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Adaptive Output Feedback Control for Switched Stochastic Nonlinear Systems with Time-varying Parameters and Unknown Output Functions

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Abstract: In this paper, we discuss the adaptive output feedback control problem for switched stochastic nonlinear systems which involve uncertain time-varying parameters and unknown output functions. The drift terms together with diffusion terms meet the conditions for linear growth with unknown rate. Firstly, an adaptive output feedback controller is proposed based on the backstepping method. Then, by using the stochastic Lyapunov stability theorem, all signals of the closed-loop system are proven to be bounded in probability and the system states are almost certain to reach the origin under arbitrary switching. Finally, a numerical example is provided to test the reliability of the proposed method.

Keywords: Adaptive control, nonlinear systems, output feedback, switched stochastic systems.

1. INTRODUCTION

In recent years, research on nonlinear systems has been attracted considerable attention [1–7] and the references therein. Switched systems as a significant branch of hybrid systems. Switched stochastic nonlinear systems are considered as a major player in physical as well as engineering systems that involve stochastic disturbances [8]. With this regard, stability theory was proposed in [9], and further discussion on the stabilizer design was given in [10]. Subsequently, the problem of global stabilization for switched stochastic nonlinear lower-triangular systems subject to arbitrary switching was addressed in [11]. As it well known, the use of backstepping design is an essential part of the global output-feedback stabilization for nonlinear systems, such as [12]. When the growth rate of nonlinearities is unknown, [13] designed an adaptive observer and controller by using this method. [14] proposed a new adaptive output feedback controller by using the methods of backstepping and universal control. [15] proposed a novel universal adaptive control scheme for nonlinear systems under lower-triangular and upper-triangular homogenous growth condition with unknown growth rates. For switched stochastic nonlinear systems, it is important to build a common Lyapunov function for all subsystems subject to arbitrary switching.

However, the above works require the precise output functions. In the presence of unknown output functions, how to handle global output feedback control problem? With this issue, an approach of global stabilization was proposed for non-switched nonlinear systems in [12], where the upper and lower bounds conditions were given for the partial derivative of the output function. [16] developed an adaptive output feedback controller for a class of stochastic nonlinear systems with unknown output gain and growth rate. [17] investigated the problem of global output feedback stabilization for a class of switched nonlinear systems with unknown control coefficients. To the best of authors' knowledge, no results are available on the control for switched stochastic nonlinear systems that involve uncertain output functions. In this paper, we will consider the problem of switched stochastic nonlinear systems described by

$$\begin{aligned} dx_i &= \ell_i x_{i+1}dt + f_{i\sigma(t)}(t,x,d(t))dt \\ &\quad + g_{i\sigma(t)}(t,x,d(t))d\omega, \ \ i=1,\dots,n-1, \\ dx_n &= \ell_n u_{\sigma(t)}dt + f_{n\sigma(t)}(t,x,d(t))dt \\ &\quad + g_{n\sigma(t)}(t,x,d(t))d\omega, \\ y &= h_{\sigma(t)}(x_1), \end{aligned} \tag{1}$$

where $x=(x_1,\dots,x_n)^T \in \mathbb{R}^n$ is system state, $u\in\mathbb{R}$ is control input, and $y\in\mathbb{R}$ is measured output, respectively. The control coefficients $\ell_i > 0$, $i=1,\dots,n$ are unknown con-

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stants. $d:\mathbb{R}\to\mathbb{R}^s$ is a continuous function denoting a parameter or disturbance that varies with time. $\sigma(t)$ is the switching signal, with its values taken in a finite set $M=\{1,\dots,m\}$ and m being the number of subsystems. The uncertain drift functions $f_{ik}:\mathbb{R}^+\times\mathbb{R}^n\times\mathbb{R}^s\to\mathbb{R}$ and the diffusion functions $g_{ik}:\mathbb{R}^+\times\mathbb{R}^n\times\mathbb{R}^s\to\mathbb{R}$ are Borel measurable and continuous functions, where $f_{ik}(t,0,d(t))=0, g_{ik}(t,0,d(t))=0$ for $i=1,\dots,n$ and $k\in M$. The uncertain function $h_k:\mathbb{R}\to\mathbb{R}$, $k\in M$, is $\mathcal{C}^1$ and $h_k(0)=0$. ω is a standard Wiener process which defined on the complete probability space $(\Omega,\mathcal{F},P)$(Ω is the sample space, $\mathcal{F}$ is the σ-field and P is probability measure). In addition, assuming no state-jump of system (1) occurs at the moments of switching, which suggests that the trajectory $x(t)$ is uninterrupted. To construct an adaptive controller for system (1), a new common coordinate is first introduced with a dynamic gain, and a novel observer is constructed without using the unmeasurable state information. Then, by using the backstepping design method, an adaptive output feedback controller is designed to globally regulate the states of the switched stochastic nonlinear systems to the origin. The main contribution of this paper are highlighted as follows: (i) Different from [12, 18, 19], in spite of unknown growth condition and unknown control coefficients, an entirely new dynamic high-gain observer is created. (ii) Compared with [3, 17], a different adaptive controller is designed to deal with the unknown output function.

2. PRELIMINARIES AND PROBLEM STATEMENT

The following notations will be used throughout the paper: $\mathbb{R}^n$ stands for the real n-dimensional space; $\mathbb{R}^+$ denotes the set of nonnegative real numbers; $\mathcal{C}^i$ denotes the set of all functions that have continuous ith partial derivatives. For consistent denotation, $\prod_{l=j}^{i}(\cdot)=1$ is taken for $j>i$. $|X|$ is the absolute value of scalar X; X^T denotes its transpose, $Tr\{X\}$ represents its trace when X is square, and $\|X\|$ is the Euclidean norm of a vector X. In $A\in\mathbb{R}^{n\times m}$, $|A|$ is the Frobenius norm, described by $|A|=(\sum_{i=1}^{n}\sum_{j=1}^{m}A_{ij}^2)^{1/2}$. $\mathcal{K}$ denotes the set of all functions: $\mathbb{R}^+\to\mathbb{R}^+$, which are continuous, and increase and vanish strictly at zero. $\mathcal{K}_\infty$ stands for the set of all class-$\mathcal{K}$ and unbounded functions. $\mathcal{KL}$ is the set of all functions $\beta(s,t):\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}^+$: these are class $\mathcal{K}$ functions for each fixed t, and they decrease to zero as $t\to\infty$ for each fixed s.

Consider the following stochastic nonlinear system:

$$\mathrm{d}x=f(x)dt+g(x)^Td\omega,\ \ x_0\in\mathbb{R}^n, \tag{2}$$

where $x\in\mathbb{R}^n$ is the system state, $\omega\in\mathbb{R}^r$ is an r-dimensional independent standard Wiener process defined on the complete probability space $(\Omega,\mathcal{F},P)$. $f:\mathbb{R}^n\to\mathbb{R}^n$ and $g^T:\mathbb{R}^n\to\mathbb{R}^{n\times r}$ are Borel measurable and continuous functions satisfying $f(0)=0$, $g(0)=0$, $\forall t\geq 0$ and x_0 is the initial value.

The following definitions and a lemma are introduced, which play important roles in this paper.

Definition 1 [20]**:** For any given $V(x)\in\mathcal{C}^2$, associated with stochastic system (2), the infinitesimal generator $\mathcal{L}$ is given by

$$\mathcal{L}V(x)=\frac{\partial V}{\partial x}f(x)+\frac{1}{2}Tr\Big\{g^T(x)\frac{\partial^2 V}{\partial x^2}g(x)\Big\}, \tag{3}$$

where $Tr\Big\{g^T(x)\frac{\partial^2 V}{\partial x^2}g(x)\Big\}$ is the Hessian term of $\mathcal{L}$.

Definition 2 [20]**:** The equilibrium $x(t)=0$ of system (2) with $f(0)=0$, $g(0)=0$ is featured by global asymptotic stability in probability, provided that there is a class $\mathcal{KL}$ function $\beta(\cdot)$ for if any $\varepsilon>0$, in which case $P\{x(t)<\beta(|x_0|)\}\geq 1-\varepsilon$, $\forall t\geq 0$, $x_0\in\mathbb{R}^n\setminus\{0\}$.

Lemma 1 [20]**:** For system (2), if there is a $\mathcal{C}^2$ function $V(x)$, class $\mathcal{K}_\infty$ functions α_1 and α_2, constants $c_1>0$, $c_2\geq 0$, and a nonnegative function $W(x)$ satisfy

$$\alpha_1(|x|)\leq V(x)\leq\alpha_2(|x|),\ \mathcal{L}V\leq -c_1W(x)+c_2, \tag{4}$$

then

(i) For system (2), a unique solution on $[0,\infty)$ for each $x_0\in\mathbb{R}^n$ is almost sure to exist;

(ii) When $c_2=0$, $f(0)=0$, $g(0)=0$, and $W(x)=\alpha_3(|x|)$ is a class $\mathcal{K}$ function, the equilibrium equation $x(t)=0$ is globally asymptotically stable in the sense of probability and $P\{\lim_{t\to\infty}|x(t)|=0\}=1$.

This paper aims to develop an output feedback controller for system (1) so as to perform global regulation of each state to origin from any initial condition. To do that, the following scenarios are assumed.

Assumption 1: For $i=1,\dots,n$, control coefficients ℓ_i satisfy $\underline{\ell}\leq\ell_i\leq\bar{\ell}$, where $\underline{\ell}$ and $\bar{\ell}$ are known positive constants.

Assumption 2: For $i=1,\dots,n$ and $k\in M$, there are unknown nonnegative constants $\tilde{\lambda}_{1k}$ and $\tilde{\lambda}_{2k}$ exist, such that

$$\begin{aligned}&|f_{ik}(t,x,d(t))|\leq\tilde{\lambda}_{1k}(|x_1|+\cdots+|x_i|),\\&|g_{ik}(t,x,d(t))|\leq\tilde{\lambda}_{2k}(|x_1|+\cdots+|x_i|).\end{aligned} \tag{5}$$

Assumption 3: There are known positive constants $\underline{\mu}_k$ and $\bar{\mu}_k$, $k\in M$ such that

$$\underline{\mu}_k\leq\frac{\partial h_k(x_1)}{\partial x_1}\leq\bar{\mu}_k,\ \ \forall x_1\in\mathbb{R}. \tag{6}$$

Remark 1: Assumption 1 suggests that the control coefficients are limited by the positive constants. From Assumption 2, it is shown that the drift and diffusion terms rely on unknown growth rate and unmeasurable states, which is a general linear growth condition. As shown in

[21], the sensor output y is a nonlinear uncertain function of the real displacement x_1 in the working region. However, the derivative of the nonlinear function $h(x_1)$ actually is bounded, which implies that Assumption 3 is a natural assumption. For example, some nonlinear output functions with bounded first derivative, such as $h(x_1) = 3x_1 + 2\sin(x_1)$, satisfy Assumption 3 as well.

