International Journal of Control, Automation and Systems 18(4) (2020) 1045-1052
http://dx.doi.org/10.1007/s12555-019-0094-5
ISSN:1598-6446 eISSN:2005-4092
http://www.springer.com/12555

A PD-type Iterative Learning Control Algorithm for One-dimension Linear Wave Equation

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Abstract: Many applications can be described by the wave equation, as a kind of important second order partial differential equations. This paper suggests applying PD-type iterative learning control (ILC) scheme with initial state learning (ISL) to a class of linear one-dimensional wave equation. A sufficient condition is given to insure the convergence of the tracking errors. Finally, a numerical simulation is presented to illustrate the efficiency of the proposed method.

Keywords: Control of Wave equation, initial learning state, iterative learning control, PD-learning scheme.

1. INTRODUCTION

Partial Differential Equations (PDE) are found in many engineering applications [1]. As one of the most important second order PDE, the wave equation has been used in different domain, such as physics, biology and engineering [1–5], where it is commonly used to define waves, i.e., the transmission of electric signals in a cable, the vibrations of a string, water waves, the propagation of electromagnetic and sound waves [6]. Many efforts had been made toward to control of wave equations [7–9]. However, due to uncertainties causing from engineering applications, e.g. measuring errors and/or unknown disturbances, it is difficult to find a precise mathematical model of the plant. Therefore, some control methods with weakly dependent-model or model free are presented in the literature. The Iterative Learning Control (ILC) is becoming an alternative method to solve this problem because of its weak model-dependence. A better control system performance can be obtained by repeating the action and learning trail by trail. The ILC was proposed for the first time by Arimoto in 1984 [10]. In recent years, the ILC has become a hot area and has attracted broad attention. This control method was proved to be a more efficient method to get the optimal control problem based on differential equations [11–22]. However, a few contributions on the ILC for partial differential equation can be found in [23–27]. By using the semi-group theory, the P-type and D-type ILC were considered for the parabolic PDEs [23–25]. In [23], a D-type anticipatory iterative learning control scheme is proposed to solve the problem of the boundary control of inhomogeneous heat equations, in which the PDE system is transformed into its integral form and thus the direct input-output relationship is utilized for convergence analysis. A P-type ILC is applied to a class of discrete parabolic distributed parameter systems described by partial differential equations in [24]. A discrete D-type ILC algorithm was proposed for a system governed by a parabolic partial differential equation in [25] where the systems have no direct channel between the input and the output. The ILC was also considered for the hyperbolic PDEs. In [26], the authors study the application of the iterative learning control for a class of mixed hyperbolic-parabolic distributed parameter systems. Further, the application of the P-type iterative learning control to the wave equation in case of boundary control was presented in [27]. The PD controller, a particular kind of common PID controller with an open-loop feedback mechanism broadly used in industrial control systems and a variety of other applications requiring continuously a modulated control to design iterative learning controller. As a combination between the D-type and the P-type, so his learning functions consist of a proportional and derivative gain on the error, also, the PD-type controller has the advantage of both types. The PD-type learning function and its various variations are arguably the widest design techniques and commonly used in the practical system and non-linear systems, see [28] and [29]. These learning functions rely on tuning law, not requiring an accurate model for implementation. There are a few contributions to apply the

Manuscript received February 9, 2019; revised June 16, 2019 and August 8,2019; accepted September 2, 2019. Recommended by Associate Editor Guangdeng Zong under the direction of Editor Guang-Hong Yang. This work is partly financial supported by the High-tech Research and Development Program of China (2014AA041802).

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PD-type iterative learning control to the partial differential equations. In [30], a combined ILC and a proportional control algorithm are proposed to suppress the unknown periodic speed variation of a stretched string system. Similarly, a PD-controller is provided with an ILC algorithm in [31] to handle the periodic uncertainties of an axially moving material system. In [32], the PD-type ILC algorithm was proven to be feasible and effective when it is imposed on the linear switched systems with arbitrarily switching rules. In [33], the PD-type iterative learning control scheme is applied to a class of affine non-linear time-delay systems with external disturbances. In [34], the PD-type iterative learning control scheme was used to study the state tracking for a class of discrete singular systems. The mutual assumption in the ILC is that the initial states in every repetition must be equal or the same as the desired output, this assumption is essential for the stability analysis, some efforts have been made to change this assumption and to eliminate the effect of the initialization errors, one of the methods proposed is using the initial state learning to overcome this problem. Motivated by the above assertion, in this paper, a PD-type open-loop iterative learning scheme with initial state learning is proposed for the one-dimension wave equation, and the proof of the method converges in L^2 space in sense of λ-norm is also given. The main contributions of this work are:

