into the following form:

$$\Xi_2 = I_{n+r-3} + \Xi_3, \tag{14}$$

with

$$\Xi_3 = \begin{bmatrix} -\frac{\alpha T^2}{2}\overline{\mathcal{L}}_{11} & -\frac{\alpha T^2}{2}\overline{\mathcal{L}}_{12} & (T-\frac{\beta T^2}{2})I_{r-1} \\ -\alpha T\overline{\mathcal{L}}_{21} & -\alpha T\overline{\mathcal{L}}_{22} & \mathbf{0} \\ -\alpha T\overline{\mathcal{L}}_{11} & -\alpha T\overline{\mathcal{L}}_{12} & -\beta T I_{r-1} \end{bmatrix}.$$

Then one has $\lambda(\Xi_2) = 1+\lambda(\Xi_3)$. Now we analyze the eigenvalue of matrix Ξ_3 as follows: Let $|\lambda I_{n+r-3} - \Xi_3| = 0$. That is

$$\begin{vmatrix} \lambda I_{r-1}+\frac{\alpha T^2}{2}\overline{\mathcal{L}}_{11} & \frac{\alpha T^2}{2}\overline{\mathcal{L}}_{12} & -(T-\frac{\beta T^2}{2})I_{r-1} \\ \alpha T\overline{\mathcal{L}}_{21} & \lambda I_{n-r-1}+\alpha T\overline{\mathcal{L}}_{22} & \mathbf{0} \\ \alpha T\overline{\mathcal{L}}_{11} & \alpha T\overline{\mathcal{L}}_{12} & (\lambda+\beta T)I_{r-1} \end{vmatrix} = \mathbf{0}. \tag{15}$$

Define the following elementary matrices:

$$P_1 = \begin{bmatrix} I_{r-1} & \mathbf{0} & -\frac{T}{2}I_{r-1} \\ \mathbf{0} & I_{n-r-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_{r-1} \end{bmatrix},$$

$$Q_1 = \begin{bmatrix} I_{r-1} & \mathbf{0} & \frac{T}{2}I_{r-1} \\ \mathbf{0} & I_{n-r-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_{r-1} \end{bmatrix},$$

$$Q_2 = \begin{bmatrix} I_{r-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I_{n-r-1} & \mathbf{0} \\ \frac{\lambda}{T}I_{r-1} & \mathbf{0} & I_{r-1} \end{bmatrix},$$

$$Q_3 = \begin{bmatrix} \mathbf{0} & \mathbf{0} & I_{r-1} \\ \mathbf{0} & I_{n-r-1} & \mathbf{0} \\ I_{r-1} & \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

It follows that $|P_1(\lambda I_{n+r-3} - \Xi_3)Q_1Q_2Q_3| = 0$. Then, according to primary transformation of matrix, one has

$$\begin{vmatrix} -TI_{r-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \lambda I_{n-r-1}+\alpha T\overline{\mathcal{L}}_{22} & \mathbf{0} \\ W_{31} & \alpha T\overline{\mathcal{L}}_{12} & W_{33} \end{vmatrix} = \mathbf{0} \tag{16}$$

where

$$W_{31} = (\lambda+\beta T)I_{r-1} + \frac{\alpha T^2}{2}\overline{\mathcal{L}}_{11},$$

$$W_{33} = \alpha T\overline{\mathcal{L}}_{11} + \frac{\lambda}{T}[\frac{\alpha T^2}{2}\overline{\mathcal{L}}_{11} + (\lambda+\beta T)I_{r-1}].$$

Then, we have

$$\begin{aligned} &|(\lambda I_{n-r-1}+\alpha T\overline{\mathcal{L}}_{22})| \\ &\times\left|(\frac{\lambda^2}{T}+\lambda\beta)I_{r-1}+(\frac{\alpha T}{2}\lambda+\alpha T)\overline{\mathcal{L}}_{11}\right| = 0. \end{aligned} \tag{17}$$

Denote the jth eigenvalue of $\overline{\mathcal{L}}_{11}$ as μ_j $(j = 1, \dots, r-1)$ and the ith eigenvalue of $\overline{\mathcal{L}}_{22}$ as η_i $(i = 1, \dots, n-r-1)$, respectively. According to the Lemma 1, one has $Re(\mu_j) > 0$, and $Re(\eta_i) > 0$. Then we have

$$\prod_{i=1}^{n-r-1}(\lambda+\alpha T\eta_i)\prod_{j=1}^{r-1}[\frac{1}{T}\lambda^2+\beta\lambda+(\frac{\alpha T}{2}\lambda+\alpha T)\mu_j] = 0. \tag{18}$$