Given the above, the scaling transformation is introduced:

$$z_i = \frac{\underline{\ell}^n}{\prod_{j=i}^n \ell_j} x_i, \ \ i = 1, \dots, n. \tag{7}$$

Based on the transformation (7), system (1) becomes

$$\begin{aligned} dz_i &= z_{i+1}dt + \varphi_{i\sigma(t)}(t,z,d(t))dt \\ &\quad + \phi^T_{i\sigma(t)}(t,z,d(t))d\omega, \ \ i = 1,\dots,n-1, \\ dz_n &= \underline{\ell}^n u_{\sigma(t)}dt + \varphi_{n\sigma(t)}(t,z,d(t))dt \\ &\quad + \phi^T_{n\sigma(t)}(t,z,d(t))d\omega, \\ y &= h_{\sigma(t)}(\ell z_1), \end{aligned} \tag{8}$$

where $z = (z_1, \dots, z_n)^T$, $\ell = (\prod_{j=1}^n \ell_j)/\underline{\ell}^n$, $\varphi_{ik} = (\underline{\ell}^n/\prod_{j=i}^n \ell_j) f_{ik}$, and $\phi_{ik} = (\underline{\ell}^n/\prod_{j=i}^n \ell_j) g_{ik}$, $i = 1, \dots, n$, $k \in M$.

From Assumptions 2 and 3, it can be found unknown positive constants λ_{1k} and λ_{2k}, and positive constants $\underline{c}_k$, $\bar{c}_k$, such that

$$\begin{aligned} &|\varphi_{ik}(t,z,d(t))| \le \lambda_{1k}(|z_1| + \cdots + |z_i|), \\ &|\phi_{ik}(t,z,d(t))| \le \lambda_{2k}(|z_1| + \cdots + |z_i|), \end{aligned} \tag{9}$$

$$\underline{c}_k \le \frac{\partial h_k(\ell z_1)}{\partial z_1} \le \bar{c}_k, \ \forall z_1 \in \mathbb{R}, \tag{10}$$

where $\underline{c}_k = \underline{\mu}_k$ and $\bar{c}_k = (\bar{\ell}^n/\underline{\ell}^n)\bar{\mu}_k$.

3. MAIN RESULTS

In this section, an adaptive output feedback controller will be developed to globally stabilize system (1). The main result is outlined in the following theorem.

Theorem 1: Under Assumptions 1-3, system (1) under arbitrary switching undergoes global adaptive regulation via the observer and controller:

$$\begin{aligned} d\hat{z}_i &= (\hat{z}_{i+1} - L^i a_i \hat{z}_1)dt, \ \ i = 1, \dots, n-1, \\ d\hat{z}_n &= (\underline{\ell}^n u - L^n a_n \hat{z}_1)dt, \\ dL &= ({\varepsilon_1}^2 + \eta_1^2 + \varepsilon_n^2)dt, \ \ L(0) = 1, \\ u &= -\underline{\ell}^{-n} L^{n+1} b_n \varepsilon_n, \end{aligned} \tag{11}$$

where $\hat{z} = (\hat{z}_1, \dots, \hat{z}_n)^T$. $a_i > 0$, $i = 1, \dots, n$ is the coefficient of the Hurwitz polynomial $p(s) = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n$. L is a dynamic high gain. $\varepsilon_i, i = 1, \dots, n$ are given by

$$\varepsilon_1 = \frac{y}{L}, \ \ \varepsilon_{i+1} = \eta_{i+1} - \alpha_i,$$

$$\alpha_i = -b_i \varepsilon_i, \ \ \eta_i = \frac{\hat{z}_i}{L^i}, \ \ i = 1, \dots, n \tag{12}$$

with $b_1, \dots, b_n$ being positive constants.

Proof: For system (8), we introduce the following the change of coordinates:

$$e_i = \frac{z_i - \hat{z}_i}{L^i}, \ \ i = 1, \dots, n. \tag{13}$$

Then, it can be verified that

$$\begin{aligned} de &= (LAe + \Phi_k(\cdot) + \Psi_k(\cdot) + \Upsilon_k(\cdot))dt - \frac{De}{L}dL, \\ d\eta &= (LA\eta + \frac{\underline{\ell}^n}{L^n}Bu)dt - \frac{D\eta}{L}dL, \end{aligned} \tag{14}$$

where

$$\begin{aligned} &e = (e_1, \dots, e_n)^T, \ \Phi_k(\cdot) = (\varphi_{1k}/L, \dots, \varphi_{n,k}/L^n)^T, \\ &\eta = (\eta_1, \dots, \eta_n)^T, \Upsilon_k = (a_1 z_1, \dots, a_n z_n)^T, \\ &\Psi_k(\cdot) = (\phi_{1k}/L, \dots, \phi_{n,k}/L^n)^T, \end{aligned}$$

and

$$A = \begin{pmatrix} -a_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n-1} & 0 & \cdots & 1 \\ -a_n & 0 & \cdots & 0 \end{pmatrix}_{n\times n}, \ \ B = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}_{n\times 1},$$

$$D = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & n \end{pmatrix}_{n\times n}.$$

Clearly, A is Hurwitz matrix through its construction. Thus, there exists a positive definite matrix P satisfying $A^TP + PA \le -I$ and $DP + PD \ge 0$.

By constructing the Lyapunov function $V_0 = e^TPe$ with a simple calculation, one has

$$\begin{aligned} \mathcal{L}V_0 =& Le^T(PA + A^TP)e - \frac{e^T(PD + DP)e}{L}dL \\ &+ 2e^TP(\Phi_k + \Upsilon_k) + \frac{1}{2}tr\{\Psi_k^T(e)\frac{\partial^2 V_0}{\partial e^2}\Psi_k(e)\} \\ \le& -L\|e\|^2 + 2e^TP\Phi_k + 2e^TP\Upsilon_k \\ &+ \frac{1}{2}\sum_{i=1}^n\sum_{j=1}^n \frac{\partial^2 V_0}{\partial e_i \partial e_j}\Psi_k^T(e)\Psi_k(e) \\ \le& -L\|e\|^2 + 2e^TP\Phi_k + 2e^TP\Upsilon_k + 2e^TP\Psi_k. \end{aligned} \tag{15}$$

Since $\dot{L}(t) \ge 0$, $L(0) = 1$, then $L(t) \ge 1$ for $\forall t \ge 0$. For $i = 1, \dots, n$, $k \in M$, it follows from (9) that

$$\begin{aligned} &|\frac{\varphi_{ik}}{L^i}| \le \frac{\lambda_{1k}}{L^i}(|z_1| + \cdots + |z_i|) \le \lambda_{1k}\sum_{j=1}^i \frac{|z_j|}{L^j}, \\ &|\frac{\phi_{ik}}{L^i}| \le \frac{\lambda_{2k}}{L^i}(|z_1| + \cdots + |z_i|) \le \lambda_{2k}\sum_{j=1}^i \frac{|z_j|}{L^j}, \end{aligned}$$

which leads to

$$\begin{aligned}\|\Phi_k(\cdot)\| &\leq \frac{|\varphi_{1k}|}{L}+\frac{|\varphi_{2k}|}{L^2}+\cdots+\frac{|\varphi_{nk}|}{L^n}\\ &\leq n\lambda_{1k}\sum_{j=1}^{n}\frac{|z_j|}{L^j},\ \ k\in M,\\ \|\Psi_k(\cdot)\| &\leq \frac{|\phi_{1k}|}{L}+\frac{|\phi_{2k}|}{L^2}+\cdots+\frac{|\phi_{nk}|}{L^n}\\ &\leq n\lambda_{2k}\sum_{j=1}^{n}\frac{|z_j|}{L^j},\ \ k\in M. \end{aligned}\tag{16}$$

Based on the definitions of e_i and η_i, it can be obtained that

$$\sum_{j=1}^{n}\frac{|z_j|}{L^j}\leq\frac{|z_1|}{L}+\sum_{i=2}^{n}|\eta_i|+\sqrt{n}\|e\|.\tag{17}$$

According to (10), it can be deduced that

$$\underline{c}_k|z_1|\leq|y|\leq\bar{c}_k|z_1|,\ \ k\in M.\tag{18}$$

Combining (16), (17) with (18), it yields

$$\begin{aligned}2e^TP\Phi_k \leq& 2\lambda_{1k}\|P\|\|e\|\big(n\frac{|\varepsilon_1|}{\underline{c}_k}+n\sum_{i=2}^{n}|\eta_i|+n\sqrt{n}\|e\|\big)\\ \leq&\frac{1}{2}\varepsilon_1^2+\frac{1}{4}\sum_{i=2}^{n}\eta_i^2+\|e\|^2\big(\frac{2\lambda_{1k}^2n^2\|P\|^2}{\underline{c}_k^2}\\ &+2\lambda_{1k}n\sqrt{n}\|P\|+4\lambda_{1k}^2n^2(n-1)\|P\|^2\big),\\ 2e^TP\Upsilon_k\leq&2\|e\|\|Pa\||\frac{L\varepsilon_1}{\underline{c}_k}|\leq\frac{L}{2}\|e\|^2+\frac{2L\|Pa\|^2}{\underline{c}_k^2}\varepsilon_1^2,\\ 2e^TP\Psi_k\leq&2\lambda_{2k}\|P\|\|e\|\big(n\frac{|\varepsilon_1|}{\underline{c}_k}+n\sum_{i=2}^{n}|\eta_i|+n\sqrt{n}\|e\|\big)\\ \leq&\frac{1}{2}\varepsilon_1^2+\frac{1}{4}\sum_{i=2}^{n}\eta_i^2+\|e\|^2\big(\frac{2\lambda_{2k}^2n^2\|P\|^2}{\underline{c}_k^2}\\ &+\lambda_{2k}n(n+\sqrt{n})\|P\|+4\lambda_{2k}^2n^2(n-1)\|P\|^2\big).\end{aligned}\tag{19}$$

Substituting (19) into (15) yields

$$\mathcal{L}V_0\leq-\big(\frac{L}{2}-\rho_k\big)\|e\|^2+\frac{1}{4}\sum_{i=2}^{n}\eta_i^2+\big(\frac{1}{2}+\frac{2L\|Pa\|^2}{\underline{c}_k^2}\big)\varepsilon_1^2,\tag{20}$$

where $\rho_k=2n^2(\lambda_{1k}^2+\lambda_{2k}^2)\|P\|^2/\underline{c}_k^2+2(\lambda_{1k}+\lambda_{2k})n\sqrt{n}\|P\|+4n^2(n-1)(\lambda_{1k}^2+\lambda_{2k}^2)\|P\|^2$ is an unknown constant dependent on λ_{1k} and λ_{2k}.