- The system considered in this paper is a system governed by a one-dimensional wave equation.
- The control method proposed in this work is a PD-type ILC, the convergence theorem of the error in the iteration domain is given.
- Instead of considering the boundary control as in [27], in this paper we consider the distributed control of a system described by a one-dimensional wave equation.
- In this paper, rather than to consider that the initial value conditions are fixed as in [26], the initial state will change according to an iterative learning updating law based on the error.
- To prove the convergence of the method, some basic mathematical tools are used instead of the form of the exact solution.

The remaining parts of this work are organized as below: Section 2 we give a description of our problem. In Section 3 we give the theorem of convergence of the method in L^2 space in sense of λ-norm, the proof of this theorem is also presented in this section. In Section 4, we give a numerical simulation that illustrate the efficacy of the method. Finally, in Section 5, we will present the conclusion of this work. Throughout this paper, for a function $y(x,t):(a,b)\times(0,1)\to\mathbb{R}$, $a,b\in\mathbb{R}_+$, take the norm $||y(\cdot,t)||_{L^2}=\sqrt{\int_a^b y(x,t)^2dx}$, and the λ-norm of the function is given by $\|y\|_\lambda=\max_{t\in(0,1)}(e^{-\lambda t}\|y(\cdot,t)\|_{L^2}^2)$.

2. PROBLEM STATEMENT

Firstly, we give a small description of the linear inhomogeneous wave equation with Neumann boundary conditions and inhomogeneous initial conditions.

$$\begin{cases}\dfrac{\partial^2 y}{\partial t^2}(x;t)-\sum\limits_0^n\dfrac{\partial^2 y}{\partial x_i^2}(x;t)=h(x,t),\\ x=(x_1;\ldots;x_n)\in\Omega;\;\; t\in[0;1]\\ \dfrac{\partial y}{\partial\vec{n}}(x;t)=0;\;\; x\in\partial\Omega;\;\; t\in[0;1],\\ y(x;0)=f(x);\;\; \dfrac{\partial y}{\partial t}(x;0)=g(x);\;\; x\in\Omega.\end{cases}\tag{1}$$

$\Omega\subset\mathbb{R}^n,\vec{n}$ is the normal vector to Ω, $\partial\Omega$ is the boundary of Ω. $h:\Omega\times(0,1)\to\mathbb{R}$, $f,g:\Omega\to\mathbb{R},h,f$ and g are smooth functions.

Remark 1: Many non-linear problems can be linearised, so in this work, we choose to study the application of the iterative learning control to the linear wave equation. Some problems cannot be linearised, we will focus on this in our next work.

Remark 2: The controllability, the existence and the uniqueness of the solution of the problem (1) are given in [35].

Remark 3: In this paper, we will apply the iterative learning control PD-type to the linear wave PDE governed by (1). For convenience we take $n=1$. When $n>1$, the PD-type ILC design would be different, and will be discussed in our future work.

Now the plant (1) is excited by input $u(x,t)$, instead of $h(x,t)$ with an output measurement $z(x,t)$, which is described by the following:

$$\begin{cases}\dfrac{\partial^2 y}{\partial t^2}(x,t)-\dfrac{\partial^2 y}{\partial x^2}(x,t)=u(x,t),\\ x\in[a,b],\;\; t\in[0,1],\\ z(x,t)=Ay(x,t)+B\displaystyle\int_0^t u(x,s)ds,\\ x\in[a,b],\;\; t\in[0,1],\;\; A,\ B>0,\\ \dfrac{\partial y}{\partial x}(x,0)=r(t);\;\; \dfrac{\partial y}{\partial x}(b,t)=v(t);\;\; t\in[0,1],\\ y(x,0)=f(x);\;\; \dfrac{\partial y}{\partial t}(x,0)=g(x);\;\; x\in[a,b].\end{cases}\tag{2}$$

$a,\ b\in\mathbb{R}$, $y(x,t)$ represents the state, $u(x,t)$ represents the control input and $z(x,t)$ represents the measured output.

Remark 4: The form of the output is frequently used in case of D-type ILC, refer to [36]. In this research, A and B are positive constant.

