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H_∞ Control Problem of Discrete 2-D Switched Mixed Delayed Systems Using the Improved Lyapunov-Krasovskii Functional

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Abstract: This paper deals with the problem of exponential stability and H_∞ control of two-dimensional (2-D) switched discrete systems with mixed time-varying delays. Firstly, this work suggests some improvements to Lyapunov-Krasovskii functional (LKF) discussed in the previous literature. Such improvements have been achieved by introducing some new terms containing the summations of state vector in single and double forms in an effort to capture the extra information related to time delays. Secondly, delay-dependent conditions based on the improved LKF are derived for the exponential stability and H_∞ performance of 2-D switched systems in the form of linear matrix inequalities (LMIs) by virtue of the average dwell time approach. Thirdly, a state-feedback controller is designed to ensure the exponential stability of the overall closed-loop system under consideration with a desirable H_∞ disturbance attenuation level γ. Finally, a suitable example is provided which highlights the benefits of the proposed results by comparing them with the results available in literature both in terms of conservativeness and computational burden.

Keywords: Average dwell-time approach, exponential stability, H_∞ control, switched systems, time delays, 2-D systems.

1. INTRODUCTION

2-D systems have many applications in several fields such as geographical data processing, electrical circuit networks, power systems, energy exchange processes, multi-body mechanics, process control, aerospace engineering, and chemical and physical processes [1–3]. Most commonly utilized state-space models for the representation of 2-D systems are the Roesser model, the Fornasini-Marchesini (FM) local model and the Attasi model [1, 4, 5].

Switched systems are the type of hybrid systems that contain multiple subsystems and a switching rule. These systems have applications in many vital areas such as motor engine control, the automotive industry, and networked control systems [6–8]. In recent years, 2-D switched systems have attracted the attention of many theoretical scientists who have made the significant contributions in investigating the stability analysis and controller design problems. In this regard, the first contribution appeared in [9, 10], where the concept of switching among 2-D systems was discussed in details and some solvable conditions were derived in terms of inequalities by utilizing the common quadratic Lyapunov functional and the multiple Lyapunov functional approaches. Later, the concept of dwell time approach was extended to 2-D switched systems in [11], where the exponential stability and the controller synthesis problems were discussed.

During the analysis and synthesis problems, the main challenge and complexity are due to the inherent existence of time delays in practical systems. These delays can affect the system's performance and can even lead to the instability. There exist many useful studies in the literature that have investigated the stability problem for 2-D switched discrete systems in the presence of time delays [12–18]. It is essential to state here that the time delay can be classified into the discrete and the distributed ones, in general [19, 20]. It is also quite possible that these delays possess the properties which are not identical. The distributed time delays are incorporated into the system dynamics in conjunction with discrete delays especially in communication-related applications to represent the delays in parallel pathways that have different axon sizes and lengths [21]. Some relevant interesting studies can be

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found in [22–32].

On the contrary, the H_∞ technique is well known in control theory that helps the system to achieve the desired performance even in the presence of an exogenous input. This technique has been utilized extensively for 2-D switched systems. For instance, one can refer to the results derived in [33–35], where H_∞ control problem for such systems are studied. It is important to highlight an observation at this point that most of the aforementioned 2-D switched system results have not considered the mixed delays during their analysis and synthesis. However, a few 2-D systems results have also appeared in the literature that have considered mixed delays, for instance authors in [36], solved H_∞ filter design problem for 2-D mixed-delayed systems with saturation and nonlinearities. The authors in [37] solved the robust H_∞ filter design problem for the 2-D fuzzy systems in the presence of mixed delays. Recently, the authors in [38] have studied the stability analysis problem of 2-D descriptor systems in the presence of mixed delays, where an improved LKF was proposed in order to deal with the inherent conservativeness of the Jensen-type inequalities. It is important to mention that the reduction of the conservativeness of the stability conditions has always been an essential issue in control engineering applications.

The above-stated observation related to 2-D switched systems and the conservativeness fact associated with the Jensen-type inequalities are the main motivating factors behind the current study. The results in this paper improve the LKF proposed in [38] and investigate the H_∞ controller synthesis problem for the 2-D discrete switched systems in the presence of the mixed delays via average dwell time approach which to the best of our knowledge have not been directly investigated in the literature till date.

The main contributions of this paper are encapsulated as follows: Firstly, by proposing a more improved LKF than the one considered in [38], we study the exponential stability problem for 2-D switched systems in the presence of mixed delays via average dwell time approach. Secondly, based upon the exponential stability results, some sufficient conditions for the H_∞ performance analysis of such systems have been derived. Thirdly, some inequalities based solvable conditions have been derived for designing a state feedback controller that promises the exponential stability of the closed-loop system under investigation with a desired H_∞ performance level γ. Finally, in the example section, we show that the improved LKF based results enjoy less conservativeness without the substantially increasing the computational burden.

Notations: $\mathbb{Z}$ denotes the set of integers, $\mathbb{Z}[a,b] \triangleq \{a, a+1, \ldots, b\}$ for $a,b \in \mathbb{Z}$, $a \le b$. $\mathbb{R}^{n\times m}$ denotes the set of $n\times m$ real matrices and $diag\{A,B\} \triangleq \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ represents two matrices A, and B of appropriate dimensions. A matrix $M \in \mathbb{R}^{n\times n}$ is semi-positive definite, $M \ge 0$, if $x^\top Mx \ge 0$, $\forall x \in \mathbb{R}^n$; M is positive definite, $M > 0$, if $x^\top Mx > 0$, $\forall x \in \mathbb{R}^n$, $x \ne 0$.

2. PROBLEM FORMULATION AND PRELIMINARIES

Consider the following 2-D switched Roesser model with mixed state delays, which is commonly used to represent the dynamics of many practical systems such as transmission lines, heat exchangers, chemical reactors, pipe furnaces [2] and also has broad applications in image enhancement, signal processing, and image deblurring applications [39]:

$$x_{\iota+1,\kappa+1} = A^{\sigma(\iota,\kappa)}x_{\iota,\kappa} + A_\tau^{\sigma(\iota,\kappa)}x_{\tau(\iota,k)} + A_d^{\sigma(\iota,\kappa)}x_{d(\iota,\kappa)} + B^{\sigma(\iota,\kappa)}w_{\iota,\kappa} + C^{\sigma(\iota,\kappa)}u_{\iota,\kappa}, \tag{1a}$$

$$z_{\iota,\kappa} = D^{\sigma(\iota,\kappa)}x_{\iota,\kappa} + D_\tau^{\sigma(\iota,\kappa)}x_{\tau(\iota,k)} + E^{\sigma(\iota,\kappa)}w_{\iota,\kappa}, \tag{1b}$$

with $x_{\iota+1,\kappa+1} = \left[x^{hT}_{\iota+1,\kappa} \;\; x^{vT}_{\iota,\kappa+1}\right]^T$, $x_{\iota,\kappa} = \left[x^{hT}_{\iota,\kappa} \;\; x^{vT}_{\iota,\kappa}\right]^T$, $x_{\tau(\iota,k)} = \left[x^{hT}_{\iota-\tau_h(\iota),\kappa} \;\; x^{vT}_{\iota,\kappa-\tau_v(\kappa)}\right]^T$, $x_{d(\iota,\kappa)} = \left[\left(\sum_{s=1}^{d_h(\iota)} x^h_{\iota-s,\kappa}\right)^T \left(\sum_{r=1}^{d_v(\kappa)} x^v_{\iota,\kappa-r}\right)^T\right]^T$, and where $\iota,\kappa \in \mathbb{Z}^+$; $x^h_{\iota,\kappa} \in \mathbb{R}^{n_h}$ and $x^v_{\iota,\kappa} \in \mathbb{R}^{n_v}$ are the horizontal state vector and the vertical state vectors, respectively; $u_{\iota,\kappa} \in \mathbb{R}^m$, $z_{\iota,\kappa} \in \mathbb{R}^p$ and $w_{\iota,\kappa} \in \mathbb{R}^q$ represent the control input, the controlled output and the disturbance which belongs to $l_2\{[0,\infty),[0,\infty)\}$, respectively; $\sigma(.,.) : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Z}[1,N]$ represents the switching signal. $A^k, A^k_\tau, A^k_d, B^k, C^k, D^k, D^k_\tau$ and E^k, $\forall k \in \mathbb{Z}$ are known constant matrices with appropriate dimensions. $\tau_h(\iota)$, $d_h(\iota)$ and $\tau_v(\kappa)$, $d_v(\kappa)$ are the time-varying delays along the horizontal and vertical directions respectively, satisfying:

$$\tau_{hL} \le \tau_h(\iota) \le \tau_{hU}, \quad \tau_{vL} \le \tau_v(\kappa) \le \tau_{vU}, \tag{2a}$$

$$d_{hL} \le d_h(\iota) \le d_{hU}, \quad d_{vL} \le \tau_v(\kappa) \le d_{vU}, \tag{2b}$$

where $\tau_{hL}, \tau_{hU}, \tau_{vL}, \tau_{vU}, d_{hL}, d_{hU}, d_{vL}$ and d_{vU} are known non-negative integers. Denoting $\mu_h = max(\tau_{hU}, d_{hU})$ and $\mu_v = max(\tau_{vU}, d_{vU})$, the initial conditions of system (1) are defined as:

$$\begin{aligned} &x^h_{\iota,\kappa} = \phi_{\iota,\kappa}, \;\; \iota \in \mathbb{Z}[-\mu_h, 0], \;\; 0 \le \kappa \le z_1, \\ &x^v_{\iota,\kappa} = \psi_{\iota,\kappa}, \;\; \kappa \in \mathbb{Z}[-\mu_v, 0], \;\; 0 \le \iota \le z_2, \\ &x^h_{\iota,\kappa} = 0, \;\; \kappa > z_1; \;\; x^v_{\iota,\kappa} = 0, \;\; \iota > z_2, \end{aligned} \tag{3}$$

where $\phi_{\iota,.} \in l_2(\mathbb{Z}^+), \forall \iota \in \mathbb{Z}[-\mu_h, 0]$ and $\psi_{.,\kappa} \in l_2(\mathbb{Z}^+), \forall \kappa \in \mathbb{Z}[-\mu_v, 0]$; $z_1 < \infty$ and $z_2 < \infty$ are the non-negative integers. This paper assumes that the switch occurs only at each sampling point of ι or κ, and the switching sequence is described as $((\iota_0,\kappa_0), \sigma(\iota_0,\kappa_0))$, $((\iota_1,\kappa_1), \sigma(\iota_1,\kappa_1))$, ..., $((\iota_\chi,\kappa_\chi), \sigma(\iota_\chi,\kappa_\chi))$, ...; where

$(\iota_\chi, \kappa_\chi)$ represents the instant at which switch occurs that in this case is χ_{th}. It is essential to mention at this stage that the value of $\sigma(\iota,\kappa)$ is only dependent on the value of $\iota+\kappa$ (please see [11], for details).

Remark 1: If we assume that there is only one subsystem then the switched system (1) will reduce to the following system:

$$x_{\iota+1,\kappa+1} = Ax_{\iota,\kappa} + A_\tau x_{\tau(\iota,k)} + A_d x_{d(\iota,\kappa)} + Bw_{\iota,\kappa} + Cu_{\iota,\kappa},$$
$$z_{\iota,\kappa} = Dx_{\iota,\kappa} + D_\tau x_{\tau(\iota,k)} + Ew_{\iota,\kappa},\ \iota,\kappa \in \mathbb{Z}^+.$$

Definition 1 [40]**:** For switching signal $\sigma(\iota,\kappa)$ and any $\Gamma > \kappa_0$, let $N_{\sigma(\iota,\kappa)}(\kappa_0,\Gamma)$ be the switching numbers of $\sigma(\iota,\kappa)$ over the interval (κ_0,Γ). For any given $N_0 \geq 0$ and $t_a > 0$, if

$$N_{\sigma(\iota,\kappa)}(\kappa_0,\Gamma) \leq N_0 + \frac{\Gamma-\kappa_0}{t_a}, \tag{4}$$

then the constant t_a is called the average dwell time and N_0 is the chatter bound. Here, we assume $N_0 = 0$ for simplicity which is very common assumption adopted in the literature.

Definition 2 [12]**:** System (1) is said to be exponentially stable under the switching signal $\sigma(\iota,\kappa)$, if for a given $\kappa_0 \geq 0$, there exist positive constants c and η such that $\sum_{\iota+\kappa=\Gamma}\|x_{\iota,\kappa}\|^2 \leq \eta e^{-c(\Gamma-\kappa_0)}\sum_{\iota+\kappa=\kappa_0}\|x_{\iota,\kappa}\|_C^2$, holds for all $\Gamma \geq \kappa_0$, and $\sum\limits_{\iota+\kappa=\kappa_0}\|x_{\iota,\kappa}\|_C^2 \triangleq \sup\limits_{\substack{-\mu_h\leq\Theta_h\leq 0\\ -\mu_v\leq\Theta_v\leq 0}}\sum\limits_{\iota+\kappa=\kappa_0}\left\{\left\|x^h_{\iota-\Theta_h,\kappa}\right\|^2 + \left\|x^v_{\iota,\kappa-\Theta_v}\right\|^2, \left\|\eta^h_{\iota-\Theta_h,\kappa}\right\|^2 + \left\|\eta^v_{\iota,\kappa-\Theta_v}\right\|^2\right\}$, where $\eta^h_{\iota-\Theta_h,\kappa} = x^h_{\iota-\Theta_h+1,\kappa} - x^h_{\iota-\Theta_h,\kappa}$, $\eta^v_{\iota,\kappa-\Theta_v} = x^v_{\iota,\kappa-\Theta_v+1} - x^v_{\iota,\kappa-\Theta_v}$.