3.1. Controller design

Step 1: Choose the Lyapunov function $V_1(e,\eta_1,\varepsilon_1)=V_0(e)+\frac{\eta_1^2}{2L}+\frac{\varepsilon_1^2}{2}$. From the definitions of ε_1 and η_1, one has

$$\begin{aligned}\dot{\varepsilon}_1=&\frac{\partial h_k}{\partial z_1}\big(Le_2+L\eta_2+\frac{\varphi_{1k}}{L}\big)+\frac{1}{2}\big(\frac{\partial h_k}{\partial z_1}\big)^2\phi_{1k}^T\phi_{1k}\\ &+\frac{1}{2}\varepsilon_1\frac{\partial^2 h_k}{\partial {z_1}^2}\phi_{1k}^T\phi_{1k}-\frac{\varepsilon_1}{L}\dot{L},\end{aligned}$$

$$\dot{\eta}_1=L\eta_2-La_1\eta_1-\frac{\eta_1}{L}\dot{L}.\tag{21}$$

By a simple calculation, one has

$$\begin{aligned}\mathcal{L}V_1=&\mathcal{L}V_0+\varepsilon_1\dot{\varepsilon}_1+\frac{\eta_1}{L}\dot{\eta}_1-\frac{\dot{L}}{2L^2}\eta_1^2\\ \leq&-\big(\frac{L}{2}-\rho_k\big)\|e\|^2+\big(\frac{1}{2}+\frac{2L\|Pa\|^2}{\underline{c}_k^2}\big)\varepsilon_1^2\\ &+\frac{1}{4}\sum_{i=2}^{n}\eta_i^2+\frac{\partial h_k}{\partial z_1}\big(Le_2+L\eta_2+\frac{\varphi_{1k}}{L}\big)\varepsilon_1\\ &+\big(\frac{1}{2}\big(\frac{\partial h_k}{\partial z_1}\big)^2\phi_{1k}^T\phi_{1k}+\frac{1}{2}\varepsilon_1\frac{\partial^2 h_k}{\partial {z_1}^2}\phi_{1k}^T\phi_{1k}\big)\varepsilon_1\\ &-\frac{\dot{L}}{L}\varepsilon_1^2+\eta_1\big(\eta_2-a_1\eta_1-\frac{\dot{L}}{L^2}\eta_1\big)-\frac{\dot{L}}{2L^2}\eta_1^2.\end{aligned}\tag{22}$$

By completion of square and (9)-(10), one has

$$\begin{aligned}&\frac{\partial h_k}{\partial z_1}L\varepsilon_1e_2\leq\bar{c}_kL|\varepsilon_1e_2|\leq\frac{L}{4}e_2^2+\bar{c}_k^2L\varepsilon_1^2\\ &\qquad\leq\frac{L}{4}\|e\|^2+\bar{c}_k^2L\varepsilon_1^2,\\ &\frac{\partial h_k}{\partial z_1}\varepsilon_1\frac{\varphi_{1k}}{L}\leq\frac{\bar{c}_k\lambda_{1k}}{\underline{c}_k}\varepsilon_1^2,\\ &\eta_1\eta_2\leq\frac{a_1}{2}\eta_1^2+\frac{1}{2a_1}\eta_2^2,\\ &\frac{1}{2}\big(\frac{\partial h_k}{\partial z_1}\big)^2\phi_{1k}^T\phi_{1k}\varepsilon_1\leq\frac{1}{2}\frac{\bar{c}_k\lambda_{2k}}{\underline{c}_k}\varepsilon_1^2,\\ &\frac{1}{2}\varepsilon_1^2\frac{\partial^2 h_k}{\partial {z_1}^2}\phi_{1k}^T\phi_{1k}\leq\frac{1}{2}\frac{\bar{c}_k\lambda_{2k}}{\underline{c}_k}\varepsilon_1^2.\end{aligned}\tag{23}$$

By construction, one has $-\frac{3\dot{L}}{2L^2}\eta_1^2\leq 0$. Then, substituting (23) into (22), one has

$$\begin{aligned}\mathcal{L}V_1\leq&-\big(\frac{L}{4}-\rho_k\big)\|e\|^2+\Big(\frac{1}{2}+\frac{\bar{c}_k\lambda_{1k}+\lambda_{2k}}{\underline{c}_k}\\ &+\big(\frac{2\|Pa\|^2}{\underline{c}_k^2}+\bar{c}_k^2\big)L\Big)\varepsilon_1^2-\frac{\dot{L}}{L}\varepsilon_1^2+\frac{1}{4}\sum_{i=2}^{n}\eta_i^2\\ &-\frac{a_1}{2}\eta_1^2+\frac{1}{2a_1}\eta_2^2+\frac{\partial h_k}{\partial z_1}L\varepsilon_1\alpha_1\\ &+\frac{\partial h_k}{\partial z_1}L\varepsilon_1(\eta_2-\alpha_1).\end{aligned}\tag{24}$$

Choose the virtual controller as

$$\alpha_1=-b_1\varepsilon_1,\ b_1\geq\max_{k\in M}\big\{\frac{1}{\underline{c}_k}\big(1+\frac{2\|Pa\|^2}{\underline{c}_k^2}+\bar{c}_k^2\big)\big\},\tag{25}$$

which leads to

$$\begin{aligned}\mathcal{L}V_1\leq&-\big(\frac{L}{4}-\rho_k\big)\|e\|^2-\big(L-\frac{1}{2}-\frac{\bar{c}_k(\lambda_{1k}+\lambda_{2k})}{\underline{c}_k}\big)\varepsilon_1^2\\ &-\frac{\dot{L}}{L}\varepsilon_1^2+\frac{1}{4}\sum_{i=2}^{n}\eta_i^2-\frac{a_1}{2}\eta_1^2+\frac{1}{2a_1}\eta_2^2\end{aligned}$$

$$+\frac{\partial h_k}{\partial x_1}L\varepsilon_1(\eta_2-\alpha_1). \quad (26)$$

Due to $\varepsilon_2=\eta_2-\alpha_1=\eta_2+b_1\varepsilon_1$, one has $\eta_2^2\le 2b_1^2\varepsilon_1^2+2\varepsilon_2^2$. Hence

$$\begin{aligned}\mathcal{L}V_1\le&-\big(\frac{L}{4}-\rho_k\big)\|e\|^2-\big(L-\iota_{1k}\big)\varepsilon_1^2+\big(\frac{1}{2}+\frac{1}{a_1}\big)\varepsilon_2^2\\&+\frac{1}{4}\sum_{i=3}^{n}\eta_i^2-\frac{a_1}{2}\eta_1^2-\frac{\dot L}{L}\varepsilon_1^2+\frac{\partial h_k}{\partial x_1}L\varepsilon_1\varepsilon_2, \quad (27)\end{aligned}$$

where $\iota_{1k}=1/2+\bar c_k(\lambda_{1k}+\lambda_{2k})/\underline{c}_k+(1/2+1/a_1)b_1^2$ is a constant but unknown.

Step 2: Select the Lyapunov function $V_2(e,\eta_1,\varepsilon_1,\varepsilon_2)=\sigma_1V_1(e,\eta_1,\varepsilon_1)+\frac{1}{2}\varepsilon_2^2$, where $\sigma_1\ge 1$ is a constant to be determined later. Based on the definition of ε_2, one has

$$\begin{aligned}\dot\varepsilon_2=&b_1\frac{\partial h_k}{\partial z_1}\big(Le_2+L\eta_2+\frac{\varphi_{1k}}{L}\big)+\frac{1}{2}b_1\big((\frac{\partial h_k}{\partial z_1})^2\phi_{1k}^T\phi_{1k}\\&+\varepsilon_1\frac{\partial^2 h_k}{\partial {z_1}^2}\phi_{1k}^T\phi_{1k}\big)+L\eta_3+a_2(Le_1-z_1)\\&-\frac{2\dot L}{L}\varepsilon_2+b_1\frac{\dot L}{L}\varepsilon_1. \quad (28)\end{aligned}$$

By a direct calculation, one has

$$\begin{aligned}\mathcal{L}V_2\le&\sigma_1\big(-(\frac{L}{4}-\rho_k)\|e\|^2-\big(L-\iota_{1k}\big)\varepsilon_1^2-\frac{\dot L}{L}\varepsilon_1^2\\&+(\frac{1}{2}+\frac{1}{a_1})\varepsilon_2^2+\frac{1}{4}\sum_{i=3}^{n}\eta_i^2-\frac{a_1}{2}\eta_1^2+\frac{\partial h_k}{\partial z_1}L\varepsilon_1\varepsilon_2\big)\\&+\varepsilon_2\Big(b_1\frac{\partial h_k}{\partial z_1}\big(Le_2+L\eta_2+\frac{\varphi_{1k}}{L}\big)\\&+\frac{1}{2}b_1\big((\frac{\partial h_k}{\partial z_1})^2\phi_{1k}^T\phi_{1k}+\varepsilon_1\frac{\partial^2 h_k}{\partial {z_1}^2}\phi_{1k}^T\phi_{1k}\big)\\&+L\eta_3+a_2(Le_1-z_1)-\frac{2\dot L}{L}\varepsilon_2+b_1\frac{\dot L}{L}\varepsilon_1\Big). \quad (29)\end{aligned}$$