The system (1) is repeatable over $t\in[0,1]$. Then the system (2) can be rewritten at iterations as

$$\begin{cases} \frac{\partial^2 y_k}{\partial t^2}(x,t) - \frac{\partial^2 y_k}{\partial x^2}(x,t) = u_k(x,t), \\ z_k(x,t) = Ay_k(x,t) + B\int_0^t u_k(x,s)ds, \\ x \in [a,b], \ t \in [0,1], \ A, \ B > 0. \end{cases} \quad (3)$$

The PD-type iterative learning control scheme in this paper is considered as follows:

$$\begin{cases} u_{k+1}(x,t) = u_k(x,t) + L_1 e_k(x,t) + L_2 \frac{\partial e_k}{\partial t}(x,t), \\ y_{k+1}(x,0) = y_k(x,0) + L_3 e_k(x,0). \end{cases} \quad (4)$$

$k = 1,2,...$ represent the iteration numbers. $L_1, L_2 0$. L_1 is the the proportional gains and L_2 differential gains. L_1, L_2 and L_3 are the designed parameters and will be given in the sequel. Now the PD-type iterative learning control scheme (4) will be applied to the plant (3), where the controller doesn't depend precise model of (1).
$e_k(x,t)$ is the tracking error and it is defined as follows: $e_k(x,t) = z_d(x,t) - z_k(x,t)$, where $z_d(x,t)$ is the desired output, $z_k(x,t)$ is the system output at the $k^t h$ the iteration, respectively. The subsequent analysis of this paper is based on the following assumptions.

Assumption 1: The plant (2) is referred to realizable, i.e., for the desired output z_d (x,t)there exist $y_d(x,t)$ and $u_d(x,t)$ such that

$$\frac{\partial^2 y_d}{\partial t^2}(x,t) - \frac{\partial^2 y_d}{\partial x^2}(x,t) = u_d(x,t),$$
$$z_d(x,t) = Ay_d(x,t) + B\int_0^t u_d(x,s)ds.$$

Assumption 2: The following boundary conditions hold for all iterations $k = 0,1,2,\ldots$ i.e.,

$$\frac{\partial y_k}{\partial x}(a,t) = r(t), \ \ \frac{\partial y_k}{\partial x}(b,t) = v(t), \ \ t \in [0,1].$$

Remark 5: Assumption 1 means that for the desired output there exist realizable a pair of input and state output such that the plant (2) can reach to the desired system output. Assumption 2 means that every iteration will be kept on boundary.

The main contribution is to construct an iterative learning control law of PD (4) for the one-dimensional linear wave equation (2) or an iterative form (3), and it is proven theoretically that for the desired target z_d the tracking error $e_k(x,t) = z_d(x,t) - z_k(x,t)$ will tend to zero and uk will tend to the desired input ud as k increases by trails. Such a convergence analysis will be made in the sequel.
A preliminary lemma is given as follows.

Lemma 1: suppose that a_k and b_k are non-negative real sequence satisfying, $a_(k+1) \leq \rho a_k + b_k$. If $0 \leq \rho < 1$ and $b_k \to 0$ when $k \to \infty$, then $a_k \to 0$ when $k \to \infty$.

The proof is easy to complete.

3. CONVERGENCE ANALYSIS

Theorem 1: Suppose that Assumptions 1 and 2 are satisfied. If

$$(1 - BL_2)^2 + (AL_2)^2 < \frac{1}{3},$$

and

$$|1 - AL_3| < 1,$$

then the iterative process of the system (3) is convergent, under the effect of the control law, i.e.,

$$||e_k||_\lambda \to 0 \ \text{ when } \ k \to \infty$$

to prove this theorem, we need the following lemma.

Lemma 2: If

$$|1 - AL_3| < 1,$$

then for all and arbitrary initial input $u_0(x,t)$, the open-loop PD-type ILC updating law (4) guarantees that

$$\lim_{k\to\infty} ||e_k||_{L^2} = 0.$$

Proof: $e_{k+1}(x,0) = z_d(x,0) - z_{k+1}(x,0) = e_k(x,0) + A(y_k(x,0) - y_{k+1(x,0)}) = (1 - AL_3)e_k(x,0)$, $||e_{k+1}(\cdot,0)||_{L^2} \leq |1 - AL_3|^2 ||e_k(\cdot,0)||_{L^2}$, According to (5) and Lemma 1 with $b_k = 0$,