Definition 3: [40] If the following conditions are satisfied, then 2-D switched system (1) under any switching signal $\sigma(.,.)$, is said to have a desired H_∞ disturbance attenuation performance γ, for any given scalar $\gamma > 0$:

(i) If $w_{\iota,\kappa} = 0$ then the system (1) is asymptotically or exponentially stable.

(ii) The inequality $\sum\limits_{\iota=0}^{\infty}\sum\limits_{\kappa=0}^{\infty}(\beta^{\iota+\kappa})\|z_{\iota,\kappa}\|_2^2 < \gamma^2\sum\limits_{\iota=0}^{\infty}\sum\limits_{\kappa=0}^{\infty}\|w_{\iota,\kappa}\|_2^2$, $\forall 0 \neq w \in l_2\{[0,\infty),[0,\infty)\}$, and $0<\beta<1$ holds under the zero boundary condition.

Lemma 1 [41]**:** For a symmetric positive definite matrix $R \in \mathbb{R}^{n\times n}$, positive integers h,v and a function $x : \mathbb{Z}[\iota-h,\kappa]\times[\iota-v,\kappa] \to \mathbb{R}^n$, $\iota,\kappa \in \mathbb{Z}^+$, the inequalities $\sum\limits_{l=\iota-d_h}^{\iota-1}\eta^{hT}_{l,\kappa}R\eta^h_{l,\kappa} \geq \frac{1}{d_h}[x_{\iota,\kappa}-x_{\iota-d_h,\kappa}]^T R[x_{\iota,\kappa}-x_{\iota-d_h,\kappa}]$, and $\sum\limits_{l=\kappa-d_v}^{\kappa-1}\eta^{vT}_{\iota,l}R\eta^v_{\iota,l} \geq \frac{1}{d_v}[x_{\iota,\kappa}-x_{\iota,\kappa-d_v}]^T R[x_{\iota,\kappa}-x_{\iota,\kappa-d_v}]$ hold, where $\eta^h_{l,\kappa} = x_{l+1,\kappa}-x_{l,\kappa}$, $\eta^v_{\iota,l} = x_{\iota,l+1}-x_{\iota,l}$.

Lemma 2 [23]**:** For any vector $\varpi_t \in \mathbb{R}^n$, two positive integers l_1 and l_2, and a symmetric positive definite matrix $H \in \mathbb{R}^{n\times n}$, the inequality $-(l_2-l_1-1)\sum\limits_{t=l_2}^{l_1}\varpi_t^T H\varpi_t \geq \left[\sum\limits_{t=l_2}^{l_1}\varpi_t^T\right] H \left[\sum\limits_{t=l_2}^{l_1}\varpi_t\right]$ holds.

3. MAIN RESULTS

In this section, firstly we shall discuss the stability analysis problem of 2-D switched systems with mixed delays. Then, the controller synthesis problem will be solved such that resultant closed-loop system is not only exponentially stable but also possesses a desired H_∞ performance level γ.

3.1. Stability and H_∞ performance analysis

Let us consider the system model of the form:

$$\begin{aligned} x_{\iota+1,\kappa+1} =& A^{\sigma(\iota,\kappa)}x_{\iota,\kappa} + A_\tau^{\sigma(\iota,\kappa)}x_{\tau(\iota,k)} \\ &+ A_d^{\sigma(\iota,\kappa)}x_{d(\iota,\kappa)},\ \iota,\ \kappa \in \mathbb{Z}^+. \end{aligned} \tag{5}$$

The stability analysis problem for the system of the form (5) is being presented in the Theorem 1 stated below.

Theorem 1: For some given positive constants τ_{hL}, τ_{vL}, τ_{hU}, τ_{vU}, d_{hL}, d_{vL}, d_{hU}, and d_{vU}, if there exist symmetric positive definite matrices $\wp^k = diag\{\wp^{kh},\wp^{kv}\}$, $Q_1^k = diag\{Q_1^{kh},Q_1^{kv}\}$, $Q_2^k = diag\{Q_2^{kh},Q_2^{kv}\}$, $Q_3^k = diag\{Q_3^{kh},Q_3^{kv}\}$, $R_1^k = diag\{R_1^{kh},R_1^{kv}\}$, $R_2^k = diag\{R_2^{kh},R_2^{kv}\}$, $W_1^k = diag\{W_1^{kh},W_1^{kv}\}$, $W_2^k = diag\{W_2^{kh},W_2^{kv}\}$, $W_3^k = diag\{W_3^{kh},W_3^{kv}\}$ with appropriate dimensions and $0<\beta<1$, such that with $k \in \mathbb{Z}[1,N]$, the following inequality holds:

$$\bar{\Pi}_0 = \begin{bmatrix} \Phi^k & \bar{A}^{kT}\wp^k & D^{kT}\Xi \\ * & -\wp^k & 0 \\ * & * & -\Psi \end{bmatrix} < 0,\ \forall k \in \mathbb{Z}[1,N], \tag{6}$$

where $\Phi^k = \begin{bmatrix} \Phi_{11}^k & 0 & \bar{\beta}_L W_1^k & \bar{\beta}_U W_2^k & 0 \\ * & \Phi_{22}^k & \bar{\beta}_U W_3^k & \bar{\beta}_U W_3^k & 0 \\ * & * & \Phi_{33}^k & 0 & 0 \\ * & * & * & \Phi_{44}^k & 0 \\ * & * & * & * & \Phi_{55}^k \end{bmatrix}$, $\Xi = [\Xi_1\ \ \Xi_2\ \ \Xi_3]$, $\Psi = diag\{\Xi_1,\Xi_2,\Xi_3\}$, and $\Phi_{11}^k = \bar{\beta}(-\wp^k + Q_1^k + Q_2^k + Q_3^k + r_d^{(1)}R_1^k + r_d^{(2)}R_2^k + r_\tau Q_2^k) + \bar{\beta}_L(-W_1^k - W_2^k)$, $\Phi_{22}^k = \bar{\beta}_U(-Q_2^k - 2W_3^k)$, $\Phi_{33}^k = \bar{\beta}_L(-Q_1^k - W_1^k - W_3^k)$, $\Phi_{44}^k = \bar{\beta}_U(-Q_3^k - W_2^k - W_3^k)$, $\Phi_{55}^k = \bar{\beta}_{dL}(-R_1^k - R_2^k)$, $r_{dh}^{(1)} = d_{hU}.d_{hL}$, $r_{dv}^{(1)} = d_{vU}.d_{vL}$, $r_\tau = diag\{r_{\tau h}I_{n_h}, r_{\tau v}I_{n_v}\}$, $\tau_L = diag\{\tau_{hL}I_{n_h},\tau_{vL}I_{n_v}\}$, $\tau_U = diag\{\tau_{hU}I_{n_h},\tau_{vU}I_{n_v}\}$, $r_d^{(1)} = diag\{r_{dh}^{(1)}I_{n_h}, r_{dv}^{(1)}I_{n_v}\}$, $r_d^{(2)} = diag\{r_{dh}^{(2)}I_{n_h}, r_{dv}^{(2)}I_{n_v}\}$, $\Xi_1 = \tau_L^2\bar{\beta}W_1^k$, $\Xi_2 = \tau_U^2\bar{\beta}W_2^k$, $\Xi_3 = (r_\tau)^2\bar{\beta}W_3^k$, $\bar{\beta}_{dL} = diag\{\beta^{1+d_{hL}}I_{n_h},\beta^{1+d_{vL}}I_{n_v}\}$, $\bar{\beta}_U = diag\{\beta^{1+\tau_{hU}}I_{n_h},\beta^{1+\tau_{vU}}I_{n_v}\}$, $\bar{\beta} = diag\{\beta I_{n_h},\beta I_{n_v}\}$, $\bar{\beta}_L = diag\{\beta^{1+\tau_{hL}}I_{n_h},\beta^{1+\tau_{vL}}I_{n_v}\}$, $r_{dh}^{(2)} = \frac{d_{hU}(d_{hU}+d_{hL})(d_{hU}-d_{hL}+1)}{2}$, $r_{dv}^{(2)} = \frac{d_{vU}(d_{vU}+d_{vL})(d_{vU}-d_{vL}+1)}{2}$, $\bar{A}^k = [A^k\ A_\tau^k\ 0\ 0\ A_d^k]$, $D^k = [A^k - I\ A_\tau^k\ 0\ 0\ A_d^k]$. Then, the exponential stability of system (5) can be guaranteed for any switching signals with average dwell time satisfying:

$$t_a > t_a^* = \frac{\ln\lambda}{-\ln\beta}, \tag{7}$$

where $\lambda \geq 1$, satisfies

$$\begin{aligned} &\wp^k \leq \lambda\wp^l,\ Q_1^k \leq \lambda Q_1^l,\ Q_2^k \leq \lambda Q_2^l,\ Q_3^k \leq \lambda Q_3^l,\\ &R_1^k \leq \lambda R_1^l,\ R_2^k \leq \lambda R_2^l,\ W_1^k \leq \lambda W_1^l,\ W_2^k \leq \lambda W_2^l,\\ &W_3^k \leq \lambda W_3^l,\ \forall k,l \in \mathbb{Z}[1,N]. \end{aligned} \tag{8}$$

Proof: With no loss of generality, it is supposed that the k_{th} subsystem is in active state, for which the following type of new suitable LKF is considered:

$$V^k_{x(\iota,\kappa)} = V^{kh}_{x(\iota,\kappa)} + V^{kv}_{x(\iota,\kappa)}, \tag{9}$$

where $V^{kh}_{x(\iota,\kappa)} = x^{hT}_{\iota,\kappa}\wp^{kh}x^h_{\iota,\kappa} + \sum_{s=\iota-\tau_{hL}}^{\iota-1} x^{hT}_{s,\kappa}Q_1^{kh}x^h_{s,\kappa}\beta^{\iota-s} + \sum_{s=\iota-\tau_h(\iota)}^{\iota-1} x^{hT}_{s,\kappa}Q_2^{kh}x^h_{s,\kappa}\beta^{\iota-s} + \sum_{m=-\tau_{hU}+1}^{-\tau_{hL}}\sum_{s=\iota+m}^{\iota-1} x^{hT}_{s,\kappa}Q_2^{kh}x^h_{s,\kappa}\beta^{\iota-s} + \sum_{s=\iota-\tau_{hU}}^{\iota-1} x^{hT}_{s,\kappa}Q_3^{kh}x^h_{s,\kappa}\beta^{\iota-s} + \tau_{hL}\sum_{m=-\tau_{hL}}^{-1}\sum_{s=\iota+m}^{\iota-1}\eta^{hT}_{s,\kappa}W_1^{kh}\eta^h_{s,\kappa}\beta^{\iota-s} + \tau_{hU}\sum_{m=-\tau_{hU}}^{-1}\sum_{s=\iota+m}^{\iota-1}\eta^{hT}_{s,\kappa}W_2^{kh}\eta^h_{s,\kappa}\beta^{\iota-s} + r_{\tau h}\sum_{m=-\tau_{hU}}^{-\tau_{hL}-1}\sum_{s=\iota+m}^{\iota-1}\eta^{hT}_{s,\kappa}W_3^{kh}\eta^h_{s,\kappa}\beta^{\iota-s} + d_{hU}\sum_{s=1}^{d_h(\iota)}\sum_{m=\iota-s}^{\iota-1}x^{hT}_{m,\kappa}R_1^{kh}x^h_{m,\kappa}\beta^{\iota-m} + d_{hU}\sum_{m=d_{hL}}^{d_{hU}}\sum_{s=1}^{m}\sum_{p=\iota-s}^{\iota-1} x^{hT}_{p,\kappa}R_2^{kh}x^h_{p,\kappa}\beta^{\iota-p}$, and $V^{kv}_{x(\iota,\kappa)} = x^{vT}_{\iota,\kappa}\wp^{kv}x^v_{\iota,\kappa} + \sum_{r=\kappa-\tau_{vL}}^{\kappa-1} x^{vT}_{\iota,r}Q_1^{kv}x^v_{\iota,r}\beta^{\kappa-r} + \sum_{r=\kappa-\tau_v(\kappa)}^{\kappa-1} x^{vT}_{\iota,r}Q_2^{kv}x^v_{\iota,r}\beta^{\kappa-r} + \sum_{m=-\tau_{vU}+1}^{-\tau_{vL}}\sum_{r=\kappa+m}^{\kappa-1} x^{vT}_{\iota,r}Q_2^{kv}x^v_{\iota,r}\beta^{\kappa-r} + \sum_{r=\kappa-\tau_{vU}}^{\kappa-1} x^{vT}_{\iota,r}Q_3^{kv}x^v_{\iota,r}\beta^{\kappa-r} + \tau_{vL}\sum_{m=-\tau_{vL}}^{-1}\sum_{r=\kappa+m}^{\kappa-1}\eta^{vT}_{\iota,r}W_1^{kv}\eta^v_{\iota,r}\beta^{\kappa-r} + \tau_{vU}\sum_{m=-\tau_{vU}}^{-1}\sum_{r=\kappa+m}^{\kappa-1}\eta^{vT}_{\iota,r}W_2^{kv}\eta^v_{\iota,r}\beta^{\kappa-r} + r_{\tau v}\sum_{m=-\tau_{vU}}^{-\tau_{vL}-1}\sum_{r=\kappa+m}^{\kappa-1}\eta^{vT}_{\iota,r}W_3^{kv}\eta^v_{\iota,r}\beta^{\kappa-r} + d_{vU}\sum_{r=1}^{d_v(\kappa)}\sum_{m=\kappa-r}^{\kappa-1}x^{vT}_{\iota,m}R_1^{kv}x^v_{\iota,m}\beta^{\kappa-m} + d_{vU}\sum_{m=d_{vL}}^{d_{vU}}\sum_{r=1}^{m}\sum_{p=\kappa-r}^{\kappa-1}x^{vT}_{\iota,p}R_2^{kv}x^v_{\iota,p}\beta^{\kappa-p}$, where $r_{\tau h} = (\tau_{hU}-\tau_{hL})$, $r_{\tau v} = (\tau_{vU}-\tau_{vL})$, $\eta^h_{\iota,\kappa} = x^h_{\iota+1,\kappa} - x^h_{\iota,\kappa}$ and $\eta^v_{\iota,\kappa} = x^v_{\iota,\kappa+1} - x^v_{\iota,\kappa}$. Clearly, $V^k_{x(\iota,\kappa)} \geq 0, \iota,\kappa \in \mathbb{Z}^+$. With respect to 2-D system model (5), $\Delta V^k_{x(\iota,\kappa)}$ is defined as follows:

$$\Delta V^k_{x(\iota,\kappa)} \triangleq \underbrace{V^{kh}_{x(\iota+1,\kappa)} - \beta V^{kh}_{x(\iota,\kappa)}}_{\Delta V^{kh}_{x(\iota,\kappa)}} + \underbrace{V^{kv}_{x(\iota,\kappa+1)} - \beta V^{kv}_{x(\iota,\kappa)}}_{\Delta V^{kv}_{x(\iota,\kappa)}}. \tag{10}$$

By using (10), the following inequality can be obtained from (9):

$$\begin{aligned} \Delta V^{kh}_{x(\iota,\kappa)} \leq\ & x^{hT}_{\iota+1,\kappa}\wp^{kh}x^h_{\iota+1,\kappa} - \beta x^{hT}_{\iota,\kappa}\wp^{kh}x^h_{\iota,\kappa} + \beta x^{hT}_{\iota,\kappa}Q_1^{kh}x^h_{\iota,\kappa}\\ &- \beta^{1+\tau_{hL}}x^{hT}_{\iota-\tau_{hL},\kappa}Q_1^{kh}x^h_{\iota-\tau_{hL},\kappa} + \beta x^{hT}_{\iota,\kappa}Q_2^{kh}x^h_{\iota,\kappa}\\ &- \beta^{1+\tau_{hU}}x^{hT}_{\iota-\tau_h(\iota),\kappa}Q_2^{kh}x^h_{\iota-\tau_h(\iota),\kappa}\\ &+ \sum_{s=\iota+1-\tau_{hU}}^{\iota-\tau_{hL}} x^{hT}_{s,\kappa}Q_2^{kh}x^h_{s,\kappa}\beta^{\iota+1-s} + \beta r_{\tau h}x^{hT}_{\iota,\kappa}Q_2^{kh}x^h_{\iota,\kappa}\\ &- \sum_{s=\iota+1-\tau_{hU}}^{\iota-\tau_{hL}} x^{hT}_{s,\kappa}Q_2^{kh}x^h_{s,\kappa}\beta^{\iota+1-s} + \beta x^{hT}_{\iota,\kappa}Q_3^{kh}x^h_{\iota,\kappa}\\ &- \beta^{1+\tau_{hU}}x^{hT}_{\iota-\tau_{hU},\kappa}Q_3^{kh}x^h_{\iota-\tau_{hU},\kappa} + \tau_{hL}^2\beta\eta^{hT}_{\iota,\kappa}W_1^{kh}\eta^h_{\iota,\kappa}\\ &- \tau_{hL}\beta^{1+\tau_{hL}}\sum_{s=\iota-\tau_{hL}}^{\iota-1}\eta^{hT}_{s,\kappa}W_1^{kh}\eta^h_{s,\kappa}\\ &+ \tau_{hU}^2\beta\eta^{hT}_{\iota,\kappa}W_2^{kh}\eta^h_{\iota,\kappa}\\ &- \tau_{hU}\beta^{1+\tau_{hU}}\sum_{s=\iota-\tau_{hU}}^{\iota-1}\eta^{hT}_{s,\kappa}W_2^{kh}\eta^h_{s,\kappa}\\ &+ (r_{\tau h})^2\beta\eta^{hT}_{\iota,\kappa}W_3^{kh}\eta^h_{\iota,\kappa}\\ &- r_{\tau h}\beta^{1+\tau_{hU}}\sum_{s=\iota-\tau_{hU}}^{\iota-\tau_{hL}-1}\eta^{hT}_{s,\kappa}W_3^{kh}\eta^h_{s,\kappa}\\ &+ r^{(1)}_{dh}x^{hT}_{\iota,\kappa}R_1^{kh}x^h_{\iota,\kappa}\beta\\ &- d_{hU}\sum_{s=1}^{d_h(\iota)}x^{hT}_{\iota-s,\kappa}R_1^{kh}\beta^{s+1}x^h_{\iota-s,\kappa}\\ &+ r^{(2)}_{dh}\beta x^{hT}_{\iota,\kappa}R_2^{kh}x^h_{\iota,\kappa}\\ &- d_{hU}\sum_{m=d_{hL}}^{d_{hU}}\sum_{s=1}^{m}x^{hT}_{\iota-s,\kappa}R_2^{kh}x^h_{\iota-s,\kappa}\beta^{s+1}, \end{aligned}$$

and

$$\begin{aligned} \Delta V^{kv}_{x(\iota,\kappa)} \leq\ & x^{vT}_{\iota,\kappa+1}\wp^{kv}x^v_{\iota,\kappa+1} - \beta x^{vT}_{\iota,\kappa}\wp^{kv}x^v_{\iota,\kappa} + \beta x^{vT}_{\iota,\kappa}Q_1^{kv}x^v_{\iota,\kappa}\\ &- \beta^{1+\tau_{vL}}x^{vT}_{\iota,\kappa-\tau_{vL}}Q_1^{kv}x^v_{\iota,\kappa-\tau_{vL}} + \beta x^{vT}_{\iota,\kappa}Q_2^{kv}x^v_{\iota,\kappa}\\ &- \beta^{1+\tau_{vU}}x^{vT}_{\iota,\kappa-\tau_v(\kappa)}Q_2^{kv}x^v_{\iota,\kappa-\tau_v(\kappa)}\\ &+ \sum_{r=\kappa+1-\tau_{vU}}^{\kappa-\tau_{vL}} x^{vT}_{\iota,r}Q_2^{kv}x^v_{\iota,r}\beta^{\kappa+1-r} + \beta r_{\tau v}x^{vT}_{\iota,\kappa}Q_2^{kv}x^v_{\iota,\kappa}\\ &- \sum_{r=\kappa+1-\tau_{vU}}^{\kappa-\tau_{vL}} x^{vT}_{\iota,r}Q_2^{kv}x^v_{\iota,r}\beta^{\kappa+1-r} + \beta x^{vT}_{\iota,\kappa}Q_3^{kv}x^v_{\iota,\kappa}\\ &- \beta^{1+\tau_{vU}}x^{vT}_{\iota,\kappa-\tau_{vU}}Q_3^{kv}x^v_{\iota,\kappa-\tau_{vU}} + \tau_{vL}^2\beta\eta^{vT}_{\iota,\kappa}W_1^{kv}\eta^v_{\iota,\kappa}\\ &- \tau_{vL}\beta^{1+\tau_{vL}}\sum_{r=\kappa-\tau_{vL}}^{\kappa-1}\eta^{vT}_{\iota,r}W_1^{kv}\eta^v_{\iota,r}\\ &+ \tau_{vU}^2\beta\eta^{vT}_{\iota,\kappa}W_2^{kv}\eta^v_{\iota,\kappa}\\ &- \tau_{vU}\beta^{1+\tau_{vU}}\sum_{r=\kappa-\tau_{vU}}^{\kappa-1}\eta^{vT}_{\iota,r}W_2^{kv}\eta^v_{\iota,r}\\ &+ (r_{\tau v})^2\beta\eta^{vT}_{\iota,\kappa}W_3^{kv}\eta^v_{\iota,\kappa}\\ &- r_{\tau v}\beta^{1+\tau_{vU}}\sum_{r=\kappa-\tau_{vU}}^{\kappa-\tau_{vL}-1}\eta^{vT}_{\iota,r}W_3^{kv}\eta^v_{\iota,r}\\ &+ r^{(1)}_{dv}x^{vT}_{\iota,\kappa}R_1^{kv}x^v_{\iota,\kappa}\beta\\ &- d_{vU}\sum_{r=1}^{d_v(\kappa)}x^{vT}_{\iota,\kappa-r}R_1^{kv}\beta^{r+1}x^v_{\iota,\kappa-r} + r^{(2)}_{dv}\beta x^{vT}_{\iota,\kappa}R_2^{kv}x^v_{\iota,\kappa} \end{aligned}$$

$$-d_{vU}\sum_{m=d_{vL}}^{d_{vU}}\sum_{r=1}^{m}x_{\iota,\kappa-r}^{vT}R_2^{kv}x_{\iota,\kappa-r}^{v}\beta^{r+1}.$$

The terms $-\tau_{hL}\beta^{1+\tau_{hL}}\sum_{s=\iota-\tau_{hL}}^{\iota-1}\eta_{s,\kappa}^{hT}W_1^{kh}\eta_{s,\kappa}^{h}$, $-\tau_{hU}\beta^{1+\tau_{hU}}\times\sum_{s=\iota-\tau_{hU}}^{\iota-1}\eta_{s,\kappa}^{hT}W_2^{kh}\eta_{s,\kappa}^{h}$, and $-\tau_{vL}\beta^{1+\tau_{vL}}\sum_{r=\kappa-\tau_{vL}}^{\kappa-1}\eta_{\iota,r}^{vT}W_1^{kv}\eta_{\iota,r}^{v}$, $-\tau_{vU}\beta^{1+\tau_{vU}}\sum_{r=\kappa-\tau_{vU}}^{\kappa-1}\eta_{\iota,r}^{vT}W_2^{kv}\eta_{\iota,r}^{v}$ in $\Delta V_{x(\iota,\kappa)}^{kh}$ and $\Delta V_{x(\iota,\kappa)}^{kv}$, can be written as the inequalities (11)-(14), respectively, by virtue of Lemma 1, which are stated below:

$$\leq x_{\tau L}^{hT}\begin{bmatrix}-\beta^{1+\tau_{hL}}W_1^{kh} & \beta^{1+\tau_{hL}}W_1^{kh}\\ * & -\beta^{1+\tau_{hL}}W_1^{kh}\end{bmatrix}x_{\tau L}^{h}, \quad (11)$$

$$\leq x_{\tau U}^{hT}\begin{bmatrix}-\beta^{1+\tau_{hU}}W_2^{kh} & \beta^{1+\tau_{hU}}W_2^{kh}\\ * & -\beta^{1+\tau_{hU}}W_2^{kh}\end{bmatrix}x_{\tau U}^{h}, \quad (12)$$

$$\leq x_{\tau L}^{vT}\begin{bmatrix}-\beta^{1+\tau_{vL}}W_1^{kv} & \beta^{1+\tau_{vL}}W_1^{kv}\\ * & -\beta^{1+\tau_{vL}}W_1^{kv}\end{bmatrix}x_{\tau L}^{v}, \quad (13)$$

$$\leq x_{\tau U}^{vT}\begin{bmatrix}-\beta^{1+\tau_{vU}}W_2^{kv} & \beta^{1+\tau_{vU}}W_2^{kv}\\ * & -\beta^{1+\tau_{vU}}W_2^{kv}\end{bmatrix}x_{\tau U}^{v}, \quad (14)$$

where $x_{\tau L}^{h}=[x_{\iota,\kappa}^{hT}\ x_{\iota-\tau_{hL},\kappa}^{hT}]^T$, $x_{\tau U}^{h}=[x_{\iota,\kappa}^{hT}\ x_{\iota-\tau_{hU},\kappa}^{hT}]^T$, $x_{\tau L}^{v}=[x_{\iota,\kappa}^{vT}\ x_{\iota,\kappa-\tau_{vL}}^{vT}]^T$, and $x_{\tau U}^{v}=[x_{\iota,\kappa}^{vT}\ x_{\iota,\kappa-\tau_{vU}}^{vT}]^T$.

The terms $-d_{hU}\sum_{s=1}^{d_h(\iota)}x_{\iota-s,\kappa}^{hT}R_1^{kh}\beta^{s+1}x_{\iota-s,\kappa}^{h}$, $-d_{hU}\sum_{m=d_{hL}}^{d_{hU}}\sum_{s=1}^{m}x_{\iota-s,\kappa}^{hT}R_2^{kh}x_{\iota-s,\kappa}^{h}\beta^{s+1}$, $-r_{\tau h}\beta^{1+\tau_{hU}}\sum_{s=\iota-\tau_{hU}}^{\iota-\tau_{hL}-1}\eta_{s,\kappa}^{hT}W_3^{kh}\eta_{s,\kappa}^{h}$, and $-d_{vU}\sum_{r=1}^{d_v(\kappa)}x_{\iota,\kappa-r}^{vT}R_1^{kv}\beta^{r+1}x_{\iota,\kappa-r}^{v}$, $-d_{vU}\sum_{m=d_{vL}}^{d_{vU}}\sum_{r=1}^{m}x_{\iota,\kappa-r}^{vT}R_2^{kv}x_{\iota,\kappa-r}^{v}\beta^{r+1}$, $-r_{\tau v}\beta^{1+\tau_{vU}}\sum_{r=\kappa-\tau_{vU}}^{\kappa-\tau_{vL}-1}\eta_{\iota,r}^{vT}W_3^{kv}\eta_{\iota,r}^{v}$ in $\Delta V_{x(\iota,\kappa)}^{kh}$ and $\Delta V_{x(\iota,\kappa)}^{kv}$, can be written in terms of inequalities (15)-(20), respectively, after utilization of Lemma 2, which are stated below:

$$\leq -d_h(\iota)\sum_{s=1}^{d_h(\iota)}x_{\iota-s,\kappa}^{hT}R_1^{kh}\beta^{s+1}x_{\iota-s,\kappa}^{h}$$

$$\leq-\left(\sum_{s=1}^{d_h(\iota)}x_{\iota-s,\kappa}^{h}\right)^T R_1^{kh}\beta^{d_{hL}+1}\left(\sum_{s=1}^{d_h(\iota)}x_{\iota-s,\kappa}^{h}\right), \quad (15)$$

$$\leq -d_h(\iota)\sum_{s=1}^{d_h(\iota)}x_{\iota-s,\kappa}^{hT}R_2^{kh}\beta^{s+1}x_{\iota-s,\kappa}^{h}$$

$$\leq-\left(\sum_{s=1}^{d_h(\iota)}x_{\iota-s,\kappa}^{h}\right)^T R_2^{kh}\beta^{d_{hL}+1}\left(\sum_{s=1}^{d_h(\iota)}x_{\iota-s,\kappa}^{h}\right), \quad (16)$$

$$\leq x_{\tau LU}^{hT}V_0\otimes\left(\beta^{1+\tau_{hU}}W_3^{kh}\right)x_{\tau LU}^{h}, \quad (17)$$

$$\leq -d_v(\kappa)\sum_{r=1}^{d_v(\kappa)}x_{\iota,\kappa-r}^{vT}R_1^{kv}\beta^{r+1}x_{\iota,\kappa-r}^{v}$$

$$\leq-\left(\sum_{r=1}^{d_v(\kappa)}x_{\iota,\kappa-r}^{vT}\right)R_1^{kv}\beta^{d_{vL}+1}\left(\sum_{r=1}^{d_v(\kappa)}x_{\iota,\kappa-r}^{v}\right), \quad (18)$$

$$\leq -d_v(\kappa)\sum_{r=1}^{d_v(\kappa)}x_{\iota,\kappa-r}^{vT}R_2^{kv}\beta^{s+1}x_{\iota,\kappa-r}^{v}$$

$$\leq-\left(\sum_{r=1}^{d_v(\kappa)}x_{\iota,\kappa-r}^{v}\right)^T R_2^{kv}\beta^{d_{vL}+1}\left(\sum_{r=1}^{d_v(\kappa)}x_{\iota,\kappa-r}^{v}\right), \quad (19)$$

$$\leq x_{\tau LU}^{vT}V_0\otimes\left(\beta^{1+\tau_{vU}}W_3^{kv}\right)x_{\tau LU}^{v}, \quad (20)$$

where $x_{\tau LU}^{h}=\left[x_{\iota-\tau_h(\iota),\kappa}^{hT}\ x_{\iota-\tau_{hL},\kappa}^{hT}\ x_{\iota-\tau_{hU},\kappa}^{hT}\right]^T$, $x_{\tau LU}^{v}=\left[x_{\iota,\kappa-\tau_v(\iota)}^{vT}\ x_{\iota,\kappa-\tau_{vL}}^{vT}\ x_{\iota,\kappa-\tau_{vU}}^{vT}\right]^T$, $V_0=\begin{bmatrix}-2 & 1 & 1\\ * & -1 & 0\\ * & * & -1\end{bmatrix}$ and the symbol $\otimes$ represents the Kronecker product. For simplicity, we denote $x_{\tau L}=\left[x_{\iota-\tau_{hL},\kappa}^{hT}\ x_{\iota,\kappa-\tau_{vL}}^{vT}\right]^T$, $x_{\tau U}=\left[x_{\iota-\tau_{hU},\kappa}^{hT}\ x_{\iota,\kappa-\tau_{vU}}^{vT}\right]^T$, $\eta_{\iota,\kappa}=\left[\eta_{\iota,\kappa}^{hT}\ \eta_{\iota,\kappa}^{vT}\right]^T$, $\Gamma_{\iota,\kappa}=\left[x_{\iota,\kappa}^T\ x_{\tau(\iota,k)}^T\ x_{\tau L}^T\ x_{\tau U}^T\ x_{d(\iota,\kappa)}^T\right]^T$.

Thus, the terms $\Delta V_{x(\iota,\kappa)}^{kh}+\Delta V_{x(\iota,\kappa)}^{kv}$, together with (11)-(20) imply that $\Delta V_{x(\iota,\kappa)}^{k}\leq x_{\iota+1,\kappa+1}^{T}\wp^k x_{\iota+1,\kappa+1}+\eta_{\iota,\kappa}^T\left(\Xi_1+\Xi_2+\Xi_3\right)\eta_{\iota,\kappa}+\Gamma_{\iota,\kappa}^T\Phi^k\Gamma_{\iota,\kappa}$. That is

$$\Delta V_{x(\iota,\kappa)}^{k}\leq\Gamma_{\iota,\kappa}^T\Pi_0\Gamma_{\iota,\kappa}, \quad (21)$$

where $\Pi_0=\bar{A}^{kT}\wp^k\bar{A}^k+D^{kT}\left(\Xi_1+\Xi_2+\Xi_3\right)D^k+\Phi^k$.

By Schur complement, the inequality (6) simply implies $\Pi_0<0$. Consequently, by (21), we obtain:

$$V_{x(\iota+1,\kappa)}^{kh}+V_{x(\iota,\kappa+1)}^{kv}\leq\beta\left(V_{x(\iota,\kappa)}^{kh}+V_{x(\iota,\kappa)}^{kv}\right), \quad (22)$$

where $\iota,\kappa\in\mathbb{Z}^+$. For any non-negative integer $\Gamma>\kappa_0=\max(z_1,z_2)$, one has that $V_{x(0,\Gamma)}^{kh}=V_{x(\Gamma,0)}^{kv}=0$. Then, summation of both sides of (22) from $\Gamma-1$ to 0 with reference to κ and from 0 to $\Gamma-1$ with reference to ι, the following can be obtained:

$$\begin{aligned}\sum_{\iota+\kappa=\Gamma}V_{x(\iota,\kappa)}^{k}&=\sum_{\iota+\kappa=\Gamma}\left(V_{x(\iota,\kappa)}^{kh}+V_{x(\iota,\kappa)}^{kv}\right)\\ &=V_{x(0,\Gamma)}^{kh}+V_{x(1,\Gamma-1)}^{kh}+\ldots+V_{x(\Gamma-1,1)}^{kh}\\ &\quad+V_{x(\Gamma,0)}^{kh}+V_{x(\Gamma,0)}^{kv}+V_{x(\Gamma-1,1)}^{kv}+\ldots\\ &\quad+V_{x(1,\Gamma-1)}^{kv}+V_{x(0,\Gamma)}^{kv}\\ &<\beta\big(0+V_{x(0,\Gamma-1)}^{kh}+V_{x(1,\Gamma-2)}^{kh}+\ldots\\ &\quad+V_{x(\Gamma-2,1)}^{kh}+V_{x(\Gamma-1,0)}^{kh}+V_{x(\Gamma-1,0)}^{kv}\\ &\quad+V_{x(\Gamma-2,1)}^{kv}+\ldots+V_{x(1,\Gamma-2)}^{kv}+V_{x(0,\Gamma-1)}^{kv}\\ &\quad+0\big)\\ &=\beta\sum_{\iota+\kappa=\Gamma-1}V_{x(\iota,\kappa)}^{k}\\ &<\ldots<\beta^{\Gamma-\kappa_0}\sum_{\iota+\kappa=\kappa_0}V_{x(\iota,\kappa)}^{k}.\end{aligned} \quad (23)$$

Now, suppose that $\vartheta=N_{\sigma(\iota,\kappa)}(\kappa_0,\Gamma)$ represents the switching number of $\sigma(\iota,\kappa)$ over the interval (κ_0,Γ), and consider $m_{\rho-\vartheta+1}<m_{\rho-\vartheta+2}<\ldots<m_{\rho-1}<m_\rho$ which actually represents the switching points over the corresponding interval, and therefore for $\Gamma\in\left[m_\rho,m_{\rho+1}\right)$, it follows from (23) that

$$\sum_{\iota+\kappa=\Gamma}V_{x(\iota,\kappa)}^{\sigma(m_\rho)}<\beta^{\Gamma-m_\rho}\sum_{\iota+\kappa=m_\rho}V_{x(\iota,\kappa)}^{\sigma(m_\rho)}. \quad (24)$$

Using (8) and (9), at switching instant $m_\rho = \iota + \kappa$, we obtain

$$\sum_{\iota+\kappa=m_\rho} V_{x(\iota,\kappa)}^{\sigma(m_\rho)} \leq \lambda \sum_{\iota+\kappa=m_\rho} V_{x(\iota,\kappa)}^{\sigma(m_{\rho-1})}. \tag{25}$$

Then, according to (4), we may write $\vartheta = N_{\sigma(\iota,\kappa)}(\kappa_0,\Gamma) \leq N_0 + \frac{\Gamma-\kappa_0}{t_a}$, and by (24) and (25), we may obtain

$$\begin{aligned}
&\sum_{\iota+\kappa=\Gamma} V_{x(\iota,\kappa)}^{\sigma(m_\rho)} \\
&< \beta^{\Gamma-m_\rho} \sum_{\iota+\kappa=m_\rho} V_{x(\iota,\kappa)}^{\sigma(m_\rho)} \\
&< \lambda\beta^{\Gamma-m_\rho}\beta^{m_\rho-m_{\rho-1}} \sum_{\iota+\kappa=m_{\rho-1}} V_{x(\iota,\kappa)}^{\sigma(m_{\rho-1})} \\
&= \lambda\beta^{\Gamma-m_{\rho-1}} \sum_{\iota+\kappa=m_{\rho-1}} V_{x(\iota,\kappa)}^{\sigma(m_{\rho-1})} \\
&\leq \ldots < \lambda^{\vartheta-1}\beta^{\Gamma-m_{\rho-\vartheta+1}} \sum_{\iota+\kappa=m_{\rho-\vartheta+1}} V_{x(\iota,\kappa)}^{\sigma(m_{\rho-\vartheta+1})} \\
&< \lambda^{\vartheta}\beta^{\Gamma-m_{\rho-\vartheta+1}}\beta^{m_{\rho-\vartheta+1}-\kappa_0} \sum_{\iota+\kappa=\kappa_0} V_{x(\iota,\kappa)}^{\sigma(\kappa_0)} \\
&\leq \lambda^{\vartheta}\beta^{\Gamma-\kappa_0} \sum_{\iota+\kappa=\kappa_0} V_{x(\iota,\kappa)}^{\sigma(\kappa_0)} \\
&\leq \lambda^{N_0} e^{\left(\frac{\ln\lambda}{t_a}+\ln\beta\right)(\Gamma-\kappa_0)} \sum_{\iota+\kappa=\kappa_0} V_{x(\iota,\kappa)}^{\sigma(\kappa_0)}.
\end{aligned} \tag{26}$$

On the contrary, two positive scalars ξ_1 and ξ_2 can be found from (9), such that the following inequality holds

$$\xi_1 \|x_{\iota,\kappa}\|^2 \leq V_{x(\iota,\kappa)}^{\sigma(\iota,\kappa)} \leq \xi_2 \|x_{\iota,\kappa}\|_C^2, \tag{27}$$

where

$$\xi_1 = \min_{k\in\mathbb{Z}[1,N]} \{\lambda_{\min}(\wp^{kh}) + \lambda_{\min}(\wp^{kv})\},$$

$$\begin{aligned}
\xi_2 = \max_{k\in\mathbb{Z}[1,N]} \Big\{ &\lambda_{\max}(\wp^{kh}) + \lambda_{\max}(\wp^{kv}) + \tau_{hL}\lambda_{\max}(Q_1^{kh}) \\
&+ \tau_{vL}\lambda_{\max}(Q_1^{kv}) + \tau_{hU}\lambda_{\max}(Q_2^{kh}) + \tau_{vU}\lambda_{\max}(Q_2^{kv}) \\
&+ r_{dh}^{(2)}\lambda_{\max}(Q_2^{kh}) + r_{dv}^{(2)}\lambda_{\max}(Q_2^{kv}) + \tau_{hU}\lambda_{\max}(Q_3^{kh}) \\
&+ \tau_{vU}\lambda_{\max}(Q_3^{kv}) + r_{dh}^{(2)}\lambda_{\max}(Q_2^{kh}) + r_{dv}^{(2)}\lambda_{\max}(Q_2^{kv}) \\
&+ \tau_{hU}\lambda_{\max}(Q_3^{kh}) + \tau_{vU}\lambda_{\max}(Q_3^{kv}) + \tau_{hL}^2\lambda_{\max}(W_1^{kh}) \\
&+ \tau_{vL}^2\lambda_{\max}(W_1^{kv}) + \tau_{hU}^2\lambda_{\max}(W_2^{kh}) + \tau_{vU}^2\lambda_{\max}(W_2^{kv}) \\
&+ r_{\tau h}^2\lambda_{\max}(W_3^{kh}) + r_{\tau v}^2\lambda_{\max}(W_3^{kv}) + d_{hU}^2\lambda_{\max}(R_1^{kh}) \\
&+ d_{vU}^2\lambda_{\max}(R_1^{kh}) + \left(r_{dh}^{(2)}\right)^2\lambda_{\max}(R_2^{kh}) \\
&+ \left(r_{dv}^{(2)}\right)^2\lambda_{\max}(R_2^{kv}) \Big\}.
\end{aligned}$$

Denoting $\eta = \frac{\xi_2\lambda^{N_0}}{\xi_1} > 0$, then the following can be deduced from (26) and (27):

$$\sum_{\iota+\kappa=\Gamma} \|x_{\iota,\kappa}\|^2 \leq \eta e^{-\left(\frac{-\ln\lambda}{t_a}-\ln\beta\right)(\Gamma-\kappa_0)} \sum_{\iota+\kappa=\kappa_0} \|x_{\iota,\kappa}\|_C^2.$$

Definition 2 implies that if the average dwell time satisfies $\left(t_a > t_a^* = \frac{\ln\lambda}{-\ln\beta}\right)$ then 2-D switched system (5) in the presence of mixed delays is exponentially stable. This concludes our proof. □

Remark 2: It is worth mentioning that the results in Theorem 1 have been obtained by improving the LKF of the results given in [38]. We have introduced the terms $\sum_{s=\iota-\tau_{hU}}^{\iota-1} x_{s,\kappa}^{hT} Q_3^h x_{s,\kappa}^h \beta^{\iota-s}$, $d_{hU}\sum_{s=1}^{d_h(\iota)}\sum_{m=\iota-s}^{\iota-1} x_{m,\kappa}^{hT} R_1^h x_{m,\kappa}^h \beta^{\iota-m}$, $\sum_{r=\kappa-\tau_{vU}}^{\kappa-1} x_{\iota,r}^{vT} Q_3^v x_{\iota,r}^v \beta^{\kappa-r}$, and $d_{vU}\sum_{r=1}^{d_v(\kappa)}\sum_{m=\kappa-r}^{\kappa-1} x_{\iota,m}^{vT} R_1^v x_{\iota,m}^v \beta^{\kappa-m}$, which contain the summations of the state vector in single and double forms and ultimately enables one to take into account the extra information related to time delays into the corresponding augmented vectors. Therefore, the obtained results are expected to enjoy less conservativeness. Now, let us exclude the aforesaid newly introduced terms from the LKF, then one may obtain the results of Corollary 1.

Corollary 1: For some given positive constants τ_{hL}, τ_{vL}, τ_{hU}, τ_{vU}, d_{hL}, d_{vL}, d_{hU}, and d_{vU}, if there exist symmetric positive definite matrices $\wp^k = diag\{\wp^{kh}, \wp^{kv}\}$, $Q_1^k = diag\{Q_1^{kh}, Q_1^{kv}\}$, $Q_2^k = diag\{Q_2^{kh}, Q_2^{kv}\}$, $R_2^k = diag\{R_2^{kh}, R_2^{kv}\}$, $W_1^k = diag\{W_1^{kh}, W_1^{kv}\}$, $W_2^k = diag\{W_2^{kh}, W_2^{kv}\}$, $W_3^k = diag\{W_3^{kh}, W_3^{kv}\}$, with appropriate dimensions and $0 < \beta < 1$, such that with $k \in \mathbb{Z}[1,N]$, the following inequality holds

$$\tilde{\Pi}_0 = \begin{bmatrix} \bar{\Phi}^k & \bar{A}^k\wp^k & D^{kT}\Xi \\ * & -\wp^k & 0 \\ * & * & -\Psi \end{bmatrix} < 0, \quad \forall k \in \mathbb{Z}[1,N] \tag{28}$$

where $\bar{\Phi}^k = \begin{bmatrix} \bar{\Phi}_{11}^k & 0 & \bar{\beta}_L W_1^k & \bar{\beta}_U W_2^k & 0 \\ * & \Phi_{22}^k & \bar{\beta}_U W_3^k & \bar{\beta}_U W_3^k & 0 \\ * & * & \Phi_{33}^k & 0 & 0 \\ * & * & * & \bar{\Phi}_{44}^k & 0 \\ * & * & * & * & \bar{\Phi}_{55}^k \end{bmatrix}$, $\bar{\Phi}_{11}^k = \bar{\beta}(-\wp^k + Q_1^k + Q_2^k + r_d^{(2)}R_2^k + r_\tau^{(1)}Q_2^k) + \bar{\beta}_L(-W_1^k - W_2^k)$, $\bar{\Phi}_{44}^k = \bar{\beta}_U(-W_2^k - W_3^k)$, $\bar{\Phi}_{55}^k = -\bar{\beta}_{dL}R_2^k$ and rest of the parameters are similar to ones defined in Theorem 1. Then, the exponential stability of system (5) can be guaranteed for any switching signals with average dwell time satisfying (7), and the following inequalities $\forall k,l \in \mathbb{Z}[1,N]$:

$$\begin{aligned}
&\wp^k \leq \lambda\wp^l, \quad Q_1^k \leq \lambda Q_1^l, \quad Q_2^k \leq \lambda Q_2^l, \quad R_2^k \leq \lambda R_2^l, \\
&W_1^k \leq \lambda W_1^l, \quad W_2^k \leq \lambda W_2^l, \quad W_3^k \leq \lambda W_3^l.
\end{aligned} \tag{29}$$

Proof: The proof of Corollary 1 is similar to the proof of Theorem 1. Thus, detailed proof is omitted. □

Remark 3: It can be noticed that Theorem 1 requires more number of decision variables (NODV) as compared to the Corollary 1. Therefore, one may state that the results of Theorem 1 are expected to enjoy less conservativeness at the cost of an increase in the computational complexity.

Thus, our aim should be to achieve less conservativeness without increasing the computational burden which is also possible if we exclude the terms $\sum_{s=\iota-\tau_{hL}}^{\iota-1} x_{s,\kappa}^{hT} Q_1^h x_{s,\kappa}^h \beta^{\iota-s}$, $\sum_{r=\kappa-\tau_{vL}}^{\kappa-1} x_{\iota,r}^{vT} Q_1^v x_{\iota,r}^v \beta^{\kappa-r}$, and take $R_1^h = R_2^h, R_1^v = R_2^v$ in LKF of Theorem 1. As a result the following corollary can be obtained.

Corollary 2: For some given positive constants τ_{hL}, τ_{vL}, τ_{hU}, τ_{vU}, d_{hL}, d_{vL}, d_{hU}, and d_{vU}, if there exist symmetric positive definite matrices $\wp^k = diag\{\wp^{kh}, \wp^{kv}\}$, $Q_2^k = diag\{Q_2^{kh}, Q_2^{kv}\}$, $Q_3^k = diag\{Q_3^{kh}, Q_3^{kv}\}$, $R_1^k = diag\{R_1^{kh}, R_1^{kv}\}$, $W_1^k = diag\{W_1^{kh}, W_1^{kv}\}$, $W_2^k = diag\{W_2^{kh}, W_2^{kv}\}$, $W_3^k = diag\{W_3^{kh}, W_3^{kv}\}$ with appropriate dimensions and $0 < \beta < 1$, such that with $k \in \mathbb{Z}[1,N]$, the following inequality holds

$$\hat{\Pi}_0 = \begin{bmatrix} \hat{\Phi}^k & \bar{A}^k \wp^k & D^{kT}\Xi \\ * & -\wp^k & 0 \\ * & * & -\Psi \end{bmatrix} < 0, \tag{30}$$

where, $\hat{\Phi}^k = \begin{bmatrix} \hat{\Phi}_{11}^k & 0 & \bar{\beta}_L W_1^k & \bar{\beta}_U W_2^k & 0 \\ * & \Phi_{22}^k & \bar{\beta}_U W_3^k & \bar{\beta}_U W_3^k & 0 \\ * & * & \hat{\Phi}_{33}^k & 0 & 0 \\ * & * & * & \Phi_{44}^k & 0 \\ * & * & * & * & \hat{\Phi}_{55}^k \end{bmatrix}$, $\hat{\Phi}_{33}^k = \bar{\beta}_L(-W_1^k - W_3^k)$, $\hat{\Phi}_{55}^k = -2\bar{\beta}_{dL} R_1^k$, $\hat{\Phi}_{11}^k = \bar{\beta}(-\wp^k + Q_2^k + Q_3^k + r_d^{(1)} R_1^k + r_d^{(2)} R_1^k + r_\tau Q_2^k) + \bar{\beta}_L(-W_1^k - W_2^k)$ and rest of the parameters are similar to ones defined in Theorem 1. Then, the exponential stability of system (5) can be guaranteed for any switching signals whose average dwell time satisfies (7), and the following inequalities $\forall k, l \in \mathbb{Z}[1,N]$:

$$\begin{aligned} &\wp^k \le \lambda \wp^l, \ Q_2^k \le \lambda Q_2^l, \ Q_3^k \le \lambda Q_3^l, \ R_1^k \le \lambda R_1^l, \\ &W_1^k \le \lambda W_1^l, \ W_2^k \le \lambda W_2^l, \ W_3^k \le \lambda W_3^l. \end{aligned} \tag{31}$$

Proof: The proof of Corollary 2 is similar to the proof of Theorem 1 and therefore the details are omitted. □

Let us consider the following system:

$$x_{\iota+1,\kappa+1} = A^{\sigma(\iota,\kappa)} x_{\iota,\kappa} + A_\tau^{\sigma(\iota,\kappa)} x_{\tau(\iota,k)}, \ \ \iota, \kappa \in \mathbb{Z}^+. \tag{32}$$

The stability analysis problem for system (32) is presented in the Corollary 3 stated below.

Corollary 3: For some given positive constants τ_{hL}, τ_{vL}, τ_{hU}, τ_{vU}, d_{hL}, d_{vL}, d_{hU}, and d_{vU}, if there exist symmetric positive definite matrices $\wp^k = diag\{\wp^{kh}, \wp^{kv}\}$, $Q_1^k = diag\{Q_1^{kh}, Q_1^{kv}\}$, $Q_2^k = diag\{Q_2^{kh}, Q_2^{kv}\}$, $Q_3^k = diag\{Q_3^{kh}, Q_3^{kv}\}$, $W_1^k = diag\{W_1^{kh}, W_1^{kv}\}$, $W_2^k = diag\{W_2^{kh}, W_2^{kv}\}$, $W_3^k = diag\{W_3^{kh}, W_3^{kv}\}$, with appropriate dimensions and $0 < \beta < 1$, such that with $k \in \mathbb{Z}[1,N]$, the following inequality holds

$$\Pi_1 = \begin{bmatrix} \tilde{\Phi}^k & \tilde{A}^k \wp^k & \tilde{D}^{kT}\Xi \\ * & -\wp^k & 0 \\ * & * & -\Psi \end{bmatrix} < 0, \forall k \in \mathbb{Z}[1,N], \tag{33}$$

where $\tilde{\Phi}^k = \begin{bmatrix} \tilde{\Phi}_{11}^k & 0 & \bar{\beta}_L W_1^k & \bar{\beta}_U W_2^k \\ * & \Phi_{22}^k & \bar{\beta}_U W_3^k & \bar{\beta}_U W_3^k \\ * & * & \Phi_{33}^k & 0 \\ * & * & * & \Phi_{44}^k \end{bmatrix}$, $\tilde{D}^k = [A^k - I \ A_\tau^k \ 0 \ 0]$, $\tilde{A}^k = [A^k \ A_\tau^k \ 0 \ 0]$, $\tilde{\Phi}_{11}^k = \bar{\beta}(-\wp^k + Q_1^k + Q_2^k + Q_3^k + r_\tau^{(1)} Q_2^k) + \bar{\beta}_L\left(-W_1^k - W_2^k\right)$, and rest of the parameters are same as those in Theorem 1. Then system (32) is exponentially stable for any switching signals with the average dwell time satisfying (7), and the following inequalities $\forall k, l \in \mathbb{Z}[1,N]$:

$$\begin{aligned} &\wp^k \le \lambda \wp^l, \ Q_1^k \le \lambda Q_1^l, \ Q_2^k \le \lambda Q_2^l, \ Q_3^k \le \lambda Q_3^l, \\ &W_1^k \le \lambda W_1^l, \ W_2^k \le \lambda W_2^l, \ W_3^k \le \lambda W_3^l. \end{aligned} \tag{34}$$

Proof: The proof of Corollary 3 is similar to the proof of Theorem 1 and therefore the detailed proof is omitted. □

Theorem 2: For some given positive constants τ_{hL}, τ_{vL}, τ_{hU}, τ_{vU}, d_{hL}, d_{vL}, d_{hU}, and d_{vU}, if there exist symmetric positive definite matrices $\wp^k = diag\{\wp^{kh}, \wp^{kv}\}$, $Q_1^k = diag\{Q_1^{kh}, Q_1^{kv}\}$, $Q_2^k = diag\{Q_2^{kh}, Q_2^{kv}\}$, $Q_3^k = diag\{Q_3^{kh}, Q_3^{kv}\}$, $R_1^k = diag\{R_1^{kh}, R_1^{kv}\}$, $R_2^k = diag\{R_2^{kh}, R_2^{kv}\}$, $W_1^k = diag\{W_1^{kh}, W_1^{kv}\}$, $W_2^k = diag\{W_2^{kh}, W_2^{kv}\}$, $W_3^k = diag\{W_3^{kh}, W_3^{kv}\}$, with appropriate dimensions and $0 < \beta < 1$, such that with $k \in \mathbb{Z}[1,N]$, the following inequality holds $\forall k$, $l \in \mathbb{Z}[1,N]$:

$$\Pi_2 = \begin{bmatrix} \Phi_1^k & A_1^{kT}\wp^k & D_1^{kT}\Xi & \Phi_2^{kT} \\ * & -\wp^k & 0 & 0 \\ * & * & -\Psi & 0 \\ * & * & * & -I \end{bmatrix} < 0, \tag{35}$$

where $\Phi_1^k = diag\{\Phi^k, -\gamma^2 I\}$, $A_1^k = [A^k \ A_\tau^k \ 0 \ 0 \ A_d^k \ B^k]$, $D_1^k = [A^k - I \ A_\tau^k \ 0 \ 0 \ A_d^k \ B^k]$, $\Phi_2^k = [D^k \ D_\tau^k \ 0 \ 0 \ 0 \ E^k]$ and rest of the parameters are similar to the ones defined in Theorem 1. Then, system (1) with $u_{\iota,\kappa} = 0$ and any non-zero disturbance satisfying $w_{\iota,\kappa} \in l_2\{[0,\infty), [0,\infty)\}$, has a desired H_∞ disturbance attenuation level γ for any switching signals with the average dwell time satisfying (7) and (8).

Proof: Now, we attempt to establish the H_∞ performance analysis for the system (1), when $u_{\iota,\kappa} = 0$ and any non-zero disturbance satisfying $w_{\iota,\kappa} \in l_2\{[0,\infty),[0,\infty)\}$. By keeping in mind (9), let us consider $\psi_{x(\iota,\kappa)} = \Delta V_{x(\iota,\kappa)}^{kh} + \Delta V_{x(\iota,\kappa)}^{kv} + z_{\iota,\kappa}^T z_{\iota,\kappa} - \gamma^2 w_{\iota,\kappa}^T w_{\iota,\kappa}, \forall k \in \mathbb{Z}[1,N]$. Now, following the proof line of Theorem 1, we obtain

$$\psi_{x(\iota,\kappa)} \le \tilde{\Gamma}_{\iota,\kappa}^T \Pi_2 \tilde{\Gamma}_{\iota,\kappa},$$

where $\Pi_2 = A_1^{kT}\wp^k A_1^k + D_1^{kT}(\Xi_1 + \Xi_2 + \Xi_3)D_1^k + \Phi_1^k + \Phi_2^{kT}\Phi_2^k$, $\tilde{\Gamma}_{\iota,\kappa} = [x_{\iota,\kappa}^T \ x_{\tau(\iota,\kappa)}^T \ x_{\tau L}^T \ x_{\tau U}^T \ x_{d(\iota,\kappa)}^T \ w_{\iota,\kappa}^T]^T$ and by the virtue of Schur complement (35) implies $\Pi_2 < 0$, which means

$$V_{x(\iota+1,\kappa)}^{kh} - \beta V_{x(\iota,\kappa)}^{kh} + V_{x(\iota,\kappa+1)}^{kv} - \beta V_{x(\iota,\kappa)}^{kv} + z_{\iota,\kappa}^T z_{\iota,\kappa}$$

$$-\gamma^2 w_{\iota,\kappa}^T w_{\iota,\kappa} < 0. \tag{36}$$

We set $\Lambda_{\iota,\kappa} = z_{\iota,\kappa}^T z_{\iota,\kappa} - \gamma^2 w_{\iota,\kappa}^T w_{\iota,\kappa}$, then (36) becomes $V_{x(\iota+1,\kappa)}^{kh} + V_{x(\iota,\kappa+1)}^{kv} < \beta\left(V_{x(\iota,\kappa)}^{kh} + V_{x(\iota,\kappa)}^{kv}\right) - \Lambda_{\iota,\kappa}$, which can be stated as follows under the zero boundary conditions:

$$\begin{aligned}
&\sum_{\iota+\kappa=\Gamma} V_{x(\iota,\kappa)}^{\sigma(m_\rho)} \\
&< \beta \sum_{\iota+\kappa=\Gamma-1} V_{x(\iota,\kappa)}^{\sigma(m_\rho)} - \sum_{\iota+\kappa=\Gamma-2} \Lambda_{\iota,\kappa} \\
&< \beta^{\Gamma-m_\rho} \sum_{\iota+\kappa=m_\rho} V_{x(\iota,\kappa)}^{\sigma(m_\rho)} - \sum_{m=m_\rho-1}^{\Gamma-2} \sum_{\iota+\kappa=m} \beta^{\Gamma-2-m} \Lambda_{\iota,\kappa} \\
&< \lambda \beta^{\Gamma-(m_\rho-1)} \sum_{\iota+\kappa=m_\rho-1} V_{x(\iota,\kappa)}^{\sigma(m_{\rho-1})} \\
&\quad - \lambda \beta^{\Gamma-m_\rho} \sum_{\iota+\kappa=m_\rho-2} \Lambda_{\iota,\kappa} - \sum_{m=m_\rho-1}^{\Gamma-2} \sum_{\iota+\kappa=m} \beta^{\Gamma-2-m} \Lambda_{\iota,\kappa} \\
&= \sum_{\iota+\kappa=m_\rho-1} \lambda^{N_{\sigma(\iota+\kappa,\Gamma)}} \beta^{\Gamma-(m_\rho-1)} V_{x(\iota,\kappa)}^{\sigma(m_{\rho-1})} \\
&\quad - \sum_{m=m_\rho-2}^{\Gamma-2} \sum_{\iota+\kappa=m} \lambda^{N_{\sigma(\iota+\kappa+1,\Gamma)}} \beta^{\Gamma-2-m} \Lambda_{\iota,\kappa} \\
&< \sum_{\iota+\kappa=m_\rho-1} \lambda^{N_{\sigma(\iota+\kappa,\Gamma)}} \beta^{\Gamma-(m_\rho-1)} V_{x(\iota,\kappa)}^{\sigma(m_{\rho-1})} \\
&\quad - \sum_{m=m_\rho-2}^{\Gamma-2} \sum_{\iota+\kappa=m} \lambda^{N_{\sigma(\iota+\kappa+1,\Gamma)}} \beta^{\Gamma-2-m} \Lambda_{\iota,\kappa} \\
&< \sum_{\iota+\kappa=m_{\rho-1}} \lambda^{N_{\sigma(\iota+\kappa,\Gamma)}} \beta^{\Gamma-m_{\rho-1}} V_{x(\iota,\kappa)}^{\sigma(m_{\rho-1})} \\
&\quad - \sum_{m=m_{\rho-1}-1}^{\Gamma-2} \sum_{\iota+\kappa=m} \lambda^{N_{\sigma(\iota+\kappa+1,\Gamma)}} \beta^{\Gamma-2-m} \Lambda_{\iota,\kappa} \\
&< \dots < \sum_{\iota+\kappa=1} \lambda^{N_{\sigma(\iota+\kappa,\Gamma)}} \beta^{\Gamma-1} V_{x(\iota,\kappa)}^{\sigma(1)} \\
&\quad - \sum_{m=0}^{\Gamma-2} \sum_{\iota+\kappa=m} \lambda^{N_{\sigma(\iota+\kappa+1,\Gamma)}} \beta^{\Gamma-2-m} \Lambda_{\iota,\kappa}.
\end{aligned}$$

Under the zero initial condition, it holds that $\sum_{\iota+\kappa=1} \lambda^{N_{\sigma(\iota+\kappa,\Gamma)}} \beta^{\Gamma-1} V_{x(\iota,\kappa)}^{\sigma(1)} = 0$. Therefore, we have $\sum_{m=0}^{\Gamma-2} \sum_{\iota+\kappa=m} \lambda^{N_{\sigma(\iota+\kappa+1,\Gamma)}} \beta^{\Gamma-2-m} \Lambda_{\iota,\kappa} < 0$. That implies

$$\begin{aligned}
&\sum_{m=0}^{\Gamma-2} \sum_{\iota+\kappa=m} \lambda^{N_{\sigma(\iota+\kappa+1,\Gamma)}} \beta^{\Gamma-2-m} \|z_{\iota,\kappa}\|_2^2 \\
&< \sum_{m=0}^{\Gamma-2} \sum_{\iota+\kappa=m} \lambda^{N_{\sigma(\iota+\kappa+1,\Gamma)}} \beta^{\Gamma-2-m} \gamma^2 \|w_{\iota,\kappa}\|_2^2.
\end{aligned} \tag{37}$$

By multiplication of $\lambda^{-N_\sigma(1,\Gamma)}$ on both sides of (37), we obtain $\sum_{m=0}^{\Gamma-2} \sum_{\iota+\kappa=m} \lambda^{-N_{\sigma(1,\iota+\kappa+1)}} \beta^{\Gamma-2-m} \|z_{\iota,\kappa}\|_2^2 < \sum_{m=0}^{\Gamma-2} \sum_{\iota+\kappa=m} \lambda^{-N_{\sigma(1,\iota+\kappa+1)}} \beta^{\Gamma-2-m} \gamma^2 \|w_{\iota,\kappa}\|_2^2$. On the other hand, from (4) it follows $N_{\sigma(\iota,\kappa)(1,\iota+\kappa+1)} \leq \frac{\iota+\kappa}{t_a}$. Therefore, we obtain $\lambda^{-N_{\sigma(\iota,\kappa)}(1,\iota+\kappa+1)} = e^{-N_{\sigma(\iota,\kappa)}(1,\iota+\kappa+1)\ln\lambda} \geq e^{-\frac{\iota+\kappa}{t_a}\ln\lambda} \geq e^{(\iota+\kappa)\ln\beta}$. Thus, $\sum_{m=0}^{\Gamma-2} \sum_{\iota+\kappa=m} e^{(\iota+\kappa)\ln\beta} \beta^{\Gamma-2-m} \|z_{\iota,\kappa}\|_2^2 < \gamma^2 \sum_{m=0}^{\Gamma-2} \sum_{\iota+\kappa=m} \lambda^{-N_{\sigma(1,\iota+\kappa+1)}\beta^{\Gamma-2-m}\|w_{\iota,\kappa}\|_2^2}$

$$\Rightarrow \sum_{m=0}^{\Gamma-2} \sum_{\iota+\kappa=m} \beta^{\Gamma-2} \|z_{\iota,\kappa}\|_2^2 < \gamma^2 \sum_{m=0}^{\Gamma-2} \sum_{\iota+\kappa=m} \beta^{\Gamma-2-m} \|w_{\iota,\kappa}\|_2^2.$$

For Γ runs from 2 to ∞, we have $\sum_{\Gamma=2}^{\infty} \sum_{m=0}^{\Gamma-2} \sum_{\iota+\kappa=m} \beta^{\Gamma-2} \|z_{\iota,\kappa}\|_2^2 < \gamma^2 \sum_{\Gamma=2}^{\infty} \sum_{m=0}^{\Gamma-2} \sum_{\iota+\kappa=m} \beta^{\Gamma-2-m} \|w_{\iota,\kappa}\|_2^2$

$$\begin{aligned}
&\Longrightarrow \sum_{m=0}^{\infty} \sum_{\iota+\kappa=m} \beta^{\iota+\kappa} \|z_{\iota,\kappa}\|_2^2 \sum_{\Gamma=2+m}^{\infty} \beta^{\Gamma-2-m} \\
&\qquad < \gamma^2 \sum_{m=0}^{\infty} \sum_{\iota+\kappa=m} \|w_{\iota,\kappa}\|_2^2 \sum_{\Gamma=2+m}^{\infty} \beta^{\Gamma-2-m} \\
&\Longrightarrow \frac{1}{1-\beta} \sum_{m=0}^{\infty} \sum_{\iota+\kappa=m} \|z_{\iota,\kappa}\|_2^2 \\
&\qquad < \frac{1}{1-\beta} \gamma^2 \sum_{m=0}^{\infty} \sum_{\iota+\kappa=m} \|w_{\iota,\kappa}\|_2^2. \\
&\Longrightarrow \sum_{m=0}^{\infty} \sum_{\iota+\kappa=m} \beta^{\iota+\kappa} \|z_{\iota,\kappa}\|_2^2 < \gamma^2 \sum_{m=0}^{\infty} \sum_{\iota+\kappa=m} \|w_{\iota,\kappa}\|_2^2.
\end{aligned}$$

That is $\sum_{\iota=0}^{\infty} \sum_{\kappa=0}^{\infty} \beta^{\iota+\kappa} \|z_{\iota,\kappa}\|_2^2 < \gamma^2 \sum_{\iota=0}^{\infty} \sum_{\kappa=0}^{\infty} \|w_{\iota,\kappa}\|_2^2$, which by Definition 3 implies that system (1) with $u_{\iota,\kappa} = 0$ is exponentially stable and has a prescribed H_∞ disturbance attenuation level γ. The proof is completed. □

3.2. Controller design

Let us consider system (1) under the state feedback controller $u_{\iota,\kappa} = K^{\sigma(\iota,\kappa)} x_{\iota,\kappa}$, which results in the following system:

$$\begin{aligned}
x_{\iota+1,\kappa+1} &= \left(A^{\sigma(\iota,\kappa)} + C^{\sigma(\iota,\kappa)} K^{\sigma(\iota,\kappa)}\right) x_{\iota,\kappa} \\
&\quad + A_\tau^{\sigma(\iota,\kappa)} x_{\tau(\iota,\kappa)} + A_d^{\sigma(\iota,\kappa)} x_{d(\iota,\kappa)} + B^{\sigma(\iota,\kappa)} w_{\iota,\kappa}, \\
z_{\iota,\kappa} &= D^{\sigma(\iota,\kappa)} x_{\iota,\kappa} + D_\tau^{\sigma(\iota,\kappa)} x_{\tau(\iota,\kappa)} + E^{\sigma(\iota,\kappa)} w_{\iota,\kappa}.
\end{aligned} \tag{38}$$

Now, we present some sufficient inequality based conditions in the following theorem for the existence of a state feedback controller $u_{\iota,\kappa} = K^{\sigma(\iota,\kappa)} x_{\iota,\kappa}$, such that the closed-loop system (38) is exponentially stable.

Theorem 3: For some given positive constants τ_{hL}, τ_{vL}, τ_{hU}, τ_{vU}, d_{hL}, d_{vL}, d_{hU}, and d_{vU}, if there exist symmetric positive definite matrices $\wp^k = diag\{\wp^{kh}, \wp^{kv}\}$, $Q_1^k = diag\{Q_1^{kh}, Q_1^{kv}\}$, $Q_2^k = diag\{Q_2^{kh}, Q_2^{kv}\}$, $Q_3^k = diag\{Q_3^{kh}, Q_3^{kv}\}$, $R_1^k = diag\{R_1^{kh}, R_1^{kv}\}$, $R_2^k = diag\{R_2^{kh}, R_2^{kv}\}$, $W_1^k = diag\{W_1^{kh}, W_1^{kv}\}$, $W_2^k = diag\{W_2^{kh}, W_2^{kv}\}$, $W_3^k = diag\{W_3^{kh},$

$W_3^{kv}\}$, and any matrices K^k, for all $k \in \mathbb{Z}[1,N]$, with appropriate dimensions and $0 < \beta < 1$, such that the following inequality holds $\forall k,\, l \in \mathbb{Z}[1,N]$:

$$\widehat{\Pi}_2 = \begin{bmatrix} \Phi_1^k & A_2^{kT} & D_2^{kT} & D_2^{kT} & D_2^{kT} & \Phi_2^{kT} \\ * & \Psi_{\wp} & 0 & 0 & 0 & 0 \\ * & * & \Psi_{\Xi_1} & 0 & 0 & 0 \\ * & * & * & \Psi_{\Xi_2} & 0 & 0 \\ * & * & * & * & \Psi_{\Xi_3} & 0 \\ * & * & * & * & * & -I \end{bmatrix} < 0, \tag{39}$$

where $A_2^k = [(A^k + C^k K^k) \;\; A_\tau^k \;\; 0 \;\; 0 \;\; A_d^k \;\; B^k]$, $D_2^k = [(A^k + C^k K^k) - I \; A_\tau^k \; 0 \; 0 \; A_d^k \; B^k]$, $\Psi_{\wp} = J_1^T \wp^k J_1 - 2J_1$, $\Psi_{\Xi_1} = J_2^T \Xi_1 J_2 - 2J_2$, $\Psi_{\Xi_2} = J_3^T \Xi_2 J_3 - 2J_3$, $\Psi_{\Xi 3} = J_4^T \Xi_3 J_4 - 2J_4$, and rest of the parameters are already defined in Theorem 1 and Theorem 2. Then, the closed-loop system (38) has a specified H_∞ disturbance attenuation level γ for any switching signal with average dwell time satisfying (7) and (8).

Proof: Replacing A^k in (35) by $A^k + C^k K^k$ and pre and post multiplying by $diag\{I,\, I,\, I,\, I,\, I,\, I,\, (\wp^k)^{-1},\, (\Xi_1)^{-1}$ $(\Xi_2)^{-1},\, (\Xi_3)^{-1},\, I\}$, results:

$$\tilde{\Pi}_2 = \begin{bmatrix} \Phi_1^k & A_2^{kT} & D_2^{kT} & D_2^{kT} & D_2^{kT} & \Phi_2^{kT} \\ * & \Phi_{\wp}^k & 0 & 0 & 0 & 0 \\ * & * & \Phi_{\Xi_1} & 0 & 0 & 0 \\ * & * & * & \Phi_{\Xi_2} & 0 & 0 \\ * & * & * & * & \Phi_{\Xi_3} & 0 \\ * & * & * & * & * & -I \end{bmatrix} < 0, \tag{40}$$

where $\Phi_{\wp}^k = -\left(\wp^k\right)^{-1}$, $\Phi_{\Xi_1} = -(\Xi_1)^{-1}$, $\Phi_{\Xi_2} = -(\Xi_2)^{-1}$, and $\Phi_{\Xi_3} = -(\Xi_3)^{-1}$. The following can be obtained for any matrices $J_1 > 0$, $J_2 > 0$, $J_3 > 0$ and $J_4 > 0$; $J_1^T \wp^k J_1 \geq 2J_1 + \Phi_{\wp}^k$, $J_2^T \Xi_1 J_2 \geq 2J_2 + \Phi_{\Xi_1}$, $J_3^T \Xi_2 J_3 \geq 2J_3 + \Phi_{\Xi_2}$, and $J_4^T \Xi_3 J_4 \geq 2J_4 + \Phi_{\Xi_3}$. Then, (39) holds if (40) is satisfied. This completes our proof. □

In what follows, we present a procedure that may prove helpful for finding the controller gains from Theorem 3.

Step 1: Input the matrices $A^{\sigma(\iota,\kappa)}$, $A_\tau^{\sigma(\iota,\kappa)}$, $A_d^{\sigma(\iota,\kappa)}$, $B^{\sigma(\iota,\kappa)}$, $C^{\sigma(\iota,\kappa)}$, $D^{\sigma(\iota,\kappa)}$, $D_\tau^{\sigma(\iota,\kappa)}$, and $E^{\sigma(\iota,\kappa)}$.

Step 2: Choose the appropriate parameters τ_{hL}, τ_{vL}, τ_{hU}, τ_{vU}, d_{hL}, d_{vL}, d_{hU}, d_{vU}, γ, then by solving the inequalitlity (39) in Theorem 3 by LMI toolbox in MATLAB, one may obtain controller gains.

Remark 4: To understand the fruitfulness of the proposed results, we present a discussion of relevant studies here. For instance, the results put forward in [33] have solved the similar problem for the 2-D switched system without delays by the common and the multiple Lyapunov functional approaches. By considering the time-varying delays, authors in [34,35] presented a solution of the similar problem by utilizing the multiple Lyapunov functional approaches. In [40], average dwell time approach was used to solve the similar problem by considering the constant time delays in the system states. In comparison to the results proposed in the studies [33–35, 40], the choice of mixed type of time delays, the improved LKF and dwell-time approach makes our results more general and different from the ones stated above. Moreover, due to the reasons stated in Remarks 2-3, the choice of an improved LKF would help in achieving less conservativeness without significantly increasing the computational burden as compared to LKF considered in [38].

Remark 5: It should be stated that the results presented in Theorem 3 may also bring some computational complexities which are evident because we need to solve $10N$ matrix inequalities to obtain $19N+1$ matrix variables.

4. AN ILLUSTRATIVE EXAMPLE

This section validates the usefulness of the proposed results with the help of an example. In practice, the Darboux partial differential equation is very commonly used in the modeling of thermal processes in pipe furnaces, chemical reactors and heat exchangers [2]. A typical representation of such operation is shown in Fig. 1. Here, we also consider a similar thermal process with multiple subsystems which can be modeled by the following partial differential equation:

$$\begin{aligned} \frac{\partial T_{s,t}}{\partial s} = &-\frac{\partial T_{s,t}}{\partial t} + a_0^{\sigma(s,t)} T_{s,t} + a_1^{\sigma(s,t)} T_{s-\tau_s,t} + a_2^{\sigma(s,t)} T_{s,t-\tau_t} \\ &+ b_1^{\sigma(s,t)} \int_0^{s_f} T_{s-x,t}\,dx + b_2^{\sigma(s,t)} \int_0^{t_f} T_{s,t-\theta}\,d\theta \\ &+ c_1^{\sigma(s,t)} w_{s,t} + c_2^{\sigma(s,t)} u_{s,t}, \end{aligned} \tag{41}$$

where $T_{s,t}$ represents an unknown function (e.g temperature) at $s \in [0,l]$ (space) and $t \in [0,\infty)$ (time); $w_{s,t}$ and $u_{s,t}$ describe the L_2-norm bounded disturbance and the control input, respectively. $a_0^{\sigma(s,t)}$, $a_1^{\sigma(s,t)}$, $a_2^{\sigma(s,t)}$, $b_1^{\sigma(s,t)}$, $b_2^{\sigma(s,t)}$, $c_1^{\sigma(s,t)}$, $c_2^{\sigma(s,t)}$ are real numbers with $\sigma(s,t)$ representing the active subsystem at (ι,κ); τ_s, s_f, and τ_t, t_f

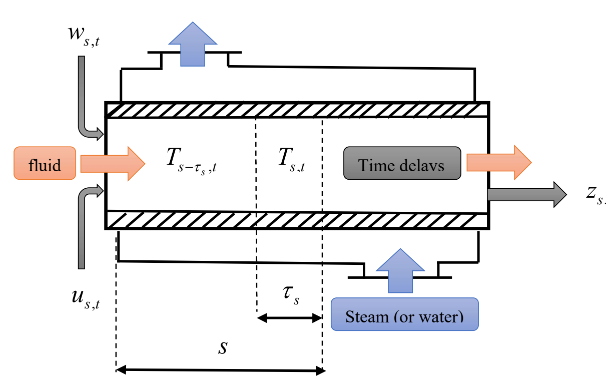

Fig. 1. Heat Exchanger [2, 41].

denote the delays in space and time, respectively. Moreover, $z_{s,t}$ in the Fig. 1 represents the output of the system. Given the increments Δs, Δt, we denote $T_{\iota,\kappa} = T_{\iota\Delta s,\kappa\Delta t}$, $w_{\iota,\kappa} = w_{\iota\Delta s,\kappa\Delta t}$, $u_{\iota,\kappa} = u_{\iota\Delta s,\kappa\Delta t}$, $\sigma(\iota,\kappa) = \sigma(\iota\Delta s, \kappa\Delta t)$ and use the approximation $\frac{\partial T_{s,t}}{\partial s} \simeq \frac{T_{\iota,\kappa}-T_{\iota-1,\kappa}}{\Delta s}$, $\frac{\partial T_{s,t}}{\partial t} \simeq \frac{T_{\iota,\kappa}-T_{\iota,\kappa-1}}{\Delta t}$ to compute the derivatives $\frac{\partial T_{s,t}}{\partial s}$ and $\frac{\partial T_{s,t}}{\partial t}$, respectively. We define $x^h_{\iota,\kappa} = T_{\iota-1,\kappa}$, $x^v_{\iota,\kappa} = T_{\iota,\kappa}$, then the following system can be obtained as the result of corresponding discretization:

$$\begin{aligned} x_{\iota+1,\kappa+1} =& A^{\sigma(\iota,\kappa)} x_{\iota,\kappa} + A_\tau^{\sigma(\iota,\kappa)} x_{\tau(\iota,k)} \\ &+ A_d^{\sigma(\iota,\kappa)} \begin{bmatrix} \sum_{s=1}^{d_h} x^h_{\iota-s,\kappa} \\ \sum_{r=1}^{d_v} x^v_{\iota,\kappa-r} \end{bmatrix} + B^{\sigma(\iota,\kappa)} w_{\iota,\kappa} \\ &+ C^{\sigma(\iota,\kappa)} u_{\iota,\kappa}, \end{aligned} \tag{42}$$

where $\iota,\kappa \in \mathbb{Z}^+$, $\tau_h(\iota) = [\tau_s/\Delta s]$, $\tau_v(\kappa) = [\tau_l/\Delta t]$, $d_h = [s_f/\Delta s]$, and $d_v = [t_f/\Delta t]$. System (42) is similar to system (1), so just for illustrative purpose let us consider the system (1) with following parameters:

Subsystem 1: $A^1 = \begin{bmatrix} 0.02 & -0.03 \\ 0 & -0.01 \end{bmatrix}$, $A^1_\tau = \begin{bmatrix} -0.3 & -0.2 \\ 0 & 0.2 \end{bmatrix}$, $A^1_d = \begin{bmatrix} -0.1 & 0 \\ 0 & -0.02 \end{bmatrix}$, $B^1 = \begin{bmatrix} 0.04 \\ 0.06 \end{bmatrix}$, $C^1 = \begin{bmatrix} 0.08 \\ 0 \end{bmatrix}$, $D^1 = \begin{bmatrix} -0.1 & 0.06 \end{bmatrix}$, $D^1_\tau = \begin{bmatrix} 0.11 & 0.02 \end{bmatrix}$, $E^1 = 1$.

Subsystem 2: $A^2 = \begin{bmatrix} 0.01 & 0.5 \\ 0 & -0.01 \end{bmatrix}$, $A^2_\tau = \begin{bmatrix} 0.21 & 0.6 \\ 0 & 0.4 \end{bmatrix}$, $A^2_d = \begin{bmatrix} 0.09 & 0.01 \\ 0 & 0.01 \end{bmatrix}$, $B^2 = \begin{bmatrix} 0.02 \\ 0.01 \end{bmatrix}$, $C^2 = \begin{bmatrix} 0.04 \\ 0.06 \end{bmatrix}$, $D^2 = \begin{bmatrix} -0.01 & -0.1 \end{bmatrix}$, $D^2_\tau = \begin{bmatrix} -0.2 & 0.1 \end{bmatrix}$, $E^2 = -0.01$.

Now, to prove the less conservativeness of the proposed results, this section is furnished with Table 1, which estimates the maximum upper bound of the delay and compares the results of Theorem 1, Corollary 1 and Corollary 2 both in terms of conservativeness and computational burden. Keeping in view Remarks 2-3, Theorem 1 presented in Table 1 suggests less conservativeness and relatively high computational effort as compared to Corollary 1. However, Corollary 2 allows us to achieve less conservativeness as compared to Corollary 1 without increasing the computational burden. Therefore, it can be concluded that the improved LKF based proposed results in Theorem 1 and Corollary 2 are better in terms of conservativeness as compared to the LKF chosen in [38]. Moreover, for system parameters stated in our example, the results established in [11, 12] cannot guarantee the feasible solution to the inequalities stated in Theorem 1 due to the presence of mixed delays while Corollary 3 in our paper can be employed to study the switched systems of the form considered in [11, 12].

In order to observe the usefulness of the results proposed in Theorem 3, we consider the same system parameters

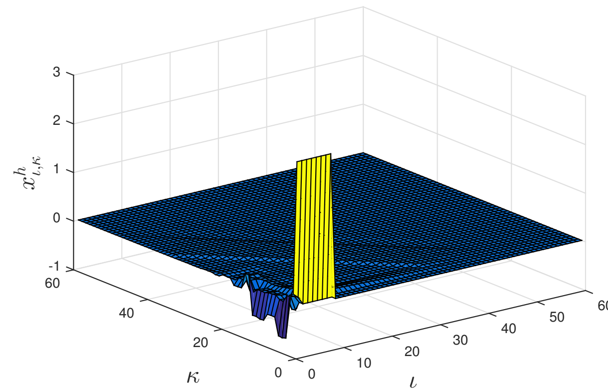

Fig. 2. The state trajectory of horizontal state.

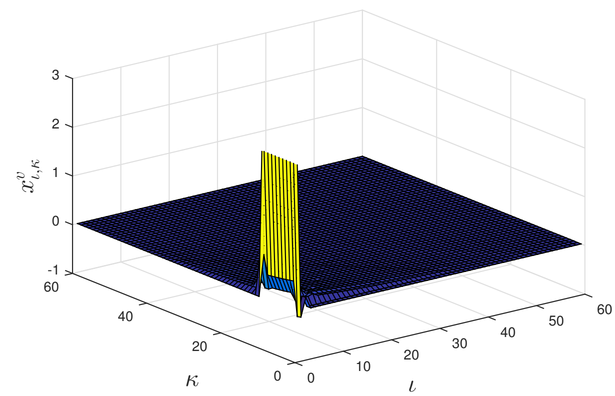

Fig. 3. The state trajectory of vertical state.

and take $\tau_{hL} = \tau_{vL} = 1$, $\tau_{hU} = \tau_{vU} = 2$, $d_{hL} = d_{vL} = 1$, $d_{hU} = d_{vU} = 2$, $\beta = 0.87$, $\gamma = 3$, $J_1 = diag\{0.11, 0.001\}$, $J_2 = diag\{14, 7\}$, $J_3 = diag\{10, 10\}$ and $J_4 = diag\{15, 10\}$. Then, by solving the inequalities in Theorem 3 we may obtain controller gains $K^1 = \begin{bmatrix} -0.0661 & 0.4749 \end{bmatrix}$, and $K^2 = \begin{bmatrix} -0.5812 & 0.4371 \end{bmatrix}$. Moreover, from (12)-(13), we can get $\lambda = 509.9796$ and $t^*_a = 44.7672$. Choose $t_a = 45$, $\tau_h(\iota) = d_h(\iota) = 1.5 + 0.5\sin\left(\frac{\pi\iota}{2}\right)$, $\tau_v(\kappa) = d_v(\kappa) = 1.5 + 0.5\sin\left(\frac{\pi\kappa}{2}\right)$, $w_{\iota,\kappa} = 5e^{(-.025\pi(\iota+\kappa))}$ and the boundary conditions $x^h_{\iota,\kappa} = 3$, $\iota \in \mathbb{Z}[-2,0]$, $0 \le \kappa \le 8$; $x^h_{\iota,\kappa} = 0, \kappa > 8$; $x^v_{\iota,\kappa} = 4$, $\kappa \in \mathbb{Z}[-2,0]$, $0 \le \iota \le 10$; $x^v_{\iota,\kappa} = 0, \iota > 10$.
We have plotted the state trajectories of the closed-loop system (38) in Figs. 2-3, and the corresponding switching signal has been shown in Fig. 4. The control input and the controlled output are depicted in Figs. 5-6, respectively. It can be noticed from Figs. 2-3 that the designed controller ensures the exponential stability of the closed-loop system (38) along with a desired H_∞ performance γ, which shows the usefulness of the established results.

5. CONCLUSION

For 2-D systems with the mixed type of state delays, the exponential stability and H_∞ control problems have been

Table 1. Maximum upper bound of delays $\tau_{hL} \leq \tau_h(\iota) \leq \tau_{hU}$, $\tau_{vL} \leq \tau_v(\kappa) \leq \tau_{vU}$, $d_{hL} \leq d_h(\iota) \leq d_{hU}$, $d_{vL} \leq \tau_v(\kappa) \leq d_{vU}$ along with the feasibility of corresponding LMIS for fixed values of $\tau_{hL} = \tau_{vL} = 1$, $d_{hL} = d_{vL} = 1.5$ and $\beta = 0.98$.

									NODV
Theorem 1	$\tau_{hU} = \tau_{vU}$	2	3	4	4.5	4.8	4.9	5	37
	$d_{hU} = d_{vU}$	2	3	4	4.5	5.2	5.3	5.4	
Corollary 1	$\tau_{hU} = \tau_{vU}$	1.4	1.8	2.2	2.6	3	3.1	3.2	29
	$d_{hU} = d_{vU}$	1.8	2.1	2.4	2.7	3.1	3.2	3.3	
Corollary 2	$\tau_{hU} = \tau_{vU}$	1.4	1.8	2.2	2.6	3.2	3.3	3.4	29
	$d_{hU} = d_{vU}$	1.9	2.3	2.7	3.1	3.6	3.7	3.8	
Feasibility		✓	✓	✓	✓	✓	×	×	

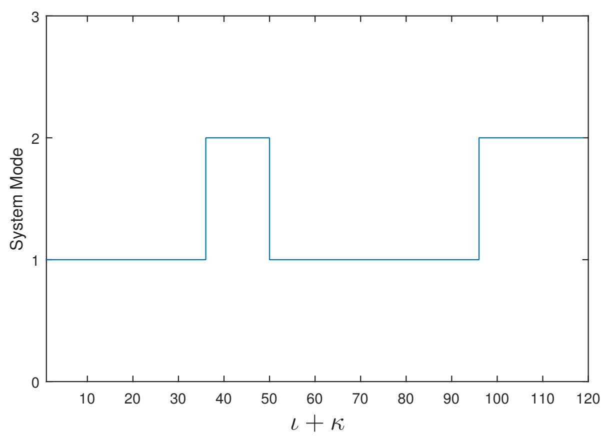

Fig. 4. Switching signal.

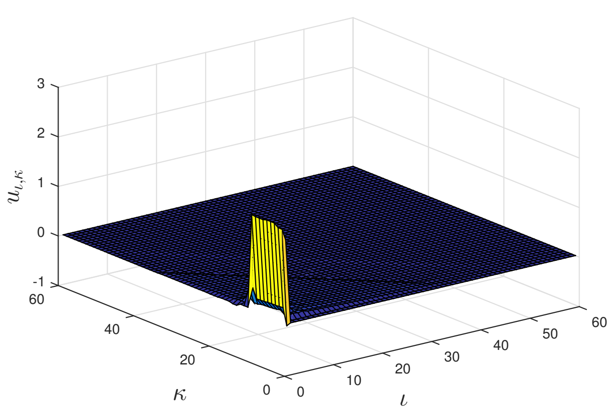

Fig. 5. The control input.

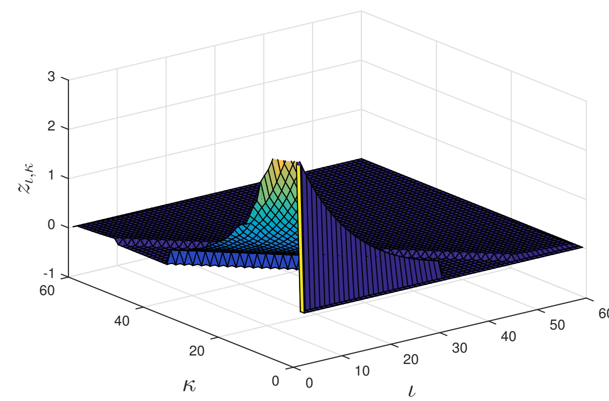

Fig. 6. The controlled output.

solved in this paper. At first, by proposing an improved LKF, some sufficient stability conditions were proposed along with the H_∞ performance analysis. Secondly, a state feedback-based controller method was put forward that promises the exponential stability and the desired H_∞ disturbance attenuation level γ for the system under consideration. Finally, the derived results based on the proposed LKF were compared with the results obtained by using the LKF considered in [38], both in terms of conservativeness and computational burden. It was shown that proposed LKF degenerates better results. One possible future extension of the derived results could be to 2-D continuous systems, for which this problem still remains unsolved. The output feedback controller design for 2-D continuous systems in presence of mixed delays could also be another interesting future problem.

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