Similar to (23), one has

$$\begin{aligned}&\sigma_1\frac{\partial h_k}{\partial z_1}L\varepsilon_1\varepsilon_2\le\sigma_1\frac{L}{6}\varepsilon_1^2+\sigma_1\frac{3\bar c_k^2L}{2}\varepsilon_2^2,\\&a_2\varepsilon_2(Le_1-x_1)\le\sigma_1\frac{L}{6}\varepsilon_1^2+\sigma_1\frac{L}{16}\|e\|^2\\&\qquad+\frac{1}{\sigma_1}\big(\frac{3a_2^2L}{2\underline{c}_k^2}+4a_2^2L\big)\varepsilon_2^2,\\&b_1\frac{\partial h_k}{\partial z_1}L\varepsilon_2e_2\le\sigma_1\frac{L}{16}\|e\|^2+\frac{4b_1^2\bar c_k^2L\varepsilon_2^2}{\sigma_1},\\&b_1\frac{\partial h_k}{\partial z_1}L\varepsilon_2\eta_2\le b_1\bar c_kL|\varepsilon_2||\varepsilon_2-b_1\varepsilon_1|\\&\qquad\le\sigma_1\frac{L}{6}\varepsilon_1^2+\big(b_1\bar c_kL+\frac{3b_1^4\bar c_k^2}{2\sigma_1}L\big)\varepsilon_2^2,\\&b_1\frac{\partial h_k}{\partial z_1}\frac{\varphi_{1k}}{L}\varepsilon_2\le\frac{b_1^2\lambda_{1k}^2\bar c_k^2}{2\sigma_1\underline{c}_k^2}\varepsilon_1^2+\frac{\sigma_1}{4}\varepsilon_2^2,\\&b_1\frac{\dot L}{L}\varepsilon_1\varepsilon_2\le\frac{b_1^2}{4}\frac{\dot L}{L}\varepsilon_1^2+\frac{\dot L}{L}\varepsilon_2^2,\end{aligned}$$

$$\begin{aligned}&\frac{\varepsilon_2}{2}b_1\big((\frac{\partial h_k}{\partial z_1})^2\phi_{1k}^T\phi_{1k}+\varepsilon_1\frac{\partial^2 h_k}{\partial {z_1}^2}\phi_{1k}^T\phi_{1k}\big)\\&\le\frac{b_1^2\lambda_{2k}^2\bar c_k^2}{2\sigma_1\underline{c}_k^2}\varepsilon_1^2+\frac{\sigma_1}{4}\varepsilon_2^2, \quad (30)\end{aligned}$$

under which (29) becomes

$$\begin{aligned}\mathcal{L}V_2\le&-\sigma_1\big(\frac{L}{8}-\rho_k\big)\|e\|^2-\big(\sigma_1-\frac{b_1^2}{4}\big)\frac{\dot L}{L}\varepsilon_1^2-\frac{\dot L}{L}\varepsilon_2^2\\&-\sigma_1\big(\frac{L}{2}-\iota_{1k}-\frac{b_1^2(\lambda_{1k}^2+\lambda_{2k}^2)\bar c_k^2}{2\underline{c}_k^2}\big)\varepsilon_1^2\\&+\frac{\sigma_1}{4}\sum_{i=3}^{n}\eta_i^2-\frac{\sigma_1a_1}{2}\eta_1^2+L\varepsilon_2\alpha_2+L\varepsilon_2(\eta_3-\alpha_2)\\&+\big(\sigma_1+\frac{\sigma_1}{a_1}+\big(\frac{3\sigma_1\bar c_k^2}{2}+\frac{3a_2^2}{2\underline{c}_k^2}+4a_2^2+4b_1^2\bar c_k^2\\&+b_1\bar c_k+\frac{3b_1^4\bar c_k^2}{2}\big)L\big)\varepsilon_2^2. \quad (31)\end{aligned}$$

Then, we can design the virtual controller

$$\begin{aligned}\alpha_2=-b_2\varepsilon_2,\ b_2\ge\max_{k\in M}\{&1+\frac{3\sigma_1\bar c_k^2}{2}+\frac{3a_2^2}{2\underline{c}_k^2}+4a_2^2\\&+4b_1^2\bar c_k^2+b_1\bar c_k+\frac{3b_1^4\bar c_k^2}{2}\}, \quad (32)\end{aligned}$$

then (31) becomes

$$\begin{aligned}\mathcal{L}V_2\le&-\sigma_1\big(\frac{L}{8}-\rho_k\big)\|e\|^2-\sigma_1\big(\frac{L}{2}-\iota_{2k}\big)\varepsilon_1^2\\&-\big(L-\sigma_1-\frac{\sigma_1}{a_1}\big)\varepsilon_2^2-\big(\sigma_1-\frac{b_1^2}{4}\big)\frac{\dot L}{L}\varepsilon_1^2\\&-\frac{\dot L}{L}\varepsilon_2^2+\frac{\sigma_1}{4}\sum_{i=3}^{n}\eta_i^2-\frac{\sigma_1a_1}{2}\eta_1^2\\&+L\varepsilon_2(\eta_3-\alpha_2), \quad (33)\end{aligned}$$

where $\iota_{2k}=\iota_{1k}+b_1^2(\lambda_{1k}^2+\lambda_{2k}^2)\bar c_k^2/(2\underline{c}_k^2)$ is an unknown positive constant.

From $\varepsilon_3=\eta_3-\alpha_2=\eta_3+b_2\varepsilon_2$, one has $\eta_3^2\le 2b_2^2\varepsilon_2^2+2\varepsilon_3^2$. Hence

$$\begin{aligned}\mathcal{L}V_2\le&-\sigma_1\big(\frac{L}{8}-\rho_k\big)\|e\|^2-\sigma_1\big(\frac{L}{2}-\iota_{2k}\big)\varepsilon_1^2\\&-\big(L-\iota_2\big)\varepsilon_2^2-\big(\sigma_1-\frac{b_1^2}{4}\big)\frac{\dot L}{L}\varepsilon_1^2-\frac{\dot L}{L}\varepsilon_2^2\\&+\frac{\sigma_1}{2}\varepsilon_3^2+\frac{\sigma_1}{4}\sum_{i=4}^{n}\eta_i^2-\frac{\sigma_1a_1}{2}\eta_1^2+L\varepsilon_2\varepsilon_3, \quad (34)\end{aligned}$$

where $\iota_2=\sigma_1+\sigma_1/a_1+\sigma_1b_2^2/2$ is a positive constant.

Inductive Step: Assume in step $i-1$, we choose the Lyapunov function $V_{i-1}=\sigma_{i-2}V_{i-2}+\frac{1}{2}\varepsilon_{i-1}^2$ with the constant $\sigma_{i-2}\ge 1$ and virtual controllers $\alpha_1,\ldots,\alpha_{i-1}$ are expressed as follows:

$$\begin{aligned}&\alpha_1=-b_1\varepsilon_1,\ \ \varepsilon_2=\eta_2-\alpha_1,\\&\alpha_2=-b_2\varepsilon_2,\ \ \varepsilon_3=\eta_3-\alpha_2,\end{aligned}$$

$$\vdots$$

$$\alpha_{i-1} = -b_{i-1}\varepsilon_{i-1}, \ \ \varepsilon_i = \eta_i - \alpha_{i-1}, \tag{35}$$

where $b_1 > 0, \ldots, b_{i-1} > 0$, and thus we have

$$\begin{aligned}
\mathcal{L}V_{i-1} \le & -\prod_{j=1}^{i-2}\sigma_j\big(\frac{L}{2^i} - \rho_k\big)\|e\|^2 \\
& -\prod_{j=1}^{i-2}\sigma_j\big(\frac{L}{2^{i-2}} - \iota_{(i-1)k}\big)\varepsilon_1^2 \\
& -\sum_{j=2}^{i-1}\prod_{l=j}^{i-2}\sigma_l\big(\frac{L}{2^{i-1-j}} - \iota_j\big)\varepsilon_j^2 \\
& -\prod_{j=2}^{i-2}\sigma_j\Big(\sigma_1 - \frac{b_1^2}{4}\big(1+\sum_{p=2}^{i-2}\prod_{l=2}^{p}b_l^2\big)\Big)\frac{\dot{L}}{L}\varepsilon_1^2 \\
& -\sum_{j=2}^{i-2}\prod_{l=j+1}^{i-2}\sigma_l\big(\sigma_j - \frac{1}{4}\sum_{p=j}^{i-2}\prod_{l=j}^{p}b_l^2\big)\frac{\dot{L}}{L}\varepsilon_j^2 \\
& -\frac{\dot{L}}{L}\varepsilon_{i-1}^2 - \frac{\prod_{l=1}^{i-2}\sigma_l a_1}{2}\eta_1^2 + \frac{\prod_{l=1}^{i-2}\sigma_l}{2}\varepsilon_i^2 \\
& +\frac{\prod_{l=1}^{i-2}\sigma_l}{4}\sum_{j=i+1}^{n}\eta_j^2 + L\varepsilon_{i-1}\varepsilon_i,
\end{aligned} \tag{36}$$

where $\iota_{(i-1)k}$ is a constant but unknown and ι_j, $j = 1,\ldots,i-1$ are constants.

In what follows, we will show that (36) also holds at step i. Select the Lyapunov function $V_i = \sigma_{i-1}V_{i-1}(e,\eta_1,\varepsilon_1,\varepsilon_2\ldots,\varepsilon_{i-1}) + \frac{1}{2}\varepsilon_i^2$. From the expression of ε_i, one has

$$\begin{aligned}
\dot{\varepsilon}_i = & L\eta_{i+1} - \big(a_i + \sum_{j=2}^{i-1}\prod_{l=j}^{i-1}b_l a_j\big)(Le_1 - z_1) \\
& + L\sum_{j=3}^{i}\prod_{l=j-1}^{i-1}b_l\eta_j + \prod_{l=1}^{i-1}b_l\frac{\partial h_k}{\partial z_1}\big(L\varepsilon_2 + Lz_2 + \frac{\varphi_{1k}}{L}\big) \\
& + \prod_{l=1}^{i-1}b_l\frac{1}{2}\big((\frac{\partial h_k}{\partial z_1})^2\phi_{1k}^T\phi_{1k} + \varepsilon_1\frac{\partial^2 h_k}{\partial z_1{}^2}\phi_{1k}^T\phi_{1k}\big) \\
& - i\frac{\dot{L}}{L}\varepsilon_i + \sum_{j=1}^{i-1}\prod_{l=j}^{i-1}b_l\frac{\dot{L}}{L}\varepsilon_j.
\end{aligned} \tag{37}$$