$$\lim_{k\to\infty} ||e_k||_{L^2} = 0.$$

This ends the proof of Lemma 2. □

Proof of Theorem 1: To prove Theorem 1, we need to prove that

$$\lim_{k\to\infty} ||e_k||_\lambda = 0.$$

Let $\Delta u_k(x,t) = u_(k+1)(x,t) - u_k(x,t), w_k(x,t) = y_{k+1}(x,t) - y_k(x,t)$,

$$\begin{aligned} \frac{d}{dt}||e_k||_{L^2}^2 &= \frac{d}{dt}\int_a^b e_k(x,t)^2 dx \\ &= \int_a^b \frac{\partial e_k}{\partial t}(x,t) e_k(x,t) dx \\ &\leq || \frac{\partial e_k}{\partial t}(\cdot,t)||_{L^2}^2 + ||e_k(\cdot,t)||_{L^2}^2. \end{aligned} \quad (5)$$

Appling the Gronwall lemma, we get

$$\frac{d}{dt}||e_k||_{L^2}^2 \leq || \frac{\partial e_k}{\partial t}(\cdot,t)||_{L^2}^2 + \int_0^t || \frac{\partial e_k}{\partial t}(\cdot,\mu)||_{L^2}^2 e^{t-\mu} d\mu, \quad (6)$$

$$||e_k(\cdot,t)||_{L^2}^2 + \int_0^t \int_0^s || \frac{\partial e_k}{\partial t}(.,\mu)||_{L^2}^2 e^{s-\mu} d\mu ds$$

$$\leq \int_0^t ||\frac{\partial e_k}{\partial t}(.,\mu)||^2_{L^2} d\mu + ||e_k(\cdot,0)||^2_{L^2}$$
$$\leq ||\frac{\partial e_k}{\partial t}||_\lambda \int_0^t e^{\lambda\mu} d\mu$$
$$+ ||\frac{\partial e_k}{\partial t}||_\lambda \int_0^t \int_0^s e^{\lambda\mu} e^{s-\mu} d\mu ds$$
$$+ ||e_k(\cdot,0)||^2_{L^2} \leq ||\frac{\partial e_k}{\partial t}||_\lambda (\frac{e^{\lambda t}-1}{\lambda}$$
$$- \frac{e^t - 1}{\lambda - 1} - \frac{e^{\lambda t}-1}{\lambda(\lambda-1)}) + ||e_k(\cdot,0)||^2_{L^2}, \tag{7}$$

$$e^{-\lambda t}||e_k(\cdot,t)||^2_{L^2} \leq ||\frac{\partial e_k}{\partial t}||_\lambda \Big(\frac{1-e^{-\lambda t}}{\lambda} - \frac{e^{t-\lambda t} - e^{-\lambda t}}{\lambda - 1} + \frac{1-e^{-\lambda t}}{\lambda(\lambda-1)}\Big) + e^{-\lambda t}||e_k(\cdot,0)||^2_{L^2}, \tag{8}$$

$$||e_k||_\lambda \leq ||\frac{\partial e_k}{\partial t}||_\lambda \max_{t\in(0,1)} (\frac{1-e^{-\lambda t}}{\lambda} - \frac{e^{t-\lambda t} - e^{-\lambda t}}{\lambda - 1} + \frac{1-e^{-\lambda t}}{\lambda(\lambda-1)}) + ||e_k(\cdot,0)||^2_{L^2}, \tag{9}$$

$$||e_k||_\lambda \leq ||\frac{\partial e_k}{\partial t}||_\lambda (\frac{1-e^{-\lambda}}{\lambda} + \frac{1-e^{-\lambda}}{\lambda(\lambda-1)}) + ||e_k(\cdot,0)||^2_{L^2}. \tag{10}$$

To prove that the method converge we must proof that $||\frac{\partial e_k}{\partial t}||_\lambda$, when $k \to \infty$.

$$e_{k+1}(x,t) = z_d(x,t) - z_{k+1}(x,t) e_k(x,t) + A(y_k(x,t) - y_{k+1}(x,t) + B\int_0^t (u_k(x,s) - u_{k+1}(x,s))ds$$
$$= e_k(x,t) - Aw_k(x,t) - B(L_2 e_k(x,t) - L_1 \int_0^t e_k(x,s)ds) + BL_2 e_k(x,0), \tag{11}$$

$$\frac{\partial e_{k+1}}{\partial t}(x,t) = (1 - BL_2)\frac{\partial e_k}{\partial t}(x,t) - A\frac{\partial w_k}{\partial t}(x,t) - BL_1 e_k(x,t). \tag{12}$$

