International Journal of Control, Automation and Systems 18(12) (2020) 3242-3246
http://dx.doi.org/10.1007/s12555-019-0947-y

ISSN:1598-6446 eISSN:2005-4092
http://www.springer.com/12555

Design of Repetitive Control Systems Using a Delayed Control Input and a State Error

Tae-Yong Doh* and **Jung Rae Ryoo**

Abstract: This paper presents a modified repetitive control scheme comprising of a state error and a control input via delayed feedback to track periodic reference trajectories and/or attenuate disturbances. The closed-loop state error dynamics can be represented using a typical neutral delay system with an exogenous input to be attenuated. The sufficient conditions to achieve overall stability and H_∞ performance to minimize state error are derived by applying a Lyapunov-Krasovskii functional and a Hamiltonian, which are expressed as an algebraic Riccati inequality (ARI) and a linear matrix inequality (LMI). Based on the derived conditions, it is shown that the repetitive controller design problem can be reformulated as an optimization problem with an LMI constraint to determine the state error feedback gain. Finally, a numerical example is presented to demonstrate the feasibility of the proposed method.

Keywords: Algebraic Riccati inequality (ARI), delayed control input, H_∞ performance, $\mathcal{L}_2$ norm, linear matrix inequality (LMI), neutral delay system, repetitive control, state error feedback.

1. INTRODUCTION

Repetitive control is a specialized control scheme used to track periodic reference commands and/or attenuate periodic exogenous disturbances. Frequency domain analysis shows that its highly accurate tracking performance can be achieved using a periodic signal generator implemented in the repetitive controller. However, the positive feedback loop to generate the periodic signal decreases the stability margin of the system. Therefore, the tradeoff between stability and tracking performance can be considered as an important factor to design the control system. Hara *et al.* [1] derived the sufficient conditions for stability of original repetitive as well as modified repetitive control systems that sacrifice tracking performance at high frequencies in favor of system stability. Doh and Ryoo dealt with the problem of a robust repetitive controller design for an uncertain feedback control system using its explicit performance information [2].

To apply repetitive control to various types of systems, several theories have been developed for repetitive control in the state space. Doh *et al.* proposed a method for designing repetitive control systems that could ensure robust stability for linear systems with time-varying uncertainties [3] and applied this repetitive controller to the track-following servo system of an optical disk drive [4]. Lucibello showed that the repetitive control of positive real systems via delayed feedback was Lyapunov asymptotically stable [5]. Q. Quan *et al.* developed a repetitive controller that was composed of a delayed control input and an output error multiplied by a time-varying gain and derived a sufficient condition for stability via a Lyapunov-Krasovskii functional and linear matrix inequalities (LMIs) [6]. Zhou *et al.* dealt with the problem of designing a robust observer-based modified repetitive-control system with a prescribed H_∞ disturbance rejection level for a class of strictly proper linear plants [7].

Many repetitive control schemes using delayed control input and output error have been suggested in previous research. However, their stability can be ensured only for positive real systems when fixed feedback gains are used. In this paper, a modified repetitive control scheme is proposed, wherein the controller is composed of a delayed control input and state error multiplied by the state error feedback gain. The closed-loop repetitive control system can be represented as a neutral delay system with an exogenous input to be attenuated. Therefore, there is a need to develop a method that not only ensures stability but also attenuates the external input of the neutral delay system representing the proposed repetitive control system. To achieve this purpose, the sufficient condition for stability and H_∞ performance of the closed-loop system is derived using a Lyapunov-Krasovskii functional and Hamiltonian. The obtained condition is expressed as an algebraic Ric-

Manuscript received November 10, 2019; revised March 3, 2020; accepted April 17, 2020. Recommended by Associate Editor Le Van Hien under the direction of Editor PooGyeon Park. This research was supported by the research fund of Hanbat National University in 2017.