Hence, one has

$$\begin{aligned}
\mathcal{L}V_i \le & -\prod_{j=1}^{i-1}\sigma_j\big(\frac{L}{2^i} - \rho_k\big)\|e\|^2 - \prod_{j=1}^{i-1}\sigma_j\big(\frac{L}{2^{i-2}} - \iota_{(i-1)k}\big)\varepsilon_1^2 \\
& -\sum_{j=2}^{i-1}\prod_{l=j}^{i-1}\sigma_l\big(\frac{L}{2^{i-1-j}} - \iota_j\big)\varepsilon_j^2 \\
& -\prod_{j=2}^{i-1}\sigma_j\Big(\sigma_1 - \frac{b_1^2}{4}\big(1+\sum_{p=2}^{i-2}\prod_{l=2}^{p}b_l^2\big)\Big)\frac{\dot{L}}{L}\varepsilon_1^2 \\
& -\sum_{j=2}^{i-2}\prod_{l=j+1}^{i-1}\sigma_l\big(\sigma_j - \frac{1}{4}\sum_{p=j}^{i-2}\prod_{l=j}^{p}b_l^2\big)\frac{\dot{L}}{L}\varepsilon_j^2 \\
& -\sigma_{i-1}\frac{\dot{L}}{L}\varepsilon_{i-1}^2 - \frac{\prod_{l=1}^{i-1}\sigma_l a_1}{2}\eta_1^2 + \frac{\prod_{l=1}^{i-1}\sigma_l}{2}\varepsilon_i^2 \\
& + \frac{\prod_{l=1}^{i-1}\sigma_l}{4}\sum_{j=i+1}^{n}\eta_j^2 + \sigma_{i-1}L\varepsilon_{i-1}\varepsilon_i \\
& + \varepsilon_i\Big(L\eta_{i+1} - \big(a_i + \sum_{j=2}^{i-1}\prod_{l=j}^{i-1}b_l a_j\big)\hat{z}_1 \\
& + L\sum_{j=3}^{i}\prod_{l=j-1}^{i-1}b_l\eta_j + \prod_{l=1}^{i-1}b_l\frac{\partial h_k}{\partial z_1}\big(L\varepsilon_2 + Lz_2 + \frac{f_{1,k}}{L}\big) \\
& + \prod_{l=1}^{i-1}b_l\frac{1}{2}\big((\frac{\partial h_k}{\partial z_1})^2\phi_{1k}^T\phi_{1k} + \varepsilon_1\frac{\partial^2 h_k}{\partial z_1{}^2}\phi_{1k}^T\phi_{1k}\big) \\
& - i\frac{\dot{L}}{L}\varepsilon_i + \sum_{j=1}^{i-1}\prod_{l=j}^{i-1}b_l\frac{\dot{L}}{L}\varepsilon_j\Big).
\end{aligned} \tag{38}$$

Similarly, due to the fact that $\sigma_l \ge 1$, $l = 1,\ldots,i-1$ and based on (9)-(13), the following is obtained by completing the square.

$$\begin{aligned}
& \sigma_{i-1}L\varepsilon_{i-1}\varepsilon_i \le \frac{\sigma_{i-1}}{6}L\varepsilon_{i-1}^2 + \frac{3\sigma_{i-1}}{2}L\varepsilon_i^2, \\
& \sum_{j=1}^{i-1}\prod_{l=j}^{i-1}b_l\frac{\dot{L}}{L}\varepsilon_j\varepsilon_i \le (i-1)\frac{\dot{L}}{L}\varepsilon_i^2 + \frac{b_1^2\prod_{l=2}^{i-1}b_l^2}{4}\frac{\dot{L}}{L}\varepsilon_1^2 \\
& \qquad + \sum_{j=2}^{i-1}\frac{\prod_{l=j}^{i-1}b_l^2}{4}\frac{\dot{L}}{L}\varepsilon_j^2, \\
& Lb_{i-1}\eta_i\varepsilon_i \le \big(b_{i-1} + \frac{3}{2}b_{i-1}^4\big)L\varepsilon_i^2 + \frac{\sigma_{i-1}}{6}L\varepsilon_{i-1}^2, \\
& \prod_{l=1}^{i-1}b_l\frac{\partial h_k}{\partial z_1}Le_2\varepsilon_i \le \frac{\prod_{l=1}^{i-1}\sigma_l}{2^{i+2}}L\|e\|^2 + 2^i\prod_{l=1}^{i-1}b_l^2\bar{c}_k^2L\varepsilon_i^2, \\
& \prod_{l=1}^{i-1}b_l\frac{\partial h_k}{\partial z_1}L\eta_2\varepsilon_i \le \frac{\prod_{l=1}^{i-1}\sigma_l}{2^i}L\varepsilon_1^2 + \frac{\prod_{l=2}^{i-1}\sigma_l}{2^{i-1}}L\varepsilon_2^2 \\
& \qquad + \big(2^{i-2}b_1^4\bar{c}_k^2 + 2^{i-3}b_1^2\bar{c}_k^2\big)\prod_{l=2}^{i-1}b_l^2L\varepsilon_i^2, \\
& \prod_{l=1}^{i-1}b_l\frac{\partial h_k}{\partial z_1}\frac{\varphi_{1k}}{L}\varepsilon_i \le \prod_{l=1}^{i-1}b_l\bar{c}_k\lambda_{1k}\frac{1}{\underline{c}_k}|\varepsilon_1||\varepsilon_i| \\
& \qquad \le \frac{\bar{c}_k^2\lambda_{1k}^2\prod_{l=1}^{i-1}b_l^2}{2\underline{c}_k^2}\varepsilon_1^2 + \frac{\prod_{l=1}^{i-1}\sigma_l}{2}\varepsilon_i^2, \\
& \prod_{l=1}^{i-1}b_l\frac{1}{2}\big((\frac{\partial h_k}{\partial z_1})^2\phi_{1k}^T\phi_{1k} + \varepsilon_1\frac{\partial^2 h_k}{\partial z_1{}^2}\phi_{1k}^T\phi_{1k}\big)\varepsilon_i \\
& \le \prod_{l=1}^{i-1}b_l\bar{c}_k\lambda_{2k}\frac{1}{\underline{c}_k}|\varepsilon_1||\varepsilon_i| \\
& \le \frac{\bar{c}_k^2\lambda_{2k}^2\prod_{l=1}^{i-1}b_l^2}{2\underline{c}_k^2}\varepsilon_1^2 + \frac{\prod_{l=1}^{i-1}\sigma_l}{2}\varepsilon_i^2, \\
& L\sum_{j=3}^{i-1}\prod_{l=j-1}^{i-1}b_l\eta_j\varepsilon_i \le \frac{1}{6}\sigma_{i-1}L\varepsilon_{i-1}^2 + \frac{\prod_{l=2}^{i-1}\sigma_l}{2^{i-1}}L\varepsilon_2^2 \\
& \qquad + 3\sum_{j=3}^{i-1}2^{i-j-1}(1+b_{j-1}^2)\prod_{l=j-1}^{i-1}b_l^2L\varepsilon_i^2 \\
& \qquad + \sum_{j=3}^{i-2}\frac{\prod_{l=j}^{i-1}\sigma_l}{2^{i-j}}L\varepsilon_j^2,
\end{aligned}$$

$$
\begin{aligned}
&-(a_i+\sum_{j=2}^{i-1}\prod_{l=j}^{i-1}b_la_j)\varepsilon_i(Le_1-x_1)\\
&\leq \big(a_i+\sum_{j=2}^{i-1}\prod_{l=j}^{i-1}b_la_j\big)|\varepsilon_i|\big(L|e_1|+\frac{L|\varepsilon_1|}{\underline{c}_k}\big)\\
&\leq \frac{\prod_{l=1}^{i-1}\sigma_lL}{2^{i+2}}\|e\|^2+\frac{\prod_{l=1}^{i-1}\sigma_lL}{2^i}\varepsilon_1^2\\
&\quad+\big(2^i+\frac{2^{i-2}}{\underline{c}_k^2}\big)\ \big(a_i+\sum_{j=2}^{i-1}\prod_{l=j}^{i-1}b_la_j\big)^2L\varepsilon_i^2. \qquad (39)
\end{aligned}
$$

Substituting (39) into (38) yields

$$
\begin{aligned}
\mathcal{L}V_i\leq&-\prod_{j=1}^{i-1}\sigma_j\big(\frac{L}{2^{i+1}}-\rho_k\big)\|e\|^2\\
&-\prod_{j=1}^{i-1}\sigma_j\big(\frac{L}{2^{i-1}}-\iota_{ik}\big)\varepsilon_1^2\\
&-\sum_{j=2}^{i-1}\prod_{l=j}^{i-1}\sigma_l\big(\frac{L}{2^{i-j}}-\iota_j\big)\varepsilon_j^2\\
&-\prod_{j=2}^{i-1}\sigma_j\Big(\sigma_1-\frac{b_1^2}{4}\big(1+\sum_{p=2}^{i-1}\prod_{l=2}^{p}b_l^2\big)\Big)\frac{\dot{L}}{L}\varepsilon_1^2\\
&-\sum_{j=2}^{i-1}\prod_{l=j+1}^{i-1}\sigma_l\big(\sigma_j-\frac{1}{4}\sum_{p=j}^{i-1}\prod_{l=j}^{p}b_l^2\big)\frac{\dot{L}}{L}\varepsilon_j^2\\
&-\frac{\dot{L}}{L}\varepsilon_i^2-\frac{\prod_{l=1}^{i-1}\sigma_la_1}{2}\eta_1^2+\frac{\prod_{l=1}^{i-1}\sigma_l}{4}\sum_{j=i+1}^{n}\eta_j^2\\
&+\varepsilon_i^2\Big(\prod_{l=1}^{i-1}\sigma_l+L\Big(\frac{3\sigma_{i-1}}{2}+\big(2^i+\frac{2^{i-2}}{\underline{c}_k^2}\big)\\
&\times\big(a_i+\sum_{j=2}^{i-1}\prod_{l=j}^{i-1}b_la_j\big)^2+b_{i-1}+\frac{3}{2}b_{i-1}^4\\
&+\big(2^{i-2}b_1^4\bar{c}_k^2+2^{i-3}b_1^2\bar{c}_k^2+2^ib_1^2\bar{c}_k^2\big)\prod_{l=2}^{i-1}b_l^2\\
&+3\sum_{j=3}^{i-1}2^{i-j-1}(1+b_{j-1}^2)\prod_{l=j-1}^{i-1}b_l^2\Big)\Big)\\
&+L\varepsilon_i\alpha_i+L\varepsilon_i\varepsilon_{i+1}, \qquad (40)
\end{aligned}
$$

where $\iota_{ik}=\iota_{(i-1)k}+\bar{c}_k^2(\lambda_{1k}^2+\lambda_{2k}^2)\prod_{l=1}^{i-1}b_l^2/(2\underline{c}_k^2)$ is an unknown constant.