Appling the Young inequality and using the L^2-norm propriety we get

$$||\frac{\partial e_{k+1}}{\partial t}(\cdot,t)||^2_{L^2} \leq 3(1-BL_2)^2 ||\frac{\partial e_k}{\partial t}(\cdot,t)||^2_{L^2} + 3A^2 ||\frac{\partial w_k}{\partial t}(\cdot,t)||^2_{L^2} + 3(BL_1)^2 ||e_k(\cdot,t)||^2_{L^2}, \tag{13}$$

$$||\frac{\partial e_{k+1}}{\partial t}||_\lambda \leq 3(1-BL_2)^2 ||\frac{\partial e_k}{\partial t}||_\lambda + 3A^2 ||\frac{\partial w_k}{\partial t}||_\lambda + 3(BL_1)^2 ||e_k||_\lambda + 3(AL_1)^2 ||\frac{\partial e_k}{\partial t}||_\lambda, \tag{14}$$

replacing (10) in (14) we get

$$||\frac{\partial e_{k+1}}{\partial t}||_\lambda \leq 3((1-BL_2)^2 + (AL_1)^2)||\frac{\partial e_k}{\partial t}||_\lambda + 3A^2||\frac{\partial w_k}{\partial t}||_\lambda + 3(BL_1)^2\Big(\frac{1-e^{-\lambda}}{\lambda} + \frac{1-e^{-\lambda}}{\lambda(\lambda-1)}\Big)||\frac{\partial e_k}{\partial t}||_\lambda + 3(BL_1)^2||e_k(\cdot,0)||^2_{L^2}. \tag{15}$$

Let $\alpha_\lambda = \frac{1-e^{-\lambda}}{\lambda} + \frac{1-e^{-\lambda}}{\lambda(\lambda-1)} = \frac{1-e^{-\lambda}}{\lambda-1}$, then we will have

$$||\frac{\partial e_{k+1}}{\partial t}||_\lambda \leq 3((1-BL_2)^2 + (AL_1)^2 + \alpha_\lambda(BL_1)^2)||\frac{\partial e_k}{\partial t}||_\lambda + 3A^2||\frac{\partial w_k}{\partial t}||_\lambda + 3(BL_1)^2||e_k(\cdot,0)||^2_{L^2}, \tag{16}$$

we also have

$$\frac{\partial^2 w_k}{\partial t^2}(x,t) - \frac{\partial^2 w_k}{\partial x^2}(x,t) = \Delta u_k(x,t).$$

Multiplying both side by $\frac{\partial w_k}{\partial t}(x,t)$ and integrating with respect to x from a to b, we get

$$\int_a^b \frac{\partial w_k}{\partial t}(x,t)\frac{\partial^2 w_k}{\partial t^2}(x,t)dx - \int_a^b \frac{\partial w_k}{\partial t}(x,t)\frac{\partial^2 w_k}{\partial x^2}(x,t)dx = \int_a^b \frac{\partial w_k}{\partial t}(x,t)\Delta u_k(x,t)dx, \tag{17}$$

while

$$\int_a^b \frac{\partial w_k}{\partial t}(x,t)\frac{\partial^2 w_k}{\partial t^2}(x,t)dx = \frac{1}{2}\frac{d}{dt}||\frac{\partial w_k}{\partial t}(\cdot,t)||^2_{L^2}, \tag{18}$$

$$\int_a^b \frac{\partial w_k}{\partial t}(x,t)\Delta u_k(x,t)dx \leq \frac{1}{2}||\frac{\partial w_k}{\partial t}(\cdot,t)||^2_{L^2} + \frac{1}{2}||\Delta u_k(\cdot,t)dx||^2_{L^2}, \tag{19}$$

$$\int_a^b \frac{\partial w_k}{\partial t}(x,t)\frac{\partial^2 w_k}{\partial x^2}(x,t)dx = -\frac{1}{2}\frac{d}{dt}\int_a^b \Big(\frac{\partial w_k}{\partial x}(x,t)\Big)^2 dx. \tag{20}$$

