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Containment Control of Multi-agent Systems with Time-delays over Heterogeneous Networks

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Abstract: Containment control problems for second-order multi-agent systems with fixed communication time-delays under heterogeneous networks topologies are investigated. Based on Lyapunov-Krasovskii functional method and the linear matrix inequality (LMI) method, sufficient conditions on the communication digraph, the feedback gains, and the allowed upper bound of the time-delays to ensure containment control of the multi-agent systems using the different containment control algorithms are given. Finally, numerical simulations are presented to demonstrate theoretical results.

Keywords: Communication time-delays, containment control, heterogeneous networks, multi-agent systems.

1. INTRODUCTION

Because of the limitation of single agent ability and enlightened by cooperative behaviors in nature, the control problem of complex system becomes research hot topic. Recently, distributed cooperative control problem of multi-agent systems has received considerable attention from control communities due to its extensive applications in formation flying of spacecraft, mobile robots, and other areas. As one of the most fundamental research topics in the field of cooperative control of multi-agent systems, consensus plays an important role. Following some pioneering works Reynolds [1], Vicsek [2], Jadbabaie [3], Moreau [4] and Ren, Beard [5, 6], investigations of consensus for multi-agent system can be found, such as leaderless finite-time consensus [7], sampled-data consensus [8], output synchronization for heterogeneous networks [9], consensus of discrete-time multi-agent system [10], leader-following consensus [11–13], consensus of multi-agent system with time-delays [14], just to mention a few. In the case of multiple leaders, the containment control problem was proposed by Ji *et al*. in [15]. The target of containment control is that the followers will all go asymptotically into the convex hull spanned by the leaders according to properly designed protocol. In recent ten years, many papers have studied the containment control problems for various multi-agent systems. Event-triggered containment control for multi-agent systems was investigated in [16]. The robust containment of uncertain linear multi-agent systems under adaptive protocols in [17] and containment control problem of switched multi-agent systems in [18] was investigated respectively. Despite of ignoring the existence of communication time-delays between agents for simplifying the theory analysis in many papers, but the delays exist because some reasons in practical multi-agent systems. Some investigations have been extensively conducted in this direction [19–22]. When the delays are constant, the frequency-domain approach has been used to give the consensus conditions [19, 20]. The containment control problem of the second-order/fractional-order multi-agent systems with time-varying delays was investigated in [21, 22], respectively. For second-order multi-agent systems, most of the paper assume that both velocity and position information communicated in the same way, resulting in homogeneous communication networks. Consensus control and containment control problem of multi-agent systems over heterogeneous networks were investigated in [23–25], respectively. However, to our limited knowledge, few research results are obtained about the containment control problem with time-delays over heterogeneous networks topology. Motivated by the above analysis and following our previous work [26, 27], in this paper, we investigate the distributed containment control of multi-agent systems with time-delays over heterogeneous networks topology. Compared with the existing works on containment control, the novel features of the current paper are three folds. Firstly, this paper investigated containment of multi-agent systems with time-delays under heterogeneous networks and the sufficient conditions in terms of

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linear matrix inequalities (LMIs) are given. In [22], distributed containment problems of continuous-time multi-agent systems with time-delays under fixed (homogeneous) networks were studied. In [24, 25], containment and consensus problems of multi-agent systems over heterogeneous networks were studied respectively, where they did not consider communication time-delays. Secondly, the allowed upper bound of the delays to ensure containment control is given. Finally, we propose the containment control protocol that makes arbitrary bounded time-delays safely be tolerated for discrete-time multi-agent systems under heterogeneous networks. The paper is organized as follows: In Section 2, we give some basic concepts in graph theory and some relative lemmas. In Section 3, the model is described and the containment control problem of second-order continuous-time/discrete-time multi-agent systems with fixed time-delays over heterogeneous networks is investigated respectively. Numerical simulations and conclusion are given in Sections 4 and 5, respectively.

Notation: The notation $diag(\omega_1, \omega_2, \ldots \omega_n,)$ denotes a diagonal matrix whose diagonal entries are $\omega_i, i = 1, 2, \ldots, n$ and the notation A^{T} denotes the transpose of matrix A. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations. The notation $P > 0 (\geq 0)$ means that P is a real symmetric positive definite (positive semi-definite) matrix. $\boldsymbol{I}$ and $\boldsymbol{0}$ represent, respectively, the identity matrix and zero matrix. The set of real numbers is denoted by $\mathbb{R}^m$. The set of real-valued $m \times n$ vectors of length m is given by $\mathbb{R}^{m\times n}$.

2. PROBLEM FORMULATIONS AND PRELIMINARIES

Let $G(\mathbf{V}, \varepsilon, \mathbf{A})$ be a directed graph of order n , with the set of nodes $\mathbf{V}=\{v_1, v_2, \ldots, v_n\}$, and the set of directed edges $\varepsilon \in V \times V$, and an adjacency matrix $\mathbf{A} = [a_{ij}] \in \mathbb{R}^{n\times n}$ with nonnegative adjacency element a_{ij} . A directed edge ε_{ij} in G is denoted by the ordered pair of node $v_i, v_j)$, where v_i is defined as the parent node and v_j is defined as the child node, which means that node v_j can receive information from node v_i . Accordingly, node v_i is a neighbor of node v_j . The adjacency elements associated with the edges are positive, that is, $\varepsilon_{ij} \in \varepsilon \Leftrightarrow a_{ij} > 0$. Moreover, we assume $a_{ii} = 0$ for all $v_i \in V$. In this paper, we use $N_i(t)$ to denote the neighbor set of agent i at the time t . Correspondingly, we defined the Laplacian matrix $\mathbf{L} = [l_{ij}] \in \mathbb{R}^{n\times n}$ of the directed graph as [28]

$$l_{ij} = \begin{cases} \sum_{j=1}^{n} a_{ij}, & i = j, \\ -a_{ij}, & i \neq j. \end{cases}$$

Definition 1 [29]**:** Let Q be a set in a real vector space $W \subseteq \mathbb{R}^n$,The set Q is called convex if, for any x and y in Q, the point $(1-\theta)x + \theta y$ is in Q for any $\theta \in [0, 1)$. The convex hull for a set of points $\mathbf{X} = \{x_1, x_2, \ldots, x_n\}$ in W is the minimal convex set containing all points in X . We use $\mathrm{Co}(\mathbf{X})$ to denote the convex hull of X. In particular,

$$\mathrm{Co}(\mathbf{X}) = \{\sum_{i=1}^{n} a_i x_i | x_i \in \mathbf{X}, a_i \in \mathbb{R}^n \geq 0, \ \sum_{i=1}^{n} a_i = 1\}.$$

For the n-agent system, an agent is called a leader if the agent has no neighbor. An agent is called a follower if the agent has a neighbor.

Definition 2 [30]**:** The containment control is achieved for multi-agent systems under a certain control input if the position and velocity(second-order system)states of the followers asymptotically converge to the convex hull formed by those of the leaders.

Assumption 1 [31]**:** For each follower, there exists at least one leader that has a directed path to the follower and communication among followers is bidirectional.

Assume that there are m leaders, where $m < n$, and $n-m$ followers. In this paper, we use $\Re$ and F to denote, respectively, the leader set and the follower set. Without loss of generality, we assume that agents 1 to $n-m(m < n)$ are followers and agents $n-m+1$ to n are leaders. Accordingly, $\mathbf{L}$ can be partitioned as

$$\mathbf{L}_x = \begin{bmatrix} \mathbf{L}_{\mathrm{x}1} & \mathbf{L}_{\mathrm{x}2} \\ 0_{m\times(n-m)} & 0_{m\times m} \end{bmatrix}, \ \mathbf{L}_v = \begin{bmatrix} \mathbf{L}_{\mathrm{v}1} & \mathbf{L}_{\mathrm{v}2} \\ 0_{m\times(n-m)} & \boldsymbol{0}_{m\times m} \end{bmatrix},$$

where $\mathbf{L}_{\mathrm{x}1}(\text{resp. } \mathbf{L}_{\mathrm{v}1}) \in \mathbb{R}^{(n-m)\times(n-m)}$ and $\mathbf{L}_{\mathrm{x}2}(\text{resp. } \mathbf{L}_{\mathrm{v}2}) \in \mathbb{R}^{(n-m)\times m}$. Under Assumption 1, the following Lemma holds true.

Lemma 1 [32]**:** Assume the communication digraph G has a directed spanning forest(same as Assumption 1 is satisfied). Then, all the eigenvalues of $\mathbf{L}_1$ have positive real parts, each element of $-\mathbf{L}_1{}^{-1}\mathbf{L}_2$ is nonnegative, and the sum of each row of $-\mathbf{L}_1{}^{-1}\mathbf{L}_2$ is 1.

From the Lemma 1, we know that $\mathbf{L}_1$ is invertible and all eigenvalues of $\mathbf{L}_1$ have positive real parts and not equal to 0. And the matrix $\mathbf{L}_1$ is symmetric matrix because that communication among followers is bidirectional. In this paper, consider the second-order containment problem for n mobile agents. Let the position information be shared along the communication network $G_x = (V, \varepsilon_x, A_x = [a_{ij}])$ and the velocity information along $G_v = (V, \varepsilon_v, A_v = [b_{ij}])$, respectively. Assume that G_x (resp.G_v) satisfy Assumption 1. Then from Lemma 1, the following results hold true.

1) All eigenvalues of $\mathbf{L}_{\mathrm{x}1}$ (resp.$\mathbf{L}_{\mathrm{v}1}$) have positive real parts and not equal to 0.

2) $-\mathbf{L}_{\mathrm{x}1}{}^{-1}\mathbf{L}_{\mathrm{x}2}1_m = 1_{n-m}$ (resp.$-\mathbf{L}_{\mathrm{v}1}{}^{-1}\mathbf{L}_{\mathrm{v}2}1_m = 1_{n-m}$).

3) The matrix $\mathbf{L}_{\mathrm{x}1}$ (resp.$\mathbf{L}_{\mathrm{v}1}$) is symmetric positive-definite matrix.

Lemma 2 [33]**:** Given a positive definite matrix $\mathbf{M} \in \mathbb{R}^{n\times n}$, two constants γ_1 and γ_2 satisfying $\gamma_1 < \gamma_2$, and a vector function $\omega : [\gamma_1, \gamma_2] \to \mathbb{R}^n$ such that the integrations concerned are well defined, the following inequality holds

$$(\int_{\gamma_1}^{\gamma_2} \omega(s)ds)^{\mathrm{T}} M (\int_{\gamma_1}^{\gamma_2} \omega(s)ds) \\ \leq (\gamma_2 - \gamma_1)(\int_{\gamma_1}^{\gamma_2} \omega^{\mathrm{T}}(s) M \omega(s)ds).$$

3. MAIN RESULTS

3.1. Containment control of continuous-time multi-agent systems with multiple stationary leaders

Consider a group of n identical agents with second-order dynamics:

$$\begin{cases} \dot{x}_i(t) = \begin{cases} 0, & i \in \Re, \\ v_i(t), & i \in F, \end{cases} \\ \dot{v}_i(t) = \begin{cases} 0, & i \in \Re, \\ u_i(t), & i \in F, \end{cases} \end{cases} \tag{1}$$

where $x_i(t) \in \mathbb{R}^n$, $v_i(t) \in \mathbb{R}^n$ and $u_i(t) \in \mathbb{R}^n$ are the position and velocity state, and control input of agent i at time t, respectively. Here we propose the control protocol as follows:

$$u_i(t) = \sum_{j \in N_i^x(t)} a_{ij}[\alpha(x_j(t-\tau) - x_i(t-\tau))] \\ + \sum_{j \in N_i^v(t)} b_{ij}[\beta(v_j(t-\tau) - v_i(t-\tau))]. \tag{2}$$

Then, the closed-loop system using (2) for (1) can be written as follows:

$$\begin{bmatrix} \dot{\mathbf{X}}(t) \\ \dot{\mathbf{V}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{n\times n} & \mathbf{I}_n \\ \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} \end{bmatrix} \begin{bmatrix} \mathbf{X}(t) \\ \mathbf{V}(t) \end{bmatrix}, \tag{3}$$

where

$$\mathbf{X}_{\mathrm{F}}(k) = [x_1(k)x_2(k)\ldots x_{n-m}(k)]^{\mathrm{T}}, \\ \mathbf{X}_{\mathrm{L}}(k) = [x_{n-m+1}(k)\ldots x_n(k)]^{\mathrm{T}}.$$

Furthermore, one can

$$\begin{bmatrix} \dot{\mathbf{X}}_{\mathrm{F}}(t) \\ \dot{\mathbf{V}}_{\mathrm{F}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{(n-m)\times(n-m)} & \mathbf{I}_{(n-m)\times(n-m)} \\ \mathbf{0}_{(n-m)\times(n-m)} & \mathbf{0}_{(n-m)\times(n-m)} \end{bmatrix} \begin{bmatrix} \mathbf{X}_{\mathrm{F}}(t) \\ \mathbf{V}_{\mathrm{F}}(t) \end{bmatrix} \\ + \begin{bmatrix} \mathbf{0}_{(n-m)\times(n-m)} & \mathbf{0}_{(n-m)\times(n-m)} \\ -\alpha \mathbf{L}_{\mathrm{x1}} & -\beta \mathbf{L}_{\mathrm{v1}} \end{bmatrix} \begin{bmatrix} \mathbf{X}_{\mathrm{F}}(t-\tau) \\ \mathbf{V}_{\mathrm{F}}(t-\tau) \end{bmatrix} \\ + \begin{bmatrix} \mathbf{0}_{(n-m)\times m} & \mathbf{0}_{(n-m)\times m} \\ -\alpha \mathbf{L}_{\mathrm{x2}} & -\beta \mathbf{L}_{\mathrm{v2}} \end{bmatrix} \begin{bmatrix} \mathbf{X}_{\mathrm{L}}(t-\tau) \\ \mathbf{V}_{\mathrm{L}}(t-\tau) \end{bmatrix}. \tag{4}$$

We define containment error function

$$E_x(t) = \mathbf{X}_{\mathrm{F}}(t) + {\mathbf{L}_{\mathrm{x1}}}^{-1} \mathbf{L}_{\mathrm{x2}} \mathbf{X}_{\mathrm{L}}(t), \\ E_v(t) = V_{\mathrm{F}}(t) + {\mathbf{L}_{\mathrm{v1}}}^{-1} \mathbf{L}_{\mathrm{v2}} V_{\mathrm{L}}(t). \tag{5}$$

Then system (1) can be converted the error system as follows:

$$\dot{E}(t) = \Phi_1 E(t) + \Phi_2 E(t-\tau), \tag{6}$$

where

$$E(t) = \begin{bmatrix} E_x(t) \\ E_v(t) \end{bmatrix}, \quad \Phi_1 = \begin{bmatrix} \mathbf{0}_{(n-m)\times(n-m)} & \mathbf{I}_{(n-m)\times(n-m)} \\ \mathbf{0}_{(n-m)\times(n-m)} & \mathbf{0}_{(n-m)\times(n-m)} \end{bmatrix}, \\ \Phi_2 = \begin{bmatrix} \mathbf{0}_{(n-m)\times(n-m)} & \mathbf{0}_{(n-m)\times(n-m)} \\ -\alpha \mathbf{L}_{\mathrm{x1}} & -\beta \mathbf{L}_{\mathrm{v1}} \end{bmatrix}.$$

By defining $\xi(t) = E(t) - E(t-\tau)$ and $\Phi = \Phi_1 + \Phi_2$, equation (6) can be rewritten as follows:

$$\dot{E}(t) = \Phi E(t) - \Phi_2 \xi(t). \tag{7}$$

From Definition 1 and Lemma 1, one can know that system(1) achieve containment control if $\lim_{t\to\infty} \mathbf{E}(t) = 0$ holds.

Lemma 3: If Assumption 1 is satisfied and $\alpha > 0$, $\beta > 0$, then the matrix Φ is Hurwitz.

Proof: It is clear that

$$\Phi = \begin{bmatrix} \mathbf{0}_{(n-m)\times(n-m)} & \mathbf{I}_{(n-m)\times(n-m)} \\ -\alpha \mathbf{L}_{\mathrm{x1}} & -\beta \mathbf{L}_{\mathrm{v1}} \end{bmatrix}.$$

We define a quadratic matrix polynomial as follows:

$$P(\lambda) = I\lambda^2 + \beta L_{v1}\lambda + \alpha L_{x1}. \tag{8}$$

Obviously, the matrix Φ corresponds to $P(\lambda)$ in (8) and the eigenvalues of Φ coincide with the eigenvalues of the quadratic matrix polynomial (8). Multiplying $P(\lambda)\omega = 0$ by ω^{T} from the left gives the quadratic equation

$$\lambda^2 \omega^{\mathrm{T}} \omega + \beta \lambda \omega^{\mathrm{T}} L_{v1} \omega + \alpha \omega^{\mathrm{T}} L_{x1} \omega = 0. \tag{9}$$

Its coeffcients are real for all ω due to the symmetry of L_{x1} and L_{v1} . We can obtain λ as the solutions of

$$\begin{aligned} \lambda &= \frac{-\beta\omega^{\mathrm{T}} L_{v1}\omega \pm \sqrt{(\beta\omega^{\mathrm{T}} L_{v1}\omega)^2 - 4*(\omega^{\mathrm{T}}\omega)*(\alpha\omega^{\mathrm{T}} L_{x1}\omega)}}{2\omega^{\mathrm{T}}\omega} \\ &= \frac{-\beta\omega^{\mathrm{T}} L_{v1}\omega \pm \sqrt{\beta^2\omega^{\mathrm{T}} L_{v1}\omega\omega^{\mathrm{T}} L_{v1}\omega - 4\alpha*(\omega^{\mathrm{T}}\omega)*(\omega^{\mathrm{T}} L_{x1}\omega)}}{2\omega^{\mathrm{T}}\omega} \\ &= \frac{-\beta\omega^{\mathrm{T}} L_{v1}\omega \pm \sqrt{\omega\omega^{\mathrm{T}}[\beta^2\omega^{\mathrm{T}} L_{v1} L_{v1}\omega - 4\alpha*(\omega^{\mathrm{T}} L_{x1}\omega)]}}{2\omega^{\mathrm{T}}\omega}, \end{aligned} \tag{10}$$

where ω is an eigenvector of $P(\lambda)$ [19]. It is clear that all solutions of λ are real, imaginary, or complex conjugate with a non-positive real part if $\alpha > 0$, $\beta > 0$ holds. The proof completed. □

Theorem 1: Consider the containment problem for system (1) under heterogeneous networks. Assume that the position information be shared along the communication network G_x and the velocity information along G_v . Protocol (2) achieves containment control asymptotically if both G_x and G_v satisfying Assumption 1 and time-delay τ satisfying

$$E_2+\tau^2 F_2<0, \tag{11}$$

where

$$E_2=\begin{bmatrix} P\Phi^{\mathrm{T}}+\Phi P & -P\Phi_2 \\ -{\Phi_2}^{\mathrm{T}}P & -R \end{bmatrix},$$

$$F_2=\begin{bmatrix} \Phi^{\mathrm{T}} \\ -{\Phi_2}^{\mathrm{T}} \end{bmatrix} R \begin{bmatrix} \Phi & -\Phi_2 \end{bmatrix}.$$

Proof: Consider the following Lyapunov function candidate

$$\begin{aligned} V(t)=&E^{\mathrm{T}}(t)PE(t) \\ &+\tau\int_{t-\tau}^{t}(s-t+\tau)\dot{E}^{\mathrm{T}}(s)R\dot{E}(s)ds, \end{aligned} \tag{12}$$

where P is a positive definite matrix and γ,τ are positive scalar and fixed time-delays, respectively. The time derivative of this Lyapunov candidate along the trajectory of system (8) is

$$\begin{aligned} \dot{V}(t)=&2E^{\mathrm{T}}(t)P\dot{E}(t)-\tau\int_{t-\tau}^{t}\dot{E}^{\mathrm{T}}(s)R\dot{E}(s)ds \\ &+\tau^2\dot{E}^{\mathrm{T}}(t)R\dot{E}(t) \\ =&2E^{\mathrm{T}}(t)P(\Phi E(t)-\Phi_2\xi(t)) \\ &-\tau\int_{t-\tau}^{t}\dot{E}^{\mathrm{T}}(s)R\dot{E}(s)ds \\ &+\tau^2(\Phi E(t)-\Phi_2\xi(t))^{\mathrm{T}}R(\Phi E(t)-\Phi_2\xi(t)). \end{aligned} \tag{13}$$

We can know that from Lemma 2

$$\begin{aligned} &\xi(t)^{\mathrm{T}}R\xi(t)\le\tau\int_{t-\tau}^{t}\dot{E}^{\mathrm{T}}(s)R\dot{E}(s)ds \\ &\Rightarrow-\tau\int_{t-\tau}^{t}\dot{E}^{\mathrm{T}}(s)\dot{E}(s)ds\le-\xi(t)^{\mathrm{T}}R\xi(t). \end{aligned} \tag{14}$$

So we have

$$\begin{aligned} \dot{V}(t)\le&\begin{bmatrix}\mathbf{E}(t)\\ \xi(t)\end{bmatrix}^{\mathrm{T}}\begin{bmatrix} P\Phi^{\mathrm{T}}+\Phi P & -P\Phi_2 \\ -{\Phi_2}^{\mathrm{T}}P & -R \end{bmatrix}\begin{bmatrix}\mathbf{E}(t)\\ \xi(t)\end{bmatrix} \\ &+\tau^2\begin{bmatrix}\mathbf{E}(t)\\ \xi(t)\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\Phi^{\mathrm{T}}\\ -{\Phi_2}^{\mathrm{T}}\end{bmatrix}R\begin{bmatrix}\Phi & -\Phi_2\end{bmatrix}\begin{bmatrix}\mathbf{E}(t)\\ \xi(t)\end{bmatrix}. \end{aligned} \tag{15}$$

The proof is completed. □

Remark 1: It is clear that $F_2\ge 0$ and $\tau^2>0$. So (11) maybe hold only when the $E_2<0$ holds. And (11) can hold when τ is sufficiently close to zero if $E_2<0$ holds. But there will be no solution for the inequality if the time-delays is too large.

Remark 2: By the Schur complement theorem, the condition that $E_2<0$ holds if and only if $P\Phi^{\mathrm{T}}+\Phi P<0$ holds. So it is worth pointing out that a necessary condition for (11) is that matrix Φ is Hurwitz.

3.2. Containment control of continuous-time multi-agent systems with multiple dynamic leaders

$$\begin{cases}\dot{x}_i(t)=v_i(t), \\ \dot{v}_i(t)=\begin{cases}0, & i\in\Re, \\ u_i(t), & i\in F.\end{cases}\end{cases} \tag{16}$$

Here we propose the following containment control protocol:

$$\begin{aligned} u_i(t)=&\sum_{j\in N_i^x(t)}a_{ij}[\alpha(x_j(t-\tau)-x_i(t-\tau))] \\ &+\sum_{j\in N_i^v(t)}b_{ij}[\beta(v_j(t-\tau)-v_i(t-\tau)). \end{aligned} \tag{17}$$

And it is defined as follows:

$$\begin{aligned} \dot{V}_{\mathrm{F}}(t)=&\mathbf{L}_{\mathrm{x1}}\mathbf{X}_{\mathrm{F}}(t-\tau)+\mathbf{L}_{\mathrm{x2}}\mathbf{X}_{\mathrm{L}}(t-\tau) \\ &+\mathbf{L}_{\mathrm{v1}}V_{\mathrm{F}}(t-\tau)+\mathbf{L}_{\mathrm{v2}}V_{\mathrm{L}}(t-\tau). \end{aligned}$$

Using the same containment error function $E_x(t),\ E_v(t)$ in section 3.1.1, one can have

$$\begin{aligned} \dot{E}_x(t)&=\dot{\mathbf{X}}_{\mathrm{F}}(t)+{\mathbf{L}_{\mathrm{x1}}}^{-1}\mathbf{L}_{\mathrm{x2}}\dot{\mathbf{X}}_{\mathrm{L}}(t) \\ &=V_{\mathrm{F}}(t)+{\mathbf{L}_{\mathrm{x1}}}^{-1}\mathbf{L}_{\mathrm{x2}}V_{\mathrm{L}}(t) \\ &=V_{\mathrm{F}}(t)+{\mathbf{L}_{\mathrm{v1}}}^{-1}\mathbf{L}_{\mathrm{v2}}V_{\mathrm{L}}(t)=E_v(t), \\ \dot{E}_v(t)&=\dot{V}_{\mathrm{F}}(t)-\dot{V}_{\mathrm{L}}(t)=\dot{V}_{\mathrm{F}}(t)-0 \\ &=\mathbf{L}_{\mathrm{x1}}E_x(t-\tau)+\mathbf{L}_{\mathrm{v1}}E_v(t-\tau). \end{aligned} \tag{18}$$

Under protocol (16), the multi-agent systems (1) can be transformed as follow

$$\dot{E}(t)=\Phi E(t)-\Phi_2\xi(t). \tag{19}$$

Theorem 2: Consider the double integrator containment problem for system (16). Let the position information be shared along the communication network G_x and the velocity information along G_v . Protocol (17) achieves containment control asymptotically if Assumption 1 holds and time-delay τ satisfying

$$\Omega+\tau^2\Psi<0, \tag{20}$$

where

$$\Omega=\begin{bmatrix} P\Phi^{\mathrm{T}}+\Phi P & -P\Phi_2 \\ -{\Phi_2}^{\mathrm{T}}P & -R \end{bmatrix},$$

$$\Psi=\begin{bmatrix} \Phi^{\mathrm{T}} \\ -{\Phi_2}^{\mathrm{T}} \end{bmatrix} R \begin{bmatrix} \Phi & -\Phi_2 \end{bmatrix}.$$

Proof: The proof procedure is quite similar to the proof of Theorem 1, and we omit it due to space limitation. □

3.3. Containment control of discrete-time multi-agent systems

In this section, we investigate the containment control problem of discrete-time multi-agent systems with fixed time-delays over heterogeneous networks. As same as in previous section, we assume that the position information be shared along the communication network G_x and the velocity information along G_y . Consider a group of n autonomous agents with double-integrator kinematics given by

$$\begin{aligned} x_i(k+1) &= x_i(k)+\varepsilon v_i(k), \\ v_i(k+1) &= v_i(k)+\begin{cases} 0, & i\in\Re, \\ \varepsilon u_i(k), & i\in F, \end{cases} \end{aligned} \tag{21}$$

where $x_i(k)\in\mathbb{R}^n$, $v_i(k)\in\mathbb{R}^n$ and $u_i(k)\in\mathbb{R}^n$ are, respectively, the position, velocity and the control input of the ith agent and the $\varepsilon>0$ is step-size. Here we propose the following containment control protocol.

$$u_i(k) = -v_i(k)+\ \chi\Delta_i^x\ [\sum_{j=1}^n a_{ij}[(x_j(k-\tau)-x_i(k))], \tag{22}$$

where $\Delta_i^x = 1/(\sum_{j\in N_i^x} a_{ij}), \Delta_i^v = 1/(\sum_{j\in N_i^v} b_{ij})$. Then, (22) can be rewritten as follows:

$$\begin{aligned} u_i(k) =&\chi\Delta_i^x[\sum_{j=1}^n a_{ij}((x_j(k-\tau)-x_i(k-\tau)) \\ &+(x_i(k-\tau)-x_i(k)))]-v_i(k) \\ &+\chi\Delta_i^v[\sum_{j=1}^n b_{ij}(v_j(k-\tau)-v_i(k-\tau))]. \end{aligned} \tag{23}$$

Furthermore, one can have

$$\begin{aligned} \mathbf{U}_{\mathrm{F}}(k) =& -\mathbf{V}_{\mathrm{F}}(k)+\chi\mathbf{X}_{\mathrm{F}}(k)-\chi\mathbf{X}_{\mathrm{F}}(k-\tau)] \\ &-\chi[\Delta_i^x\mathbf{L}_x\mathbf{X}(k-\tau)+\Delta_i^v\mathbf{L}_v\mathbf{V}(k-\tau)] \\ =& -\chi[\bar{\mathbf{L}}_{x1}\mathbf{X}_{\mathrm{F}}(k-\tau)+\bar{\mathbf{L}}_{x2}\mathbf{X}_{\mathrm{L}}(k-\tau) \\ &+\bar{\mathbf{L}}_{v1}\mathbf{V}_{\mathrm{F}}(k-\tau)+\bar{\mathbf{L}}_{v2}\mathbf{V}_{\mathrm{L}}(k-\tau)]-\mathbf{V}_{\mathrm{F}}(k) \\ &+\chi\mathbf{I}_{n-m}\mathbf{X}_{\mathrm{F}}(k)-\chi\mathbf{I}_{n-m}\mathbf{X}_{\mathrm{F}}(k-\tau), \end{aligned} \tag{24}$$

where the matrix $\bar{\mathbf{I}}_{n-m}$ is defined asdiag(Δ_i) and the matrix $\bar{\mathbf{L}}=[\bar{l}_{ij}]\in\mathbb{R}^{n\times n}$ of the directed graph is defined as $\bar{l}_{ij}$=$\Delta_i l_{ij}$. As same as the matrix $\mathbf{L}$, the matrix $\bar{\mathbf{L}}$ can be also partitioned as

$$\bar{\mathbf{L}}=\begin{bmatrix} \bar{\mathbf{L}}_1 & \bar{\mathbf{L}}_2 \\ 0_{m\times(n-m)} & 0_{m\times m} \end{bmatrix}.$$

About the matrix $\bar{\mathbf{L}}_1$ and $\bar{\mathbf{L}}_2$ we have such two lemmas:

Lemma 4: The matrix $\bar{\mathbf{L}}_{\mathrm{x1}}$ (resp.$\bar{\mathbf{L}}_{\mathrm{v1}}$) is also invertible if the matrix $\mathbf{L}_1$ is invertible.

Proof: See Appendix A.1. □

Lemma 5: $\bar{\mathbf{L}}_1^{-1}\bar{\mathbf{L}}_2 = {\mathbf{L}_1}^{-1}\mathbf{L}_2$

Proof: See Appendix A.2. □

From Lemma 4 and Lemma 5, the following results hold true.

$$\bar{\mathbf{L}}_{\mathrm{x1}}^{-1}\bar{\mathbf{L}}_{x2} = {\mathbf{L}_{\mathrm{x1}}}^{-1}\mathbf{L}_{x2},\ \ \bar{\mathbf{L}}_{\mathrm{v1}}^{-1}\bar{\mathbf{L}}_{v2} = {\mathbf{L}_{\mathrm{v1}}}^{-1}\mathbf{L}_{v2}.$$

Let

$$\begin{aligned} H=&\begin{bmatrix} \mathbf{I}_{n-m} & \varepsilon\mathbf{I}_{n-m} \\ -\chi\varepsilon\mathbf{I}_{n-m} & (1-\varepsilon)\mathbf{I}_{n-m} \end{bmatrix}, \\ \mathbf{B}_1=&\begin{bmatrix} \mathbf{0}_{n-m} & \mathbf{0}_{n-m} \\ \chi\varepsilon(\mathbf{I}_{n-m}-\bar{\mathbf{L}}_{\mathrm{x1}}) & -\chi\varepsilon\bar{\mathbf{L}}_{\mathrm{v1}} \end{bmatrix}, \\ \mathbf{B}_2=&\begin{bmatrix} \mathbf{0}_{n-m} & \mathbf{0}_{n-m} \\ -\chi\varepsilon\bar{\mathbf{L}}_{x2} & -\vec{k}\varepsilon\bar{\mathbf{L}}_{v2} \end{bmatrix}. \end{aligned}$$

Under protocol(22), the multi-agent system(21) can be written in a compact form as follows:

$$\begin{aligned} \begin{bmatrix} X_{\mathrm{F}}(k+1) \\ V_{\mathrm{F}}(k+1) \end{bmatrix} =&H\begin{bmatrix} X_{\mathrm{F}}(k) \\ V_{\mathrm{F}}(k) \end{bmatrix} \\ &+\mathbf{B}_1\begin{bmatrix} X_{\mathrm{F}}(k-\tau) \\ V_{\mathrm{F}}(k-\tau) \end{bmatrix}+\mathbf{B}_2\begin{bmatrix} X_{\mathrm{L}}(k) \\ V_{\mathrm{L}}(k) \end{bmatrix}. \end{aligned} \tag{25}$$

Let

$$\begin{aligned} \mathbf{E}_x(k) =&X_{\mathrm{F}}(k)+{\mathbf{L}_{\mathrm{x1}}}^{-1}\mathbf{L}_{\mathrm{x2}}X_{\mathrm{L}}(k), \\ \mathbf{E}_v(k) =&V_{\mathrm{F}}(k)+{\mathbf{L}_{\mathrm{v1}}}^{-1}\mathbf{L}_{\mathrm{v2}}V_{\mathrm{L}}(k), \\ \mathbf{E}(k) =& [\mathbf{E}_x(k)\ \ \mathbf{E}_v(k)]^{\mathrm{T}}. \end{aligned} \tag{26}$$

Then, (24) can be converted to error system as follows:

$$\begin{aligned} &\mathbf{E}(k+1) \\ =&H\mathbf{E}(k) \\ &+\begin{bmatrix} \mathbf{0}_{n-m} & \mathbf{0}_{n-m} \\ -\chi\varepsilon(\bar{\mathbf{L}}_{\mathrm{x1}}-\mathbf{I}_{n-m}) & -\chi\varepsilon\bar{\mathbf{L}}_{\mathrm{v1}} \end{bmatrix}\mathbf{E}(k-\tau). \end{aligned} \tag{27}$$

Let

$$\tilde{B}=\begin{bmatrix} \mathbf{0}_{n-m} & \mathbf{0}_{n-m} \\ -(\bar{\mathbf{L}}_{\mathrm{x1}}-\mathbf{I}_{n-m}) & -\bar{\mathbf{L}}_{\mathrm{v1}} \end{bmatrix},\ \ \mathbf{B}=\chi\varepsilon\tilde{B}. \tag{28}$$

Then, (27) can be rewritten

$$\mathbf{E}(k+1) = \mathbf{H}\mathbf{E}(k)+\mathbf{B}\mathbf{E}(k-\tau). \tag{29}$$

Lemma 6: For $0<\chi<1$, then $\rho(\mathbf{H})$ <1 , where $\rho(\mathbf{H})$ represents the spectral radius of matrix $\mathbf{H}$.

Proof: See Appendix A.3. □

Lemma 7: If $\rho(\mathbf{H})$ <1 , then there exist positive constants $\mathrm{K}\geq 1$ and $0<\gamma<1$ such that $||\mathbf{H}||^k \leq \mathrm{K}\gamma^k, k\geq 0$.

Lemma 8: Inequality $y^{\tau+1}-\gamma y^{\tau}-l>0$ has at least one solution $y\in(\gamma,1)$ if $1-\gamma-l>0$ holds.

Proof: See Appendix A.4. □

Lemma 9: If Assumption 1 is satisfied in the interconnection graph G_x (resp. G_v), then Equation (15) has a unique equilibrium 0.

Proof: See Appendix A.5. □

Theorem 3: If the Assumption 1 is satisfied, then under control protocol (22), for any bounded time-delays, there exist some $\chi > 0$ such that the containment for system (21) is reached asymptotically.

Proof: From (29) , the following (30) can be derived

$$\mathbf{E}(k) = \mathbf{H}^k\mathbf{E}(0) + \sum_{s=0}^{k-1}\mathbf{H}^{k-1-s}\mathbf{B}\mathbf{E}(s-\tau). \tag{30}$$

For $0 < \chi < 1$, by Lemma 6, $\rho(\mathbf{H}) < 1$. Noticing that Lemma 7, there exist constants $\mathrm{K} \geq 1$ and $0 < \gamma < 1$ such that $||\mathbf{H}||^k \leq \mathrm{K}\gamma^k, k \geq 0$. Therefore by (30) , we have

$$\begin{aligned} ||\mathbf{E}(k)|| \leq& ||\mathbf{H}^k||.||\mathbf{E}(0)|| \\ &+ \sum_{s=0}^{k-1}||\mathbf{H}^{k-1-s}||.||\mathbf{B}||.||\mathbf{E}(s-\tau)||, \end{aligned} \tag{31}$$

$$\begin{aligned} ||\mathbf{E}(k)|| \leq& K\gamma^k||\mathbf{E}(0)|| \\ &+ \sum_{s=0}^{k-1}\mathrm{K}\gamma^{k-1-s}||\mathbf{B}||.||\mathbf{E}(s-\tau)||. \end{aligned} \tag{32}$$

For $(1-\gamma) - K||\mathbf{B}|| > 0$, by Lemma 8, there exists a positive constant $\hat{\lambda}$ satisfying $\gamma < \hat{\lambda} < 1$ such that

$$\hat{\lambda}^{\tau+1} - \gamma\hat{\lambda}^{\tau} - K||\mathbf{B}|| > 0.$$

In the following, we aim to show that

$$||\mathbf{E}(k)|| \leq K||\varphi||\hat{\lambda}^k, \;\; k \geq 0, \tag{33}$$

where $||\varphi|| = sup_{k\in[-\tau,0]}||\mathbf{E}(k)|$. It is clear that

$$||\mathbf{E}(k)|| \leq K||\varphi||\hat{\lambda}^k, \;\; k \in [-\tau, 0]. \tag{34}$$

Thus, we first show for any $\eta > 1$,

$$||\mathbf{E}(k)|| < \eta K||\varphi||\hat{\lambda}^k \triangleq \phi(k), \;\; k \geq 0. \tag{35}$$

If (35) is not true, then there must exist a $k^* > 0$ such that

$$||\mathbf{E}(k)|| < \phi(k), \;\; k \in [0, k^*), \;\; ||\mathbf{E}(k^*)|| = \phi(k^*). \tag{36}$$

By (32), one can obtain that

$$\begin{aligned} \phi(k^*) =& ||\mathbf{E}(k^*)|| \leq K\gamma^{k^*}||\mathbf{E}(0)|| \\ &+ \sum_{s=0}^{k^*-1}(K\gamma^{k^*-1-s}||\mathbf{B}|| \cdot ||\mathbf{E}(s-\tau)||) \\ <& K\gamma^{k^*}\eta||\varphi|| \cdot 1 \\ &+ \sum_{s=0}^{k^*-1}(K\gamma^{k^*-1-s}||\mathbf{B}|| \cdot \eta \cdot K \cdot ||\varphi|| \cdot \hat{\lambda}^{s-\tau}) \\ =& K\gamma^{k^*}\eta||\varphi||[1 + \frac{K||\mathbf{B}||}{\gamma\lambda^{\tau}}\sum_{s=0}^{k^*-1}(\hat{\lambda}/\gamma)^s] \\ =& K\gamma^{k^*}\eta||\varphi||[1 + \frac{K||\mathbf{B}||}{\gamma\hat{\lambda}^{\tau}} \times \frac{1-(\hat{\lambda}/\gamma)^{k^*}}{1-(\hat{\lambda}/\gamma)}] \\ =& K\eta||\varphi||[\gamma^{k^*} + \frac{K||\mathbf{B}||}{\hat{\lambda}^{\tau}} \times \frac{\gamma^{k^*}-\hat{\lambda}^{k^*}}{\gamma-\hat{\lambda}}] \\ =& K\eta||\varphi||[\gamma^{k^*} + \frac{K||\mathbf{B}||}{\hat{\lambda}^{\tau}(\hat{\lambda}-\gamma)} \times (\hat{\lambda}^{k^*}-\gamma^{k^*})] \\ <& K\eta||\varphi||[\gamma^{k^*} - (\gamma^{k^*}-\hat{\lambda}^{k^*})] \\ =& K\eta||\varphi||\hat{\lambda}^{k^*} = \phi(k^*), \end{aligned}$$

which is a contradiction. Thus, for any $\eta > 1$, equation (35) holds. Let $\eta \to 1$, then (33) holds. So if Assumption 1 is satisfied in graph and respectively, for $0 < \chi < \min\{1, \frac{(1-\gamma)}{\varepsilon K||\tilde{\mathbf{B}}||}\}$, as for any initial condition invoking Lemma 9, there exist some $\chi > 0$ such that (29) is reached stable asymptotically. That means $\lim_{k\to\infty}\mathbf{E}(k) = 0$

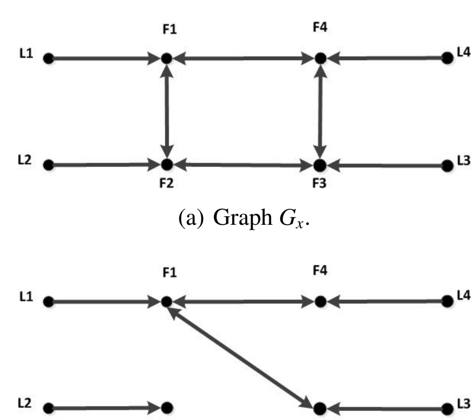
$$\Leftrightarrow \begin{cases} \lim_{k\to\infty}\mathbf{X}_F(k) = -\mathbf{L}_{\mathrm{x}1}{}^{-1}\mathbf{L}_{\mathrm{x}2}\mathbf{X}_{\mathrm{L}}(k), \\ \lim_{k\to\infty}V_F(k) = -\mathbf{L}_{\mathrm{v}1}{}^{-1}\mathbf{L}_{\mathrm{v}2}V_{\mathrm{L}}(k). \end{cases}$$

According to Definition 2 and Lemma 1, the final positions (resp. velocity) of the followers are within the convex hull formed by the stationary leaders. The proof is completed. □

4. NUMERICAL SIMULATIONS

In this section, we present several simulation results to validate the previous theoretical results. We consider a group of agents with 4 leaders and 4 followers. When the directed graph G_x and G_v are fixed as in Fig. 1(a) and (b), respectively. It can be noted that Assumption 1 is satisfied in graph G_x and graph G_v.

Example 1: The corresponding Laplacian submatrix as



(a) Graph G_x.

(b) Graph G_v.

Fig. 1. Fixed undirected network topology.

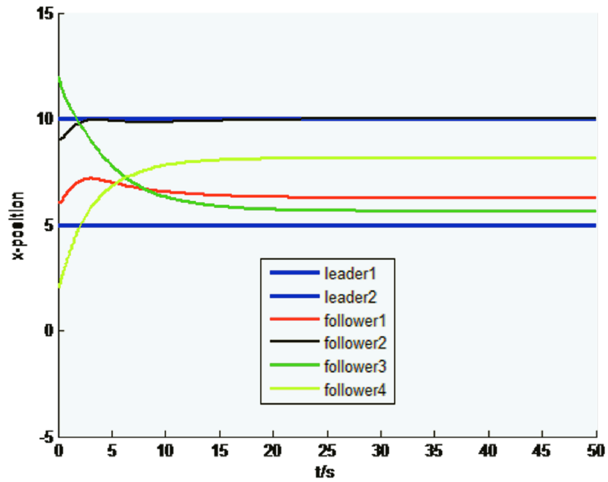


Fig. 2. State position trajectories of system (1) ($\tau = 0.1$).

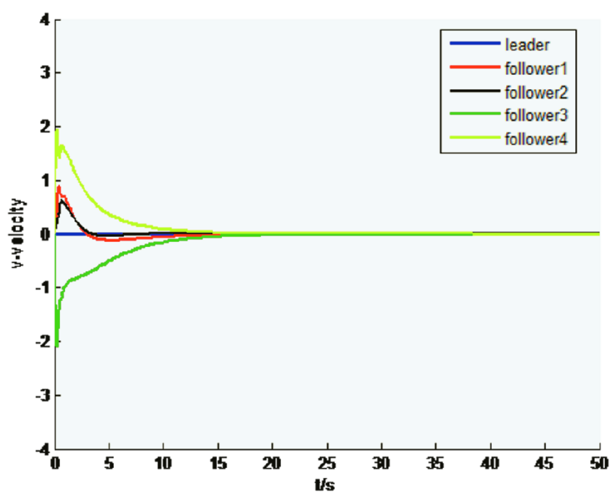


Fig. 3. State velocity trajectories of system (1)($\tau = 0.1$).

follows:

$$L_{x1} = \begin{bmatrix} 3 & -1 & 0 & -1 \\ -1 & 3 & -1 & 0 \\ 0 & -1 & 3 & -1 \\ -1 & 0 & -1 & 3 \end{bmatrix}, \quad L_{v1} = \begin{bmatrix} 3 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 2 & 0 \\ -1 & 0 & 0 & 2 \end{bmatrix}.$$

We choose the scalar $\alpha = 1$, $\beta = 2$. By using Matlab Toolbox, we can get **P**.

Solving the LMIs(11) by using the GEVP solver in Matlab's LMI Toolbox, we can get $alpha = 100.21$ and $\tau_{op} = 0.10$. The simulation result using (2) for (1) is shown in Figs. 2-4 when $\tau = 0.1, \tau = 0.2$ respectively.

We can see that position and velocity states of all followers ultimately converge to the convex hull formed by the leaders when the time-delays τ=0.1. In Theorem 2, the velocity of leaders is not zero. One assumes that $v_i(t) = 3, i \in \Re$. The simulation result as Fig. 5.

Example 2: For the discrete-time multi-agent system,

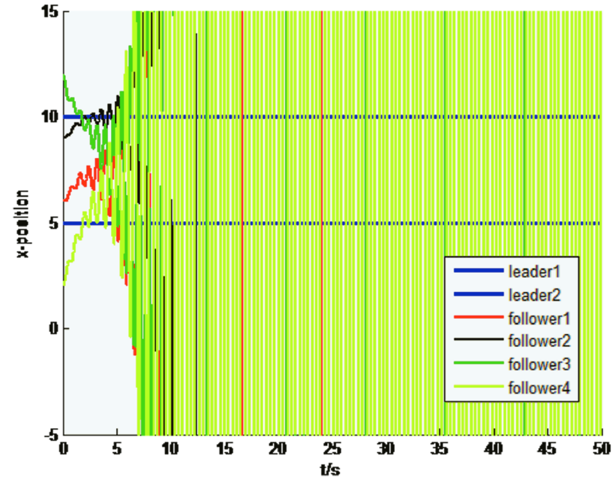


Fig. 4. State x-position trajectories of system (1) ($\tau = 0.2$).

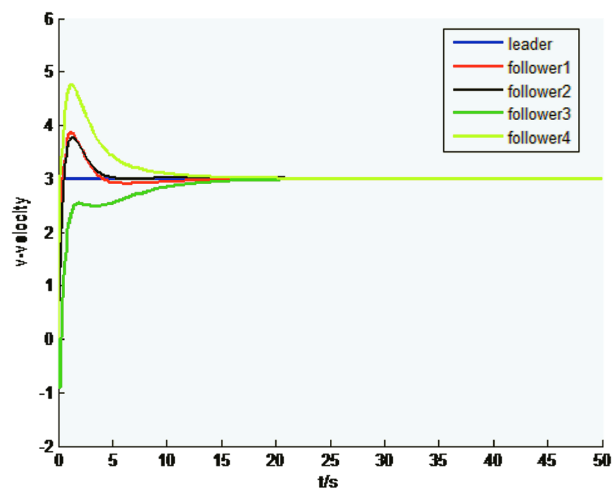


Fig. 5. State x-position trajectories of system (16) ($\tau = 0.1$).

let $\varepsilon = 0.01$. We choose control scale $\chi = 0.8$ $(0 < \chi < 1)$. The simulation result using protocol (21) for system (20) is shown in Fig. 6, Fig. 7, Fig. 8, and Fig. 9, respectively.

We can see that all position states of all followers ultimately converge to the convex hull formed by the stationary leaders when the time-delays $\tau = 10$ (resp. $\tau = 50$). The velocity states of all followers converge to zero. Furthermore, comparing Figs. 6-7 and Figs. 8-9, we can know that the convergence rate will decrease when the time-delays increase.

5. CONCLUSION

This paper studied the distributed containment control problem of second-order multi-agent systems with fixed communication time-delays under heterogeneous network

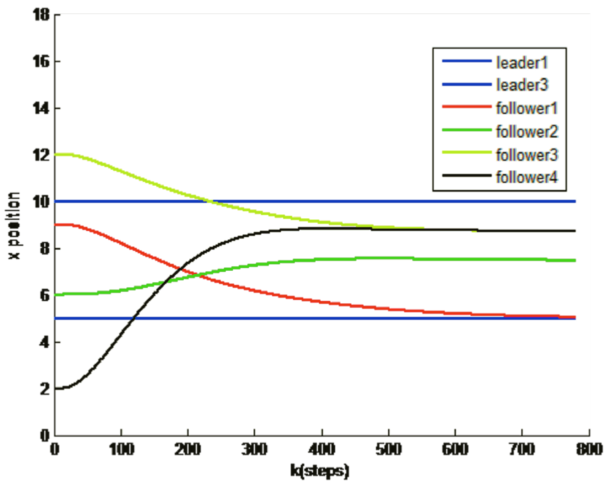


Fig. 6. Position trajectories of system (21) ($\tau = 10$).