Clearly, we can design the virtual controller as $\alpha_i=-b_i\varepsilon_i$ with

$$
\begin{aligned}
b_i\geq\max_{k\in M}\Big\{&1+\frac{3\sigma_{i-1}}{2}+\big(2^i+\frac{2^{i-2}}{\underline{c}_k^2}\big)\big(a_i+\sum_{j=2}^{i-1}\prod_{l=j}^{i-1}b_la_j\big)^2\\
&+b_{i-1}+\frac{3}{2}b_{i-1}^4+3\sum_{j=3}^{i-1}2^{i-j-1}(1+b_{j-1}^2)\prod_{l=j-1}^{i-1}b_l^2\\
&+\big(2^{i-2}b_1^4\bar{c}_k^2+2^{i-3}b_1^2\bar{c}_k^2+2^ib_1^2\bar{c}_k^2\big)\prod_{l=2}^{i-1}b_l^2\big)\Big\},
\end{aligned}
$$

and $\eta_{i+1}^2\leq 2\varepsilon_{i+1}^2+2b_i^2\varepsilon_i^2$, one has

$$
\begin{aligned}
\mathcal{L}V_i\leq&-\prod_{j=1}^{i-1}\sigma_j\big(\frac{L}{2^{i+1}}-\Theta_k\big)\|e\|^2\\
&-\prod_{j=1}^{i-1}\sigma_j\big(\frac{L}{2^{i-1}}-\iota_{ik}\big)\varepsilon_1^2\\
&-\sum_{j=2}^{i}\prod_{l=j}^{i-1}\sigma_l\big(\frac{L}{2^{i-j}}-\iota_j\big)\varepsilon_j^2\\
&-\prod_{j=2}^{i-1}\sigma_j\big(\sigma_1-\frac{b_1^2}{4}\Big(1+\sum_{p=2}^{i-1}\prod_{l=2}^{p}b_l^2\Big)\big)\frac{\dot{L}}{L}\varepsilon_1^2\\
&-\sum_{j=2}^{i-1}\prod_{l=j+1}^{i-1}\sigma_l\big(\sigma_j-\frac{1}{4}\sum_{p=j}^{i-1}\prod_{l=j}^{p}b_l^2\big)\frac{\dot{L}}{L}\varepsilon_j^2\\
&-\frac{\dot{L}}{L}\varepsilon_i^2-\frac{\prod_{l=1}^{i-1}\sigma_la_1}{2}\eta_1^2+\frac{\prod_{l=1}^{i-1}\sigma_l}{2}\varepsilon_{i+1}^2\\
&+\frac{\prod_{l=1}^{i-1}\sigma_l}{4}\sum_{j=i+2}^{n}\eta_j^2+L\varepsilon_i\varepsilon_{i+1}, \qquad (41)
\end{aligned}
$$

where $\iota_i=(1+b_i^2/2)\prod_{l=1}^{i-1}\sigma_l$.

This concludes the proof by induction.

Finally, at Step n, we choose the Lyapunov function $V_n=\sigma_{n-1}V_{n-1}+\frac{1}{2}\varepsilon_n^2$. Then, one has

$$
\begin{aligned}
\mathcal{L}V_n\leq&-\prod_{j=1}^{n-1}\sigma_j\big(\frac{L}{2^{n+1}}-\Theta_k\big)\|e\|^2\\
&-\prod_{j=1}^{n-1}\sigma_j\big(\frac{L}{2^{n-1}}-\iota_{nk}\big)\varepsilon_1^2\\
&-\sum_{j=2}^{n-1}\prod_{l=j}^{n-1}\sigma_l\big(\frac{L}{2^{n-j}}-\iota_j\big)\varepsilon_j^2\\
&-\prod_{j=2}^{n-1}\sigma_j\Big(\sigma_1-\frac{b_1^2}{4}\big(1+\sum_{p=2}^{n-1}\prod_{l=2}^{p}b_l^2\big)\Big)\frac{\dot{L}}{L}\varepsilon_1^2\\
&-\sum_{j=2}^{n-1}\prod_{l=j+1}^{n-1}\sigma_l\big(\sigma_j-\frac{1}{4}\sum_{p=j}^{n-1}\prod_{l=j}^{p}b_l^2\big)\frac{\dot{L}}{L}\varepsilon_j^2\\
&-\frac{\dot{L}}{L}\varepsilon_n^2-\frac{\prod_{l=1}^{i-1}\sigma_la_1}{2}\eta_1^2\\
&+\varepsilon_n^2\Big(\prod_{l=1}^{n-1}\sigma_l+L\Big(\frac{3\sigma_{n-1}}{2}+\big(2^n+\frac{2^{n-2}}{\underline{c}_k^2}\big)\times\\
&\big(a_n+\sum_{j=2}^{n-1}\prod_{l=j}^{i-1}b_la_j\big)^2\\
&+\big(2^{n-2}b_1^4\bar{c}_k^2+2^{n-3}b_1^2\bar{c}_k^2+2^nb_1^2\bar{c}_k^2\big)\prod_{l=2}^{n-1}b_l^2\\
&+3\sum_{j=3}^{n-1}2^{n-j-1}(1+b_{j-1}^2)\prod_{l=j-1}^{n-1}b_l^2\\
&+b_{n-1}+\frac{3}{2}b_{n-1}^4\Big)\Big)+\frac{\underline{g}^n}{L^n}\varepsilon_nu, \qquad (42)
\end{aligned}
$$

where $\iota_{nk}=\iota_{(n-1)k}+\bar{c}_k^2(\lambda_{1k}^2+\lambda_{2k}^2)\prod_{l=1}^{n-1}b_l^2/(2\underline{c}_k^2)$ is an unknown constant. Then, we design the controller

$$u = -\underline{\ell}^{-n} L^{n+1} b_n \varepsilon_n = -\underline{\ell}^{-n} \sum_{j=2}^{n} L^{n+1-j} \prod_{l=j}^{n} b_l \hat{z}_j - \underline{\ell}^{-n} L^n \prod_{l=1}^{n} b_l y, \tag{43}$$

where

$$\begin{aligned} b_n \geq \max_{k\in M} \Big\{ & 1 + \frac{3\sigma_{n-1}}{2} + b_{n-1} + \frac{3}{2} b_{n-1}^4 \\ & + \big(2^n + \frac{2^{n-2}}{\underline{c}_k^2}\big)\big(a_n + \sum_{j=2}^{n-1}\prod_{l=j}^{n-1} b_l a_j\big)^2 \\ & + \big(2^{n-2} b_1^4 \bar{c}_k^2 + 2^{n-3} b_1^2 \bar{c}_k^2 + 2^n b_1^2 \bar{c}_k^2\big) \prod_{l=2}^{n-1} b_l^2 \\ & + 3\sum_{j=3}^{n-1} 2^{n-j-1}(1 + b_{j-1}^2) \prod_{l=j-1}^{n-1} b_l^2 \Big\}, \end{aligned}$$

under which (42) becomes

$$\begin{aligned} \mathcal{L}V_n \leq & -\prod_{j=1}^{n-1} \sigma_j \big(\frac{L}{2^{n+1}} - \rho_k\big) \|e\|^2 \\ & - \prod_{j=1}^{n-1} \sigma_j \big(\frac{L}{2^{n-1}} - \iota_{nk}\big) \varepsilon_1^2 \\ & - \sum_{j=2}^{n} \prod_{l=j}^{n-1} \sigma_l \big(\frac{L}{2^{n-j}} - \iota_j\big) \varepsilon_j^2 \\ & - \sum_{j=2}^{n-1} \prod_{l=j+1}^{n-1} \sigma_l \big(\sigma_j - \frac{1}{4} \sum_{p=j}^{n-1} \prod_{l=j}^{p} b_l^2\big) \frac{\dot{L}}{L} \varepsilon_j^2 \\ & - \prod_{j=2}^{n-1} \sigma_j \big(\sigma_1 - \frac{b_1^2}{4}\Big(1 + \sum_{p=2}^{n-1}\prod_{l=2}^{p} b_l^2\Big)\big) \frac{\dot{L}}{L} \varepsilon_1^2 \\ & - \frac{\dot{L}}{L} \varepsilon_n^2 - \frac{\prod_{l=1}^{n-1} \sigma_l a_1}{2} \eta_1^2, \end{aligned} \tag{44}$$

where $\iota_n = \prod_{l=1}^{n-1} \sigma_l$.

In the end, we choose σ_j, $j = 1, \ldots, n-1$ as

$$\sigma_1 \geq \max \big\{1, \frac{b_1^2}{4}\big(1 + \sum_{p=2}^{n-1}\prod_{l=2}^{p} b_l^2\big)\big\},$$

$$\sigma_j \geq \max \Big\{1, \frac{1}{4}\sum_{p=j}^{n-1}\prod_{l=j}^{p} b_l^2\Big\}, \quad j = 2, \ldots, n-1, \tag{45}$$

and for $j = 2, \ldots, n$, define

$$\bar{\rho} = \max_{k\in M} \Big\{ \prod_{l=1}^{n-1} \sigma_l \rho_k, \prod_{l=1}^{n-1} \sigma_l \iota_{n,k}, \prod_{l=j}^{n-1} \sigma_l \iota_j \Big\},$$

$$\tau = \min \Big\{ \frac{\prod_{l=1}^{n-1} \sigma_l}{2^{n+1}}, \frac{\prod_{l=1}^{n-1} \sigma_l a_1}{2}, \frac{\prod_{l=j}^{n-1} \sigma_l}{2^{n-j}} \Big\},$$

such that (44) becomes

$$\mathcal{L}V_n \leq -(\tau L - \bar{\rho})(\|e\|^2 + \|\varepsilon\|^2) - \tau\eta_1^2. \tag{46}$$

3.2. Stability analysis

For any initial condition $(x(0), \hat{z}(0)) \in \mathbb{R}^n \times \mathbb{R}^n$ and $L(0) = 1$, we want to prove the existence and uniqueness of the solution $(x(t), \hat{z}(t), L(t))$ of the closed-loop system on $[0, \infty)$. Moreover, $\lim_{t\to+\infty}(x(t), \hat{z}(t)) = 0$ and $\lim_{t\to+\infty} L(t) = \bar{L} \in \mathbb{R}_+$.

First, we need to demonstrate the boundedness and uniqueness of the solution $(x(t), \hat{z}(t), L(t))$ on the maximal interval $[0, t_f)$ for $0 < t_f \leq +\infty$. To achieve this, a contradiction argument is applied. Suppose that $\lim_{t\to t_f} L(t) = +\infty$. As $\dot{L}(t) \geq 0$, $L(t)$ is monotonic non-decreasing function. There exists a finite time $T \in [0, t_f)$, such that $L(t) \geq (\tau + \bar{\rho})/\tau$, $\forall t \in [T, t_f)$. Thus, (46) yields

$$\mathcal{L}V_n \leq -\tau(\|e\|^2 + \|\varepsilon\|^2) - \tau\eta_1^2, \ \ \forall t \in [T, t_f). \tag{47}$$

Then, one has

$$\begin{aligned} +\infty = L(t_f) - L(T) = & \int_T^{t_f} \dot{L}(t) dt \\ \leq & -\int_T^{t_f} \frac{\mathcal{L}V(t)}{\tau} dt \leq \frac{V(T)}{\tau} < +\infty, \end{aligned}$$

which leads to a contradiction. It can be concluded that $L(t)$ is bounded on $[0, t_f)$.