From the above formula we can get

$$\frac{d}{dt}||\frac{\partial w_k}{\partial t}(\cdot,t)||^2_{L^2} + \frac{d}{dt}\int_a^b (\frac{\partial w_k}{\partial x}(x,t))^2 dx \leq ||\frac{\partial w_k}{\partial t}(\cdot,t)||^2_{L^2} + ||\Delta u_k(\cdot,t)dx||^2_{L^2}, \tag{21}$$

we have $\int_a^b (\frac{\partial w_k}{\partial x}(x,t))^2 dx \geq 0$, adding $\int_a^b (\frac{\partial w_k}{\partial x}(x,t))^2 dx$ to the second term of (21) we can get

$$\frac{d}{dt}(||\frac{\partial w_k}{\partial t}(\cdot,t)||^2_{L^2} + \int_a^b (\frac{\partial w_k}{\partial x}(x,t))^2 dx)$$

$$\leq \frac{\partial w_k}{\partial t}(\cdot,t)||_{L^2}^2 + \int_a^b (\frac{\partial w_k}{\partial x}(x,t))^2 + ||\Delta u_k(\cdot,t)dx||_{L^2}^2. \quad (22)$$

Appling the Gronwall Lemma, we get

$$\frac{d}{dt}(||\frac{\partial w_k}{\partial t}(\cdot,t)||_{L^2}^2 + \int_a^b (\frac{\partial w_k}{\partial x}(x,t))^2 dx) \leq ||\Delta u_k(\cdot,t)dx||_{L^2}^2 + \int_0^t ||\Delta u_k(.,\mu)dx||_{L^2}^2 e^{t-\mu} d\mu. \quad (23)$$

From another side we have

$$||\frac{\partial w_k}{\partial t}(\cdot,t)||_{L^2}^2 \leq ||\frac{\partial w_k}{\partial t}(\cdot,t)||_{L^2}^2 + \int_a^b (\frac{\partial w_k}{\partial x}(x,t))^2 dx. \quad (24)$$

So, we can get

$$||\frac{\partial w_k}{\partial t}||_\lambda \leq (\frac{1-e^{-\lambda}}{\lambda} + \frac{1-e^{-\lambda}}{\lambda(\lambda-1)})||\Delta u_k||_\lambda = \alpha_\lambda ||\Delta u_k||_\lambda, \quad (25)$$

we also have

$$||\Delta u_k||_\lambda = ||u_{k+1} - u_k||_\lambda = ||L_1 e_k - L_2 \frac{\partial e_k}{\partial t}||_\lambda \leq L_1^2 ||e_k||_\lambda + L_2^2 ||\frac{\partial e_k}{\partial t}||_\lambda, \quad (26)$$

from (10), (25) and (26), we get

$$||\frac{\partial w_k}{\partial t}||_\lambda \leq \alpha_\lambda (L_1^2(\alpha_\lambda ||\frac{\partial e_k}{\partial t}||_\lambda + ||e_k(\cdot,0)||_{L^2}^2) + L_2^2 ||\frac{\partial e_k}{\partial t}||_\lambda). \quad (27)$$

Replacing (27) in (16)

$$||\frac{\partial e_{k+1}}{\partial t}||_\lambda \leq 3((1-BL_2)^2 + (AL_1)^2 + \alpha_\lambda (BL_1)^2)||\frac{\partial e_k}{\partial t}||_\lambda + 3A^2\alpha_\lambda (L_1^2(\alpha_\lambda ||\frac{\partial e_k}{\partial t}||_\lambda + ||e_k(\cdot,0)||_{L^2}^2) + L_2^2 ||\frac{\partial e_k}{\partial t}||_\lambda) + 3(BL_1)^2 ||e_k(\cdot,0)||_{L^2}^2,$$

$$||\frac{\partial e_{k+1}}{\partial t}||_\lambda \leq 3((1-BL_2)^2 + (AL_1)^2 + \alpha_\lambda ((BL_1)^2 + (AL_2)^2) + (\alpha_\lambda AL_1)^2)||\frac{\partial e_k}{\partial t}||_\lambda + 3(\alpha_\lambda (AL_1)^2 + (BL_1)^2)||e_k(\cdot,0)||_{L^2}^2. \quad (28)$$

Let $\gamma = (1-BL_2)^2 + (AL_1)^2 + \alpha_\lambda((BL_1)^2 + (AL_2)^2) + (\alpha_\lambda AL_1)^2$, we get,

$$||\frac{\partial e_{k+1}}{\partial t}||_\lambda \leq 3\gamma ||\frac{\partial e_k}{\partial t}||_\lambda + 3(\alpha_\lambda (AL_1)^2 + (BL_1)^2)||e_k(\cdot,0)||_{L^2}^2. \quad (29)$$

It is observed that when $\lambda \to \infty$, $\alpha_\lambda \to 0$, and from the theorem we will have $0 \leq \gamma < 1$, so according to Lemma 1 and Lemma 2,$||\frac{\partial e_{k+1}}{\partial t}||_\lambda \to 0$ when $k \to \infty$. And form (10) we can have, $||e_k||_\lambda \to 0$,when $k \to \infty$.