Tae-Yong Doh is with the Department of Electronic and Control Engineering, Hanbat National University, 125 Dongseodaero, Yuseong-gu, Daejeon 34158, Korea (e-mail: dolerite@hanbat.ac.kr). Jung Rae Ryoo is with the Department of Electrical and Information Engineering, Seoul National University of Science and Technology, 232 Gongreungro, Nowon-gu 01811, Korea (e-mail: jrryoo@seoultech.ac.kr).
* Corresponding author.

cati inequality (ARI) and an LMI. Owing to the fundamental properties of the proposed scheme, the state error converges to a nonzero value close to zero. However, the condition for H_∞ performance to reduce the state error can also be included in the LMI condition. Therefore, the design method to find the state error feedback gain of the repetitive controller can be formulated as an optimization problem subject to the LMI constraint, which ensures not only the overall stability but also minimization of the state error in the sense of $\mathcal{L}_2$ norm. The validity of the proposed method is then verified through a numerical example.

2. PROBLEM FORMULATION

Fig. 1 depicts the proposed repetitive control scheme. Consider the following class of linear systems:

$$G: \begin{array}{rcl} \dot{x}(t) & = & Ax(t)+Bu(t), \\ y(t) & = & Cx(t), \end{array} \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}$, and $u(t) \in \mathbb{R}$ with $t \geq 0$ are the plant state, output, and control input, respectively, and $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$, and $C \in \mathbb{R}^{1\times n}$.

Assumption 1: All the eigenvalues of A are less than 0. (A, B) is controllable and (C, A) is observable.

Assumption 2 [6]**:** There exists a bounded, continuous, and periodic control input $u_d(t) = u_d(t-T)$ that allows $x(t)$ to track a desired periodic state $x_d(t) \in \mathbb{R}^n$ with period T perfectly when substituted for $u(t)$ in (1). Therefore, the reference system can be given as

$$\begin{aligned} &\dot{x}_d(t) = Ax_d(t)+Bu_d(t), \\ &y_d(t) = Cx_d(t), \end{aligned} \tag{2}$$

where $y_d(t) \in \mathbb{R}$ is a desired periodic trajectory.

Remark 1: Taking the Laplace transform of (1), $Y_d(s) = G(s)U_d(s)$ where $G(s) = C(sI-A)^{-1}B$; $Y_d(s)$ and $U_d(s)$ are Laplace transforms of $y_d(t)$ and $u_d(t)$, respectively. For any periodic, bounded, and continuous trajectory $y_d(t)$ that satisfies the appropriate differentiability condition, if $G(s)$ is invertible and $G(s)^{-1}$ is stable, then $u_d(t)$ is periodic, bounded, continuous, and unique [6], [8]. For example, suppose $G(s) = 1/(1+s)$; thus $U_d(s) = (1+s)Y_d(s)$, which implies that $u_d(t) = y_d(t) + \dot{y}_d(t)$. Hence, the appropriate differentiability condition implies that $y_d(t)$ is first-order differentiable.

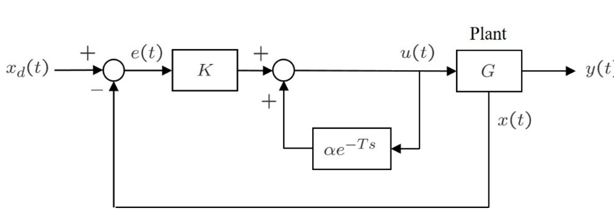

Fig. 1. Repetitive control system with delayed control input and state error.

The control objective is to track $x_d(t)$ with a period T, and a modified repetitive controller is designed for this purpose as

$$u(t) = \alpha u(t-T) + Ke(t) \tag{3}$$

with an initial condition

$$u(\theta) = \varphi(\theta),\ \theta \in [-T, 0), \tag{4}$$

where α is a constant value satisfying the condition $0 < \alpha < 1$, $K \in \mathbb{R}^{1\times n}$ is the state error feedback gain, and $e(t) = x_d(t) - x(t)$ is the state error between the desired state and the plant state.

The overall repetitive control system is as shown in Fig. 1, where the structure is almost identical to that of Hara's modified repetitive control [1], with the exception that $q(s)$ is replaced by α.

By subtracting (2) from (1), the state error dynamic system can be described as

$$\dot{e}(t) = Ae(t) + Bu_e(t), \tag{5}$$

where $u_e(t) = u_d(t) - u(t)$. Owing to the property of $u_d(t)$ in Assumption 2, i.e., $u_d(t) = u_d(t-T)$, (3) can be written as

$$\begin{aligned} u_e(t) =& u_d(t-T) - \alpha u(t-T) - Ke(t) \\ =& \alpha u_e(t-T) + (1-\alpha)u_d(t-T) - Ke(t). \end{aligned} \tag{6}$$

Combining (5) and (6) and using the periodicity of the system, i.e., $\dot{e}(t-T) = Ae(t-T) + Bu_e(t-T)$, a neutral delay system can be obtained as follows:

$$\begin{aligned} \dot{e}(t) - \alpha\dot{e}(t-T) =& (A-BK)e(t) - \alpha Ae(t-T) \\ &+ (1-\alpha)Bu_d(t-T). \end{aligned} \tag{7}$$

By denoting $A_c \triangleq A - BK$, $A_\alpha \triangleq \alpha A$, and $\nu(t) \triangleq (1-\alpha)Bu_d(t-T) \in \mathbb{R}^n$, (7) can be simply written as a neutral delay system with an equivalent bounded and periodic disturbance $\nu(t)$ as follows:

$$\dot{e}(t) - \alpha\dot{e}(t-T) = A_c e(t) - A_\alpha e(t-T) + \nu(t). \tag{8}$$

In (8), as α approaches 1, $e(t)$ decreases since $\nu(t)$ approaches 0. However, if α becomes 1, the stability of the system cannot be ensured any further, as will be shown in the next section. Moreover, since α is not equal to 1, there exists $\nu(t) \neq 0$ such that $e(t)$ does not converge to zero any longer.

Therefore, we attempt to find a method to minimize $e(t)$ over the periodic exogenous input $\nu(t)$. The closed-loop transfer function $T_{e\nu}(s)$ from $\nu(t)$ to $e(t)$ is given by

$$T_{e\nu}(s) = \left[sI\left(1-\alpha e^{-Ts}\right) - \left(A_c - A_\alpha e^{-Ts}\right)\right]^{-1}. \tag{9}$$

Since $\nu(t)$ is bounded and periodic, $\nu(t)$ is a power signal of which power is defined as the power semi-norm ($\|\cdot\|_{\mathcal{P}}$) [9] and $e(t)$ is also a power signal which is generated by the linear system (8). In order to obtain as small a state error $e(t)$ as possible for the external periodic input $\nu(t)$ in the system (8), the $\mathcal{H}_\infty$ norm of $T_{e\nu}(s)$ should be minimized, which is equivalent to minimize $e(t)$ in the sense of $\mathcal{L}_2$ norm [9]. This means that the repetitive control design problem is replaced with a problem that ensures stability and minimizes $\|T_{e\nu}\|_\infty$ of the neutral delay system (8).

3. MAIN RESULTS

The following results will play an important role in designing the repetitive control system.

Theorem 1: The closed-loop system (8) is asymptotically stable and $\|T_{e\nu}(s)\|_\infty < \gamma$ if one of the following two equivalent conditions holds:

1) There exist matrices $P = P^T \in \mathbb{R}^{n\times n} > 0$, $Q = Q^T \in \mathbb{R}^{n\times n} > 0$ and $S = S^T \in \mathbb{R}^{n\times n} > 0$ satisfying the ARI

$$A_c^T P + PA_c + Q + S + I + PP/\gamma^2 + MW^{-1}M^T + PAS_1^{-1}A^T P < 0, \tag{10}$$

where

$$W = Q/\alpha^2 - (Q+S+I) > 0, \tag{11}$$
$$S_1 = S/\alpha^2, \tag{12}$$
$$M = PA_c + Q + S + I. \tag{13}$$

2) There exist matrices $X = X^T = P^{-1} \in \mathbb{R}^{n\times n} > 0$, $Q = Q^T \in \mathbb{R}^{n\times n} > 0$ and $S = S^T \in \mathbb{R}^{n\times n} > 0$ satisfying the ARI

$$XA_c^T + A_c X + Q + S + I + XX/\gamma^2 + M_1 W^{-1} M_1^T + XA^T S_1^{-1} AX < 0, \tag{14}$$

where

$$M_1 = XA_c^T + Q + S + I. \tag{15}$$

Proof: 1) We introduce a Lyapunov-Krasovskii functional $V(e_t)$ for the system (8), which is a modified form of the Lyapunov-Krasovskii functionals used in neutral delay equations [10], [11]:

$$V(e_t) = (e(t) - \alpha e(t-T))^T P (e(t) - \alpha e(t-T)) + \int_{t-T}^{t} e(\tau)^T (Q+S) e(\tau) d\tau, \tag{16}$$

where $e_t = e(t+\theta)$, $\theta \in [-T, 0]$. It can be shown that there exist scalar $c_1 > 0$ and $c_2 > 0$ such that the following relation holds:

$$c_1 \|\mathcal{D}_\phi\|^2 \le V(\phi) \le c_2 \sup_{\theta\in[-T,0]} \|\phi(\theta)\|^2, \tag{17}$$

where $\mathcal{D}_\phi = e(\phi) - \alpha e(\phi - T)$.

In order to show that the system (8) is asymptotically stable with a disturbance attenuation γ, the associated Hamiltonian $H(e,\nu,t)$ should be satisfied [9], [11]:

$$H(e,\nu,t) = \dot V(e_t) + e(t)^T e(t) - \gamma^2 \nu(t)^T \nu(t) \le 0. \tag{18}$$

This inequality can be rewritten as

$$\begin{aligned} H&(e,\nu,t) \\ &= \mathcal{D}_t^T (A_c^T P + PA_c + Q + S + I)\mathcal{D}_t \\ &\quad + \mathcal{D}_t^T (\alpha PA_c - PA_\alpha + \alpha(Q+S) + \alpha I) e(t-T) \\ &\quad + e(t-T)^T (\alpha A_c^T P - A_\alpha^T P + \alpha(Q+S) + \alpha I)\mathcal{D}_t \\ &\quad + e(t-T)^T (\alpha^2 (Q+S) + \alpha^2 I - (Q+S)) e(t-T) \\ &\quad + \nu(t)^T P \mathcal{D}_t + \mathcal{D}_t^T P \nu(t) - \gamma^2 \nu(t)^T \nu(t) \le 0. \end{aligned} \tag{19}$$

If Ξ is denoted as $\Xi = [\mathcal{D}_t^T \ \alpha e(t-T)^T \ \nu(t)^T]^T$, then (19) can be rearranged as follows:

$$\Xi^T \Lambda \Xi \le 0, \tag{20}$$

where

$$\Lambda = \begin{bmatrix} \Pi & M - PA & P \\ M^T - A^T P & -W - S_1 & 0 \\ P & 0 & -\gamma^2 I \end{bmatrix}, \tag{21}$$
$$\Pi = A_c^T P + PA_c + Q + S + I. \tag{22}$$

The sufficient condition for (18) is then equivalent to $\Lambda < 0$. Λ can be separated as follows:

$$\Lambda = \begin{bmatrix} \Pi & M & P \\ M^T & -W & 0 \\ P & 0 & -\gamma^2 I \end{bmatrix} + \begin{bmatrix} 0 & -PA & 0 \\ -A^T P & -S_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} < 0. \tag{23}$$

Using Schur complement [12], the ARI (10) can be obtained from (23).

2) By substituting P for X^{-1} in the ARI (10) and using mathematical manipulation, the ARI (14) can be obtained. □

As explained in the previous section, if α becomes 1, the W in (10) or (14) is no longer positive definite, and the overall stability of the system cannot be ensured by Theorem 1.

Remark 2: If α is 1, then (8) becomes a critical case of the neutral delay equation and its characteristic equation has an infinite sequence of roots whose real parts approach zero, as noted in [6, 13, 14].

Remark 3: Using two positive definite matrices Q and S in (16), W can be ensured to be positive definite. If only Q or S is used, then W is always negative definite and Theorem 1 cannot be established.

The following theorem provides a solution to the H_∞ problem, i.e., $\|T_{ev}\|_\infty < \gamma$ to determine K.

Theorem 2: The closed-loop system (8) is asymptotically stable and $\|T_{ev}(s)\|_\infty < \gamma$ if there exist matrices $X > 0$, Y, $Q > 0$, and $S > 0$ satisfying the LMI

$$\begin{bmatrix} \Omega & \Sigma & XA^T & X \\ \Sigma^T & -W & 0 & 0 \\ AX & 0 & -S_1 & 0 \\ X & 0 & 0 & -\gamma^2 I \end{bmatrix} < 0, \tag{24}$$

where

$$\Omega = XA^T - Y^T B^T + AX - BY + Q + S + I, \tag{25}$$

$$\Sigma = XA^T - Y^T B^T + Q + S + I, \tag{26}$$

and the state error feedback gain K is given by

$$K = YX^{-1}. \tag{27}$$

Proof: Using Schur complement [12], the ARI (14) can be represented as an LMI

$$\begin{bmatrix} \Pi_1 & M_1 & XA^T & X \\ M_1^T & -W & 0 & 0 \\ AX & 0 & -S_1 & 0 \\ X & 0 & 0 & -\gamma^2 I \end{bmatrix} < 0, \tag{28}$$

where

$$\Pi_1 = XA_c^T + A_c X + Q + S + I. \tag{29}$$

Using the realization $A_c = A - BK$ for the closed-loop system (8), (24) readily follows from (28) along with the change of variable $K = YX^{-1}$. □

The LMI (24) is convex in variables X, Y, and γ, and the state error feedback gain K can be obtained via convex optimization. To obtain the minimum value of γ and evaluate the feasibility of the LMI condition, the problem to determine K can be formulated as a standard problem of linear objective minimization that is subject to an LMI constraint as follows:

$$\text{Minimize } \gamma \text{ over } X,\ Y \text{ satisfying (24).} \tag{30}$$

4. NUMERICAL EXAMPLE

To demonstrate the feasibility of the proposed method, an illustrative example is presented herein. It is assumed that A, B, and C of the plant (1) are given as

$$A = \begin{bmatrix} -5 & 0 \\ 0 & -10 \end{bmatrix},\ B = \begin{bmatrix} 0.2 \\ -0.2 \end{bmatrix},\ C = [1\ \ 1]. \tag{31}$$

With $\alpha = 0.9$, $Q = 5I$, and $S = 0.01I$, solutions of the optimization problem (30) using Robust Control Toolbox [15] are given as

$$X = \begin{bmatrix} 0.7538 & 0.2733 \\ 0.2733 & 0.4279 \end{bmatrix} > 0, \tag{32}$$

$$Y = \begin{bmatrix} 11.4815 & -7.0094 \end{bmatrix}, \tag{33}$$

$$\gamma = 0.625. \tag{34}$$

Note that α, Q, and S should be selected to ensure that $W > 0$. From (27), K is obtained as

$$K = [27.5547\ \ -33.9846], \tag{35}$$

using (32) and (33).

A desired state with a period $T = 2$ second is given as

$$x_d(t) = \begin{bmatrix} x_{d1}(t) \\ x_{d2}(t) \end{bmatrix} = \begin{bmatrix} 2.1\sin(\pi t + 0.3) \\ 1.2\sin(\pi t - 2.6) \end{bmatrix}, \tag{36}$$

which renders the desired trajectory $y_d(t) = \sin(\pi t)$.

In simulation, αe^{-Ts} is not connected before 4 second, i.e., the control input is given only as $u(t) = Ke(t)$. The repetitive controller is turned on at 4 second, i.e., the 3rd period. Fig. 2 and Fig. 3 show that not only $x(t)$ tracks $x_d(t)$ but also the state error diminishes abruptly once the repetitive controller is turned on.

Fig. 4 shows the root mean square (rms) values of the state error $e(t)$ versus the number of period, where the performance improvement by the added repetitive controller is clearly shown. More quantitatively, the rms values of $e_1(t)$ and $e_2(t)$ decrease steadily to approach around 0.02598 and 0.01759, which are 13.65% and 14.91% of the initial rms values 0.1903 and 0.1180, respectively, thus verifying the benefit of repetitive control. Since α is slightly less than 1 but not equal to 1, the state error does not converge to zero even if it reaches a near-zero value.

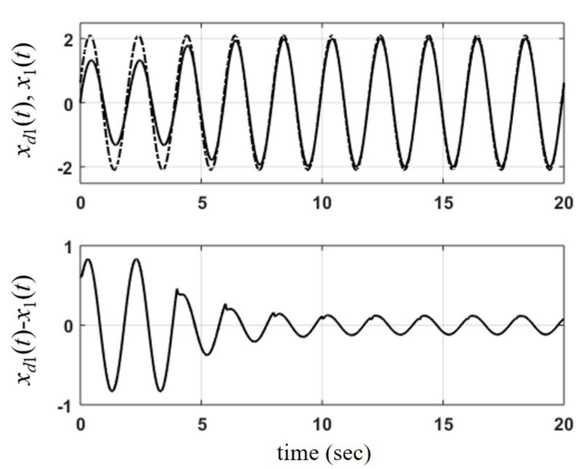

Fig. 2. Desired state $x_{d1}(t)$(dash), state $x_1(t)$(solid), and state error $e_1(t) = x_{d1}(t) - x_1(t)$.

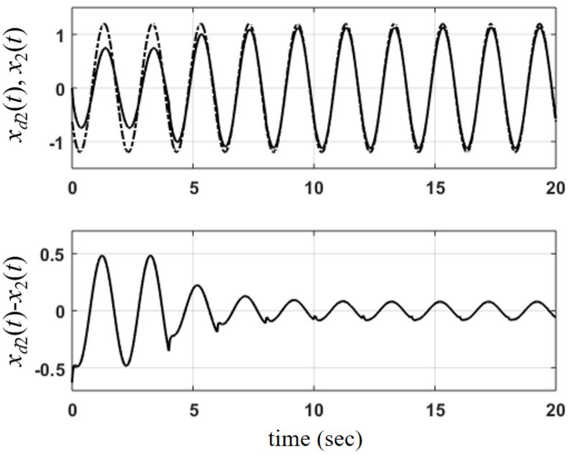

Fig. 3. Desired state $x_{d2}(t)$(dash), state $x_2(t)$(solid), and state error $e_2(t) = x_{d2}(t) - x_2(t)$.

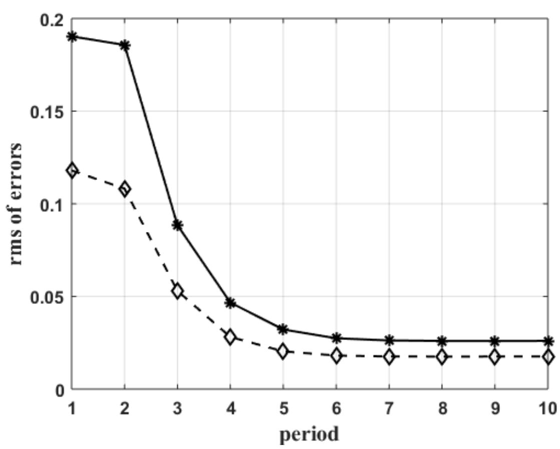

Fig. 4. Root mean square(rms) values of state error $e_1(t)$(solid) and $e_2(t)$(dash).

5. CONCLUSION

The problem of repetitive control system design using delayed control input and state error has been considered. The state error dynamics of the repetitive control system is represented as a neutral delay system with a periodic disturbance. The sufficient conditions for the stability and the state error reduction are derived, which have the forms of the ARI and LMI, respectively. We showed that the repetitive controller design problem could be reformulated as an optimization problem with an LMI constraint to find the state error feedback gain.

In further works, we will develop a repetitive control system using a delayed control input based on the output feedback, observer-based control, *etc*., by considering the case where a state cannot be measured. Moreover, for industrial applications, studies to ensure robustness to uncertainties can be performed in the design of the proposed repetitive control system.

REFERENCES

[1] S. Hara, Y. Yamamoto, T. Omata, and H. Nakano, "Repetitive control system: A new type servo system for periodic exogenous signals," *IEEE Trans. Autom. Control*, vol. 37, no. 7, pp. 659-668, 1988.

[2] T.-Y. Doh and J. R. Ryoo, "Robust approach to repetitive controller design for uncertain feedback control systems," *IET Control Theory & App.*, vol. 7, pp. 431-439, 2013.

[3] T.-Y. Doh and M. J. Chung, "Repetitive control design for linear systems with time-varying uncertainties," *IEE Proc. - Control Theory & App.*, vol. 150, pp. 427-432, 2003.

[4] T.-Y. Doh, J. R. Ryoo, and M. J. Chung, "Design of a repetitive controller: an application to the track-following servo system of optical disk drives," *IEE Proc. - Control Theory & App.*, vol. 153, pp. 323-330, 2006.

[5] P. Lucibello, "Repetitive control of positive real systems via delayed feedback is Lyapunov asymptotically stable," *IEEE Trans. Autom. Control*, vol. 52, no. 9, pp. 1748-1751, 2007.

[6] Q. Quan, D. Yang, K.-Y. Cai, and J. Jiang, "Repetitive control by output error for a class of uncertain time-delay systems," *IET Control Theory & App.*, vol. 3, no. 9, pp. 1283-1292, 2009.

[7] L. Zhou, J. She, and S. Zhou, "Robust H_∞ control of an observer-based repetitive-control system," *J. the Franklin Institute*, vol. 335, pp. 4952-4969, 2018.

[8] J. L. Schiff, *The Laplace Transform: Theory and Applications*, Springer-Verlag, Berlin, 1999.

[9] K. Zhou, J. C. Doyle, and K. Glover, *Robust and Optimal Control*, Prentice-Hall, 1996.

[10] S. Xu, J. Lam, and C. Yang, "H_∞ and positive-real control for linear neutral delay system," *IEEE Trans. Autom. Control*, vol. 46, no. 8, pp. 1321-1326, 2001.

[11] M. S. Mahmoud, *Robust Control and Filtering for Time-Delay System*, Dekker, 2000.

[12] S. Boyd, L. E. Ghaoui, E. Feron, V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, SIAM, 1994.

[13] R. Rabah, G. M. Sklyar, and A. V. Rezounenko, "Stability analysis of neutral type systems in Hilbert space," *J. Differential Equations*, vol. 214, no. 2, pp. 391-428, 2005.

[14] J. Hale, *Theory of Functional Differential Equations*, Springer-Verlag, New York, 1977.

[15] G. Balas, R. Chiang, A. Packard, and M. Safonov, *Robust Control Toolbox*, The MathWorks, Inc., 2019.