Next, we will show that η is bounded on $[0, t_f)$. With considerations given to the Lyapunov function, i.e., $V(\eta) = \eta^T P \eta$ for system (14), it can be obtained through simple calculations

$$\begin{aligned} \mathcal{L}V(z) = & L\eta^T(PA + A^T P)\eta + \frac{2}{L^n}\eta^T PBu \\ & - \frac{\dot{L}}{L} z^T (PD + DP)\eta \\ \leq & -\frac{L}{2}\|\eta\|^2 + 2Lb_n^2\|P\|^2\varepsilon_n^2 \\ \leq & -\frac{1}{2}\|\eta\|^2 + 2b_n^2\|P\|^2 L\dot{L}. \end{aligned} \tag{48}$$

Thus, for $\forall t \in [0, t_f)$, one has

$$\begin{aligned} \eta^T(t)P\eta(t) \leq & \eta^T(0)P\eta(0) - \int_0^t \frac{1}{2}\|\eta(s)\|^2 ds \\ & + b_n^2\|P\|^2(L^2(t) - L^2(0)) \\ \leq & \eta^T(0)P\eta(0) + b_n^2\|P\|^2\bar{L}^2. \end{aligned}$$

This leads to the conclusion that on $[0, t_f)$

$$\|\eta(t)\|^2 \leq \frac{1}{\lambda_{\min}(P)}\big(\eta^T(0)P\eta(0) + b_n^2\|P\|^2\bar{L}^2\big),$$

$$\int_0^t \|\eta(s)\|^2 ds \leq 2\big(\eta^T(0)P\eta(0) + b_n^2\|P\|^2\bar{L}^2\big), \tag{49}$$

which indicates $\eta(t)$ and $\int_0^t \|\eta(s)\|^2 ds$ are bounded on $[0, t_f)$.

Then, e is claimed to be bounded on $[0, t_f)$. On account of which the coordinates are changed as follows:

$$\bar{e}_i = \frac{z_i - \hat{z}_i}{L^{*i}}, \ \ i = 1, \ldots, n,$$

where $L^* = \max_{k\in M}\left\{\bar{L}, \rho_k + 3\right\}$ is a positive constant.

Then, the error dynamic system (14) becomes

$$\bar{e} = L^* A\bar{e} + L^* a\bar{e}_1 - L\Lambda_1 a\bar{e}_1 + \Lambda_2 a z_1 + \Phi_k^* + \Psi_k^*, \tag{50}$$

where $\bar{e} = (\bar{e}_1, \ldots, \bar{e}_n)^T$, $\Lambda_1 = \mathrm{diag}\{1, L/L^*, \ldots, (L/L^*)^{n-1}\}$, $\Lambda_2 = \mathrm{diag}\{L/L^*, \ldots, (L/L^*)^n\}$, $\Phi_k^* = (\varphi_{1k}/L^*, \cdots, \varphi_{nk}/L^{*n})^T$, and $\Psi_k^* = (\phi_{1k}/L^*, \cdots, \phi_{nk}/L^{*n})^T$.

Now, we choose the Lyapunov function $V(\bar{e}) = \bar{e}^T P\bar{e}$ for system (50). A direct calculation yields

$$\begin{aligned}\mathcal{L}V(\bar{e}) \leq & -L^*\|\bar{e}\|^2 + 2L^*\bar{e}^T Pa\bar{e}_1 - 2L\bar{e}^T P\Lambda_1 a\bar{e}_1 \\ & + 2\bar{e}^T P\Lambda_2 a z_1 + 2\bar{e}^T P\Phi_k^* \\ & + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2 V(\bar{e})}{\partial\bar{e}_i\partial\bar{e}_j}\Psi_k^{*T}\Psi_k^* \\ \leq & -L^*\|\bar{e}\|^2 + 2L^*\bar{e}^T Pa\bar{e}_1 - 2L\bar{e}^T P\Lambda_1 a\bar{e}_1 \\ & + 2\bar{e}^T P\Lambda_2 a z_1 + 2\bar{e}^T P\Phi_k^* + 2\bar{e}^T P\Psi_k^*.\end{aligned} \tag{51}$$

By completing the squares and $L/L^* \leq 1$, one has

$$\begin{aligned} & 2L^*\bar{e}^T Pa\bar{e}_1 \leq L^{*2}\|Pa\|^2\bar{e}_1^2 + \|\bar{e}\|^2, \\ & 2L\bar{e}^T P\Lambda_1 a\bar{e}_1 \leq L^2\|P\Lambda_1 a\|^2\bar{e}_1^2 + \|\bar{e}\|^2, \\ & 2\bar{e}^T P\Lambda_2 a z_1 \leq \frac{L^2\|P\Lambda_2 a\|^2}{\underline{c}_k^2}\varepsilon_1^2 + \|\bar{e}\|^2, \end{aligned} \tag{52}$$

$$\begin{aligned} 2\bar{e}^T P\Phi_k^* \leq & 2\lambda_{1k}\|P\|\|\bar{e}\|\big(n\frac{|\varepsilon_1|L}{\underline{c}_k L^*} + n\sum_{i=2}^{n}\Big(\frac{L}{L^*}\Big)^i|\eta_i| \\ & + n\sqrt{n}\|\bar{e}\|\big) \\ \leq & \frac{1}{4}\varepsilon_1^2 + \frac{1}{8}\sum_{i=2}^{n}\eta_i^2 + \big(\frac{\lambda_{1k}^2 n^2\|P\|^2L^2}{\underline{c}_k^2 L^{*2}} \\ & + 2\lambda_{1k}^2 n^2(n-1)\Big(\frac{L}{L^*}\Big)^{2i}\|P\|^2\big)\|\bar{e}\|^2 \\ & + \lambda_{1k}n\sqrt{n}\|P\| \leq \frac{1}{4}\varepsilon_1^2 + \frac{1}{8}\sum_{i=2}^{n}\eta_i^2 + \rho_1 k\|\bar{e}\|^2, \\ 2\bar{e}^T P\Psi_k^* \leq & 2\lambda_{2k}\|P\|\|\bar{e}\|\big(n\frac{|\varepsilon_1|L}{\underline{c}_k L^*} + n\sum_{i=2}^{n}\Big(\frac{L}{L^*}\Big)^i|\eta_i| \\ & + n\sqrt{n}\|\bar{e}\|\big) \\ \leq & \frac{1}{4}\varepsilon_1^2 + \frac{1}{8}\sum_{i=2}^{n}\eta_i^2 + \big(\frac{\lambda_{2k}^2 n^2\|P\|^2L^2}{\underline{c}_k^2 L^{*2}} + \lambda_{2k}n\sqrt{n}\|P\| \\ & + 2\lambda_{2k}^2 n^2(n-1)\Big(\frac{L}{L^*}\Big)^{2i}\|P\|^2\big)\|\bar{e}\|^2 \\ \leq & \frac{1}{4}\varepsilon_1^2 + \frac{1}{8}\sum_{i=2}^{n}\eta_i^2 + \rho_k\|\bar{e}\|^2. \end{aligned} \tag{53}$$

Substituting (52) and (53) into (51), it yields

$$\begin{aligned}\mathcal{L}V(\bar{e}) \leq & (L^* - \rho_k - 3)\|\bar{e}\|^2 + \frac{1}{4}\sum_{i=2}^{n}\eta_i^2 \\ & + \Big(\frac{1}{2} + \frac{L^2\|P\Lambda_2 a\|^2}{\underline{c}_k^2}\Big)\varepsilon_1^2 \\ & + \Big(L^{*2}\|Pa\|^2 + L^2\|P\Lambda_1 a\|^2\Big)\bar{e}_1^2 \\ \leq & -\|\bar{e}\|^2 + \Big(\frac{1}{2} + \frac{L^2\|P\Lambda_2 a\|^2}{\underline{c}_k^2}\Big)\varepsilon_1^2 \\ & + \frac{1}{4}\|\eta(t)\|^2 + \Big(L^{*2}\|Pa\|^2 + L^2\|P\Lambda_1 a\|^2\Big)\times \\ & \Big(\frac{2L^2\xi_1^2}{L^{*2}\underline{c}_k^2} + \frac{2L^2\eta_1^2}{L^{*2}}\Big) \\ \leq & -\|\bar{e}\|^2 + \frac{1}{4}\|\eta(t)\|^2 + \tilde{\rho}\varepsilon_1^2 + \tilde{\rho}\eta_1^2 \\ \leq & -\|\bar{e}\|^2 + \frac{1}{4}\|\eta(t)\|^2 + \tilde{\rho}\dot{L}, \end{aligned} \tag{54}$$

where $\tilde{\rho} = \max_{k\in M}\Big\{\frac{1}{2} + \frac{\bar{L}^2\|P\Lambda_2 a\|^2}{\underline{c}_k^2} + 2\bar{L}^2\frac{\|Pa\|^2}{\underline{c}_k^2} + 2\bar{L}^2\frac{\|P\Lambda_1 a\|^2}{\underline{c}_k^2}$, $2\bar{L}^2\|Pa\|^2 + 2\bar{L}^2\|P\Lambda_1 a\|^2\Big\}$. From (54), it follows that on $[0, t_f)$

$$\begin{aligned}\bar{e}^T(t)P\bar{e}(t) \leq & \bar{e}^T(0)P\bar{e}(0) - \int_0^t \|\bar{e}(s)\|^2 ds \\ & + \tilde{\rho}(L(t) - L(0)) + \frac{1}{4}\int_0^t \|\eta(s)\|^2 ds,\end{aligned}$$

which implies

$$\begin{aligned} \|\bar{e}(t)\|^2 \leq & \frac{1}{\lambda_{\min}(P)}\Big(\bar{e}^T(0)P\bar{e}(0) + \tilde{\rho}\bar{L} \\ & + \frac{1}{4}\int_0^t \|\eta(s)\|^2 ds\Big), \\ \int_0^t \|\bar{e}(s)\|^2 ds \leq & \bar{e}^T(0)P\bar{e}(0) + \tilde{\rho}\bar{L} \\ & + \frac{1}{4}\int_0^t \|\eta(s)\|^2 ds. \end{aligned} \tag{55}$$

Since $\eta(t)$ and $\int_0^t \|\eta(s)\|^2 ds$ are bounded on $[0, t_f)$, one can obtain $\bar{e}(t)$ and $\int_0^t \|\bar{e}(s)\|^2 ds$ are also bounded on $[0, t_f)$ from (55). From the definitions of $\bar{e}_i$, e_i, $i = 1, \ldots, n$, one obtains $e(t)$ and $\int_0^t \|e(s)\|^2 ds$ are bounded on $[0, t_f)$ as well. So far, $(\eta(t), e(t), L(t))$ has been proven to be bounded on $[0, t_f)$. With the definitions of z_i and ε_i, $i = 1, \ldots, n$, it is easy to know that $(x(t), \hat{z}(t), L(t))$ is bounded on $[0, t_f)$.