This end the proof of Theorem 1. □

4. NUMERICAL SIMULATION

In order to illustrate the effectiveness of the ILC mentioned in this paper, a specific numerical example is considered as the following system. Without losing generation taking: $a = 0$, $b = \pi$, $A = B = 1$, then we have the following system:

$$\begin{cases} \frac{\partial^2 y}{\partial t^2}(x,t) - \frac{\partial^2 y}{\partial x^2}(x,t) = u(x,t); \ x \in [0,\pi], \ t \in [0,1], \\ z(x,t) = Ay(x,t) + B\int_0^t u(x,s)ds, \\ \frac{\partial y}{\partial x}(0,t) = t - \sin t + 1; \ t \in [0,1], \\ \frac{\partial y}{\partial x}(\pi,t) = t - \sin t + 1 - 2\pi; \ t \in [0,1], \\ y(x,0) = 0; \frac{\partial y}{\partial t}(x,0) = 0; \ x \in [a,b]. \end{cases}$$

For the given desired output $z_d(x,t) = y_d(x,t) + \int_0^t u_d(x,\tau)d\tau$,such that $y_d(x,t) = xt - xsin(t) + x - x^2 - t^2$ and $u_d(x,t) = xsin(t)$. According to theorem, we must take $0 < L_1 < 0.57$ and $-\sqrt{\frac{1}{3} - L_1^2} + 1 < L_2 < 1$, so that the convergence of the method will be satisfied, in this application, we take $L_1 = 0.5$, $L_2 = 0.8$, $L_3 = 0.5$, and for the first iteration we take $u_0(x,t) = 0$, the input and the initial conditions for the first iteration are equal to zero so the existence and uniqueness of the solution of the problem is assured. The simulation results are shown in Figs. 1-6.

Fig. 1 shows the λ-norm of the error-iteration number. Fig. 2 shows the desired output and the system output at $k = 0$ (here for $x = \pi/2$). Fig. 3 shows the desired output and the system output in the 12th iteration (here for $x = \pi/2$). Fig. 4 shows the desired output and the output in the 17th iteration (here for $x = \pi/2$). From Figs. 2 and 3 we can see that the output gets closer to the desired result with the increase of iteration, what mean the effectiveness of the control method. Fig. 4 and Fig. 5 show the desired output and the output at the 17th, respectively.

5. CONCLUSION

In this work, the Iterative Learning Control with initial state learning for a class of linear inhomogeneous wave equation was proposed. By choosing the PD-type scheme the convergence theorem of this control method

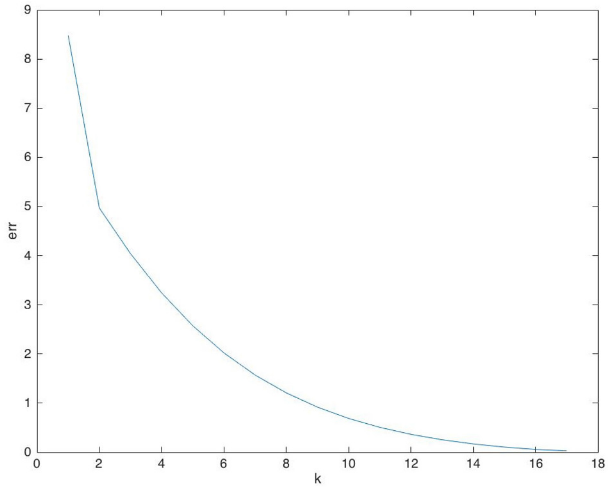

Fig. 1. λ-norm of the tracking error versus number of iterations.

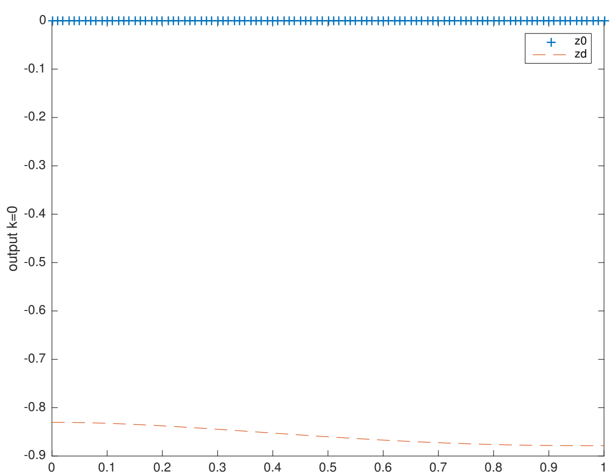

Fig. 2. the desired output with the system output at $k = 0$.