It follows that

$$\lambda+\alpha T\eta_i=0,\ \ i=1,\ldots,n-r-1, \tag{19}$$

$$2\lambda^2+(2\beta T\lambda+\alpha T^2\mu_j)\lambda+2\alpha T^2\mu_j=0,\quad j=1,\ldots,r-1. \tag{20}$$

In order to guarantee $|\lambda(\Xi_2)|<1$, it follows from (19) that

$$|1-\alpha T\eta_i|<1,\ \ i=1,\ldots,n-r-1.$$

That is

$$(1-\alpha TRe(\eta_i))^2+\alpha^2T^2Im^2(\eta_i)<1. \tag{21}$$

Note $Re(\eta_i)>0$, α, $\beta>0$, one has $\alpha < \min_{j=1,\ldots,n-r-1}\left\{\frac{2Re(\eta_i)}{T|\eta_i|^2}\right\}$, which is the condition (i) in Theorem 1.
By using the quadratic formula, it follows from (20) that

$$|4-2\beta T-\alpha T^2\mu_j\pm\sqrt{(2\beta T+\alpha T^2\mu_j)^2-16\alpha T^2\mu_j}| < 16,\ \ j=1,\ldots,r-1. \tag{22}$$

Set $\sqrt{(2\beta T+\alpha T^2\mu_j)^2-16\alpha T^2\mu_j}=c_j+id_j$, here i is the imaginary unit. By some computations, one has

$$\begin{cases} c_j^2-d_j^2=4\beta^2T^2+4\alpha T^2(\beta T-4)Re(\mu_j)\\ \qquad\qquad +\alpha^2T^4Im^2(\mu_j),\\ c_jd_j=\alpha T^2(2\beta T+\alpha T^2Re(\mu_j)-8)Im(\mu_j). \end{cases} \tag{23}$$

Denote

$$m_{1j}=\alpha^2T^4(2\beta T+\alpha T^2Re(\mu_j)-8)^2Im^2(\mu_j),$$
$$m_{2j}=4\beta^2T^2+4\alpha T^2(\beta T-4)Re(\mu_j)+\alpha^2T^4Im^2(\mu_j).$$

It is not difficult to deduce that

$$\begin{cases} c_j^2=\dfrac{\sqrt{m_{2j}^2+4m_{1j}}+m_{2j}}{2},\\ d_j^2=\dfrac{\sqrt{m_{2j}^2+4m_{1j}}-m_{2j}}{2}. \end{cases} \tag{24}$$

It follows from (22) that

$$(\alpha T^2Re(\mu_j)+2\beta T-4)^2 \mp 2c_j(\alpha T^2Re(\mu_j)+2\beta T-4) +\alpha^2T^4Im^2(\mu_j)\pm 2d_j\alpha T^2Im(\mu_j)+c_j^2+d_j^2 < 256, \tag{25}$$

which combines with (24) implies the condition (ii) in Theorem 1. □

Remark 2: The results in Theorem 1 give a sufficient condition for system (8) to reach consensus. The algebraic graph theory and matrix analysis have been used to prove the results. Compared with [1] and [2], this paper studies heterogeneous multi-agent systems. Compared with [17–23], this paper uses data sampling method to discretize continuous systems. In [15], the authors mentioned that the emphasis is on the communication constraints in consensus problems. The results in this paper gives how the control gains and the interaction topology influence the convergence problem of the system.

For the second algorithm with velocity measurement, that is system (9), the following theorem will give the consensus results.

Theorem 2: With Assumptions 1, 2, and 3, the heterogeneous multi-agent system (9) can reach consensus if α, β and T satisfy the following conditions:

$$\text{(i)}\ \ \alpha < \min_{i=1,\ldots,n-r-1}\left\{\frac{2Re(\eta_i)}{T|\eta_i|^2}\right\},$$
$$\text{(ii)}\ \ g_j(\alpha,\beta,\mu_j,T)<240,\ \ j=1,\ldots,r-1, \tag{26}$$

where

$$\begin{aligned} &g_j(\alpha,\beta,\mu_j,T)\\ &=T^2\omega^2|\mu_j|^2\pm T(4-T\omega Re(\mu_j))\\ &\quad\times(2w_{2j}+2\sqrt{w_{1j}^2+4w_{2j}})^{\frac{1}{2}}\\ &\quad\mp T^2\omega Im(\mu_j)(-2w_{2j}+2\sqrt{w_{1j}^2+4w_{2j}})^{\frac{1}{2}}\\ &\quad+T^2\sqrt{w_{1j}^2+4w_{2j}}-8T\omega Re(\mu_j),\\ &\omega=\alpha T+2\beta,\\ &w_{1j}=Im^2(\mu_j)(\omega^2Re(\mu_j)-8\alpha)^2,\\ &w_{2j}=\omega^2(Re^2(\mu_j)-Im^2(\mu_j))-16\alpha Re(\mu_j), \end{aligned}$$

and μ_j is the jth eigenvalue of $\overline{\mathcal{L}}_{11}$, η_i is the ith eigenvalue of $\overline{\mathcal{L}}_{22}$.

Proof: The similar method to the above will be used to prove this theorem. Here we need to prove $\rho(\Psi_2)<1$. The matrix Ψ_2 can be decomposed into

$$\Psi_2=I_{n+r-3}+\Psi_3, \tag{27}$$

with

$$\Psi_3=\begin{bmatrix} -\frac{\alpha T^2}{2}\overline{\mathcal{L}}_{11} & -\frac{\alpha T^2}{2}\overline{\mathcal{L}}_{12} & TI_{r-1}-\frac{\beta T^2}{2}\overline{\mathcal{L}}_r\\ -\alpha T\overline{\mathcal{L}}_{21} & -\alpha T\overline{\mathcal{L}}_{22} & \mathbf{0}\\ -\alpha T\overline{\mathcal{L}}_{11} & -\alpha T\overline{\mathcal{L}}_{12} & -\beta T\overline{\mathcal{L}}_r \end{bmatrix}.$$

Then one has $\lambda(\Psi_2)=1+\lambda(\Psi_3)$. Now we analyze the eigenvalue of matrix Ψ_3 as follows. Let $|\lambda I_{n+r-3}-\Psi_3|=0$. That is

$$\begin{vmatrix} \lambda I_{r-1}+\frac{\alpha T^2}{2}\overline{\mathcal{L}}_{11} & \frac{\alpha T^2}{2}\overline{\mathcal{L}}_{12} & \frac{\beta T^2}{2}-TI_{r-1}\\ \alpha T\overline{\mathcal{L}}_{21} & \lambda I_{n-r-1}+\alpha T\overline{\mathcal{L}}_{22} & \mathbf{0}\\ \alpha T\overline{\mathcal{L}}_{11} & \alpha T\overline{\mathcal{L}}_{12} & \lambda I_{r-1}+\beta T\overline{\mathcal{L}}_r \end{vmatrix}=\mathbf{0}. \tag{28}$$

Let P_1, Q_1, Q_2, Q_3 be the same elementary matrices as the aforementioned. According to the properties of elementary transformation of matrix, we have $|P_1(\lambda I_{n+r-3} - \Psi_3)Q_1Q_2Q_3| = 0$. By some computations, one has

$$\begin{vmatrix} -TI_{r-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \lambda I_{n-r-1} + \alpha T\overline{\mathcal{L}}_{22} & \mathbf{0} \\ \overline{W_1} & \alpha T\overline{\mathcal{L}}_{12} & \overline{W_2} \end{vmatrix} = \mathbf{0} \tag{29}$$

where

$$\overline{W_1} = \lambda I_{r-1} + \beta T\overline{\mathcal{L}}_r + \frac{\alpha T^2}{2}\overline{\mathcal{L}}_{11},$$

$$\overline{W_2} = \frac{\lambda^2}{T}I_{r-1} + \alpha T(\frac{\lambda}{2}+1)\overline{\mathcal{L}}_{11} + \beta\lambda\overline{\mathcal{L}}_r.$$

Noticing Assumption 3, we have

$$\left|(\lambda I_{n-r-1} + \alpha T\overline{\mathcal{L}}_{22})\right| \times \left|\frac{\lambda^2}{T}I_{r-1} + (\alpha T\frac{\lambda}{2} + \alpha T + \beta\lambda)\overline{\mathcal{L}}_{11}\right| = 0. \tag{30}$$

Denote the jth eigenvalue of $\overline{\mathcal{L}}_{11}$ as μ_j $(j = 1, \ldots, r-1)$ and the ith eigenvalue of $\overline{\mathcal{L}}_{22}$ as η_i $(i = 1, \ldots, n-r-1)$, respectively. By applying Lemma 1, we know $Re(\mu_j) > 0$, and $Re(\eta_i) > 0$. It follows from (30) that

$$\prod_{i=1}^{n-r-1}(\lambda + \alpha T\eta_i)\prod_{j=1}^{r-1}[\frac{1}{T}\lambda^2 + (\alpha T\frac{\lambda}{2} + \alpha T + \beta\lambda)\mu_j] = 0. \tag{31}$$

That is,

$$\lambda + \alpha T\eta_i = 0, \;\; i = 1, \ldots, n-r-1 \tag{32}$$

$$2\lambda^2 + (2\beta T + \alpha T^2)\mu_j\lambda + 2\alpha T^2\mu_j = 0, \; j = 1, \ldots, r-1. \tag{33}$$

In order to guarantee $|\lambda(\Psi_2)| < 1$, it follows from (32) that

$$|1 - \alpha T\eta_i| < 1, \;\; i = 1, \ldots, n-r-1.$$

That is

$$(1 - \alpha TRe(\eta_i))^2 + \alpha^2T^2Im^2(\eta_i) < 1. \tag{34}$$

Note $Re(\eta_i) > 0$, $\alpha, \beta > 0$, one has $\alpha < \min_{i=1,\ldots,n-r-1}\left\{\frac{2Re(\eta_i)}{T|\eta_i|^2}\right\}$, which is the condition (i) in Theorem 2.

It follows from (33) that

$$|4 - T(2\beta + \alpha T)\mu_j \pm T\sqrt{(2\beta + \alpha T)^2\mu_j^2 - 16\alpha\mu_j}| < 16, \;\; j = 1, \ldots, r-1. \tag{35}$$

Let $\sqrt{(2\beta + \alpha T)^2\mu_j^2 - 16\alpha\mu_j} = c_j + id_j$, here i is the imaginary unit. Denote $\omega = 2\beta + \alpha T$. By some computations, one has

$$\begin{cases} c_j^2 - d_j^2 = \omega^2(Re^2(\mu_j) - Im^2(\mu_j)) - 16\alpha Re(\mu_j), \\ c_jd_j = Im(\mu_j)(\omega^2Re(\mu_j) - 8\alpha). \end{cases} \tag{36}$$

Denote $w_{1j} = [Im(\mu_j)(\omega^2Re(\mu_j) - 8\alpha)]^2$, $w_{2j} = \omega^2(Re^2(\mu_j) - Im^2(\mu_j)) - 16\alpha Re(\mu_j)$. It is not difficult to obtain that

$$\begin{cases} c_j^2 = \dfrac{w_{2j} + \sqrt{w_{2j}^2 + 4w_{1j}}}{2}, \\ d_j^2 = \dfrac{-w_{2j} + \sqrt{w_{2j}^2 + 4w_{1j}}}{2}, \end{cases} \tag{37}$$

which combines with (35) yields the condition (ii) in Theorem 2. □

4. SIMULATION RESULTS

In order to show the effectiveness of the obtained results, we will give two simulation examples in this section. For simplicity, we suppose that $a_{ij} = 1$ if $(i, j) \in \mathcal{E}$, otherwise $a_{ij} = 0$ in the following two examples.

Example 1: Let the multi-agent system be made up of 8 agents. According to the Assumptions 1 and 2, we choose the interaction topology for system (10) as Fig. 1. Here agents 1 to 4 are second-order, agents 5 to 8 are first-order. Let $T = 0.1s$. According to the conditions in Theorem 1, we obtain a feasible $\alpha = 0.32$, $\beta = 0.67$ by calculation. Then we get the trajectories of the agents as shown in Figs. 2 and 3. The simulation results are consistent with the theoretical results obtained in this paper.

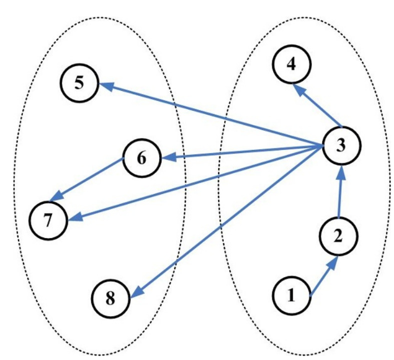

Fig. 1. Topology $\mathcal{G}$.

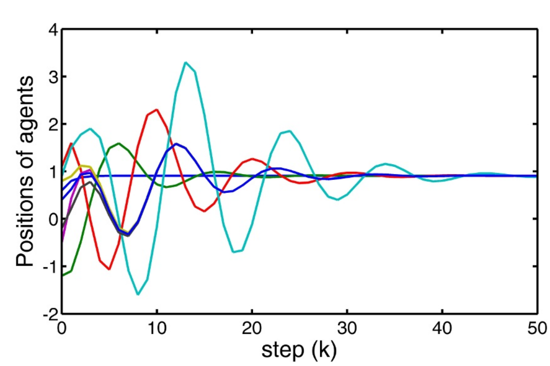

Fig. 2. Position trajectories of case 1.

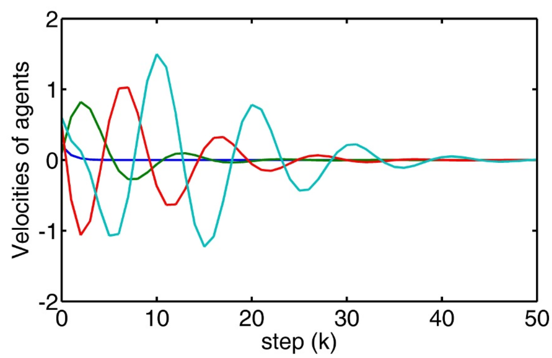

Fig. 3. Velocity trajectories of case 1.

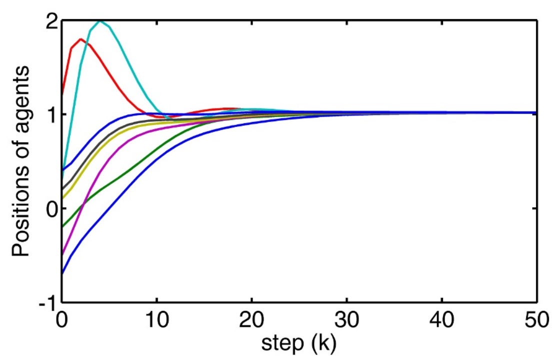

Fig. 4. Position trajectories of case 2.

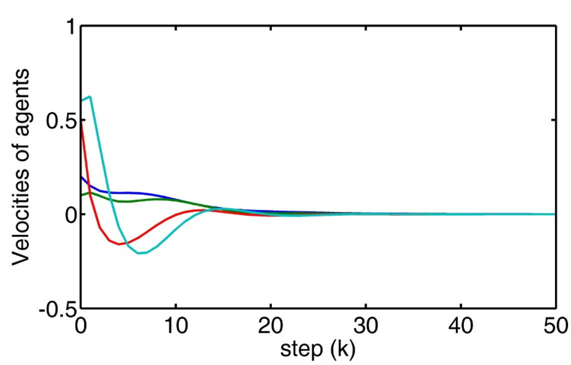

Fig. 5. Velocity trajectories of case 2.

Example 2: In this example, we also let the multi-agent systems be made up of 8 agents. The Assumption 3 is necessary to the proof of Theorem 2. However, we find it is not necessary to simulation. Hence the interaction topology can be chosen as the same as Example 1. Let $T = 0.1$ s. We calculate a feasible $\alpha = 0.45$, $\beta = 0.78$. Figs. 4 and 5 give the position and velocity trajectories of each agent respectively. The simulation shows that the position and velocity of each agent can quickly converge to the value of the consensus state.

5. CONCLUSION

In recent years, the heterogeneous multi-agent system has been becoming a hot topic in the field of control. In this paper, we have studied the problem of consensus for one kinds of heterogeneous multi-agent systems. The continuous-time model under consideration was converted into the discrete-time model by sampling. Then the discretized system has been transformed to a reduced-order error model. Two consensus algorithms have been studied respectively. We have deduced the sufficient conditions for the systems considered to reach consensus. Finally, numerical simulations have been provided to verify the obtained results. The results obtained in this paper seem too complicated. In future work, we will seek simpler results and extend them to the case of switched topologies.

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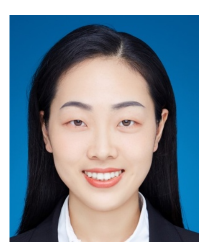

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