Moreover, $t_f = +\infty$. This can be further verified by contradiction. In the case where $t_f < +\infty$, t_f is a finite-escape time. This indicates that the component of solution $(z(t), \hat{z}(t), L(t))$ approaches infinity if $t \to t_f$. The contradiction here, however, is that the continuity of the solution ensures $(z(t), \hat{z}(t), L(t))$ is bounded when $t = t_f$. Hence, the closed-loop system involves a solution that is bounded over $[0, +\infty)$.

Finally, based on the boundedness of $(\eta(t), e(t), L(t))$ on $[0, +\infty)$, the conclusion can be drawn that $\dot{\eta}(t)$ and $\dot{e}(t)$ are bounded on $[0, +\infty)$. It should be noted that $\int_0^{+\infty} \|\eta(t)\|^2 dt < +\infty$ and $\int_0^{+\infty} \|e(t)\|^2 dt < +\infty$, by Barbalat's Lemma, one has $\lim_{t\to+\infty} z(t) = 0$ and $\lim_{t\to+\infty} \varepsilon(t) = 0$. From the definitions of $L(t)$, η_i and

e_i, $i=1,\dots,n$, it holds $\lim_{t\to+\infty}x(t)=\lim_{t\to+\infty}z(t)=0$, $\lim_{t\to+\infty}\hat{z}(t)=0$ and $\lim_{t\to+\infty}L(t)=\bar{L}\in\mathbb{R}_+$.

4. A SIMULATION EXAMPLE

Consider the switched uncertain nonlinear system

$$\begin{aligned} dx_1 &= \ell_1 x_2 dt + f_{1\sigma(t)}(t,x,d(t))dt \\ &\quad + g^T_{1\sigma(t)}(t,x,d(t))d\omega, \\ dx_2 &= \ell_2 u dt + f_{2\sigma(t)}(t,x,d(t))dt \\ &\quad + g^T_{2\sigma(t)}(t,x,d(t))d\omega, \\ y &= c_{1\sigma(t)}x_1 + c_{2\sigma(t)}\sin x_1 \end{aligned} \tag{56}$$

where $\sigma(t):[0,+\infty)\to M=\{1,2\}$, $f_{11}(t,x,d(t))=\lambda_{11}x_1+d_{11}(t)x_1\sin^2 x_2$, $f_{21}(t,x,d(t))=\lambda_{12}x_2\sin x_1+d_{12}(t)\lambda_{13}\ln(1+\lambda_{14}x_2^2)$, $f_{12}(t,x,d(t))=\lambda_{21}d_{21}(t)x_1\sin x_1$, $f_{22}(t,x,d(t))=\frac{\lambda_{22}x_2}{1+x_1^4}+d_{22}(t)x_1$, $g_{11}=0.2, g_{21}=1$, $g_{12}=0.1, g_{22}=1.2$ with $\lambda_{11},\lambda_{12},\lambda_{21},\lambda_{22}$ being unknown constants and $d_{11},d_{12},d_{21},d_{22}$ being uncertain bounded parameters. $0.4\le c_{11},c_{21}\le 1.4$, $1.1\le c_{12},c_{22}\le 2$ and $1\le \ell_1,\ell_2\le 1.5$. By verification, the switched stochastic nonlinear system (56) satisfies Assumptions 1-3. Accordingly, based on Theorem 1, we can design a dynamic high-gain observer and a universal output feedback controller as

$$\begin{aligned} d\hat{z}_1 &= \hat{z}_2 dt - 2L\hat{z}_1 dt, \\ d\hat{z}_2 &= u dt - L^2\hat{z}_1 dt, \\ u &= -12.546L\hat{z}_2 - 426.828L^2 y, \\ \dot{L} &= \frac{\hat{z}_1^2}{L^2} + \frac{y^2}{L^2} + \left(\frac{\hat{z}_2}{L^2} + b_1\frac{y}{L}\right)^2. \end{aligned} \tag{57}$$

The simulation is carried out with parameters as: $\ell_1=1, \ell_2=1.2$, $\lambda_{11}=1.5, \lambda_{12}=0.8, \lambda_{13}=0.5, \lambda_{14}=0.7, \lambda_{21}=0.5, \lambda_{22}=0.2$, $d_{11}=1, d_{12}=0.6, d_{21}=0.4, d_{22}=0.3$, $b_1=100$ and $c_{11}=0.2, c_{21}=1, c_{12}=1.1, c_{22}=1.5$. The initial condition is $(x_1(0),x_2(0))=(0.1,-0.3)$ and $(\hat{z}_1(0),\hat{z}_2(0),L(0))=(0,0,1)$. The effectiveness of the proposed control scheme is confirmed in Figs. 1-5.

5. CONCLUSION

This paper has discussed the problem of adaptive output feedback control for a class of switched stochastic nonlinear systems under arbitrary switching. A new observer with a dynamic gain has been designed to estimate the states. An adaptive output feedback controller has been proposed based on the backstepping method to guarantee that all the signals of the closed-loop system are bounded in probability and the system states converge to the origin almost surely. However, some problems need to be solved in the future, such as the design of an adaptive output feedback controller for high-order switched uncertain stochastic nonlinear systems under weaker conditions.

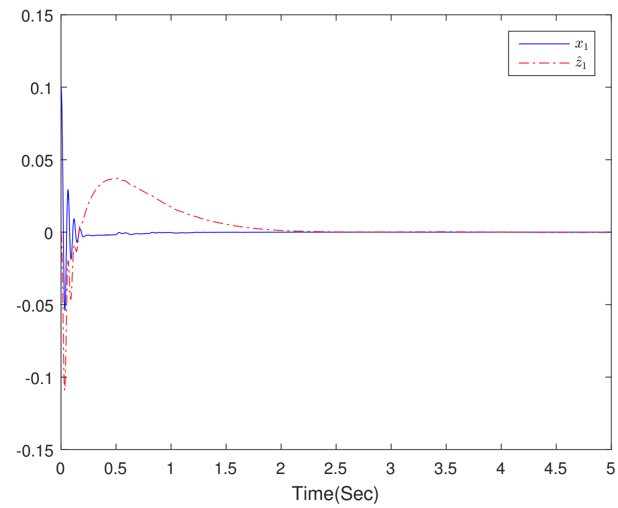

Fig. 1. Trajectories of x_1 and $\hat{z}_1$.

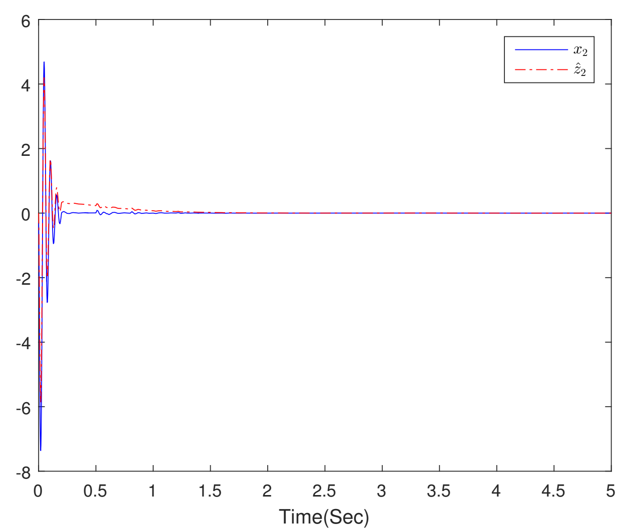

Fig. 2. Trajectories of x_2 and $\hat{z}_2$.

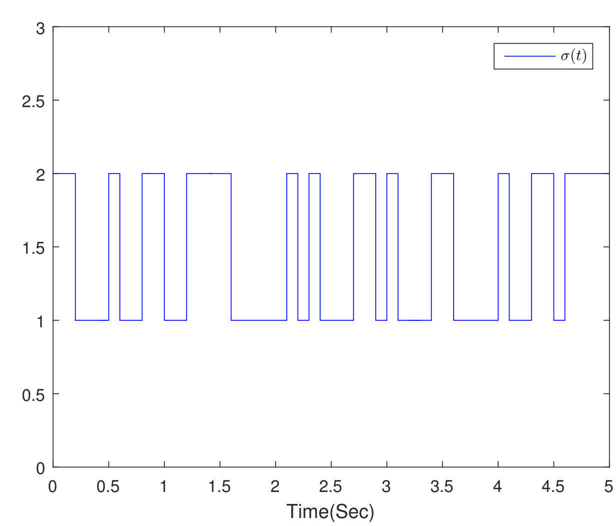

Fig. 3. The switching signal $\sigma(t)$.

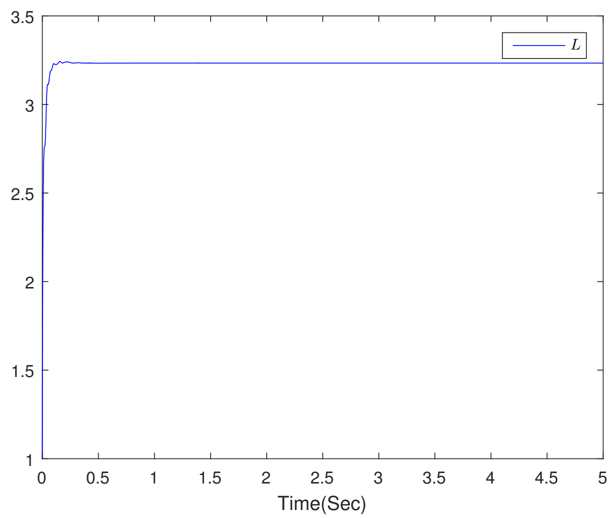

Fig. 4. The curve of dynamic gain L.

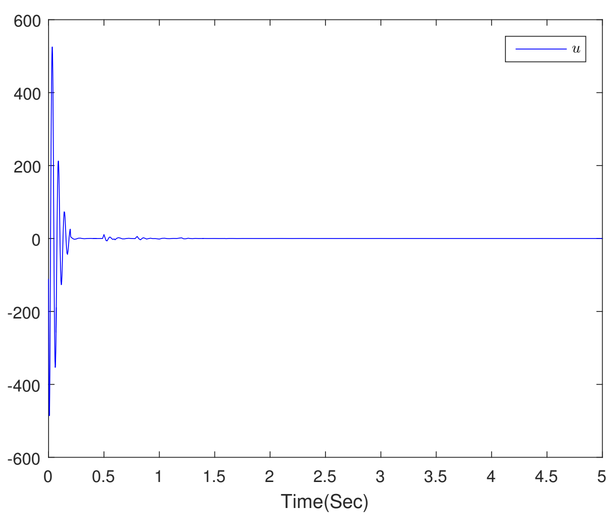

Fig. 5. The control input u.

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