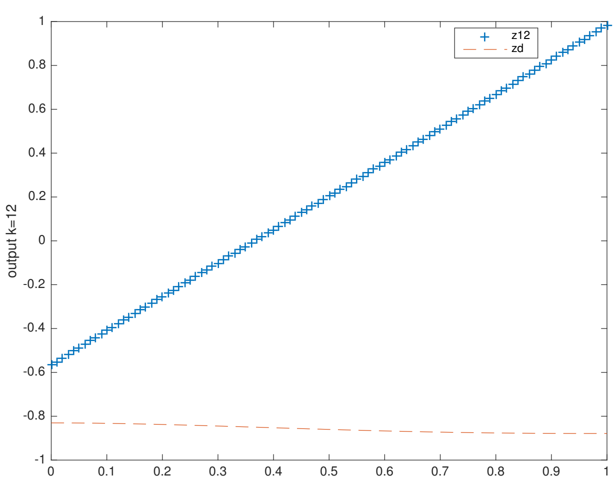

Fig. 3. the desired output with the system output at the 12th iteration.

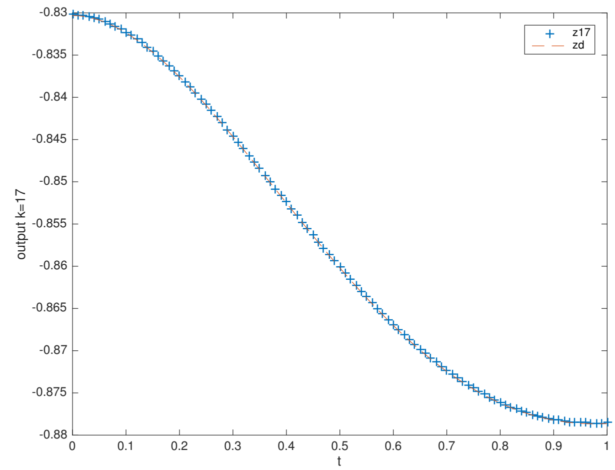

Fig. 4. the desired output with the system output at the 17th iteration.

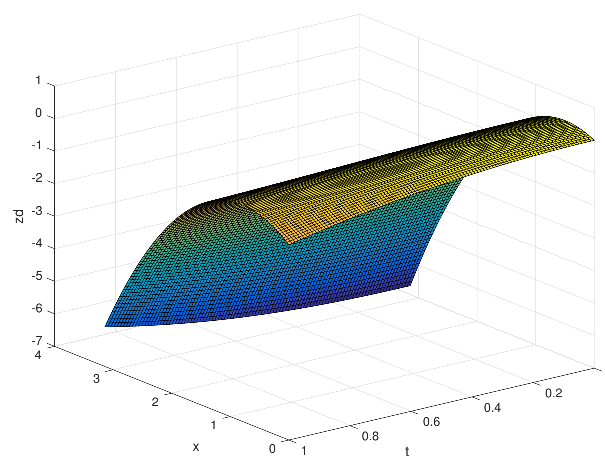

Fig. 5. the desired output.

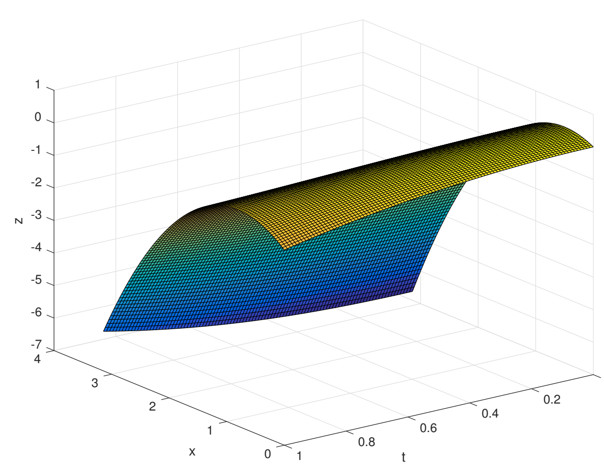

Fig. 6. the system output at the 17th iteration.

in L^2 space in the sense of λ-norm was presented and proven. The simulation result is consistent with the theoretical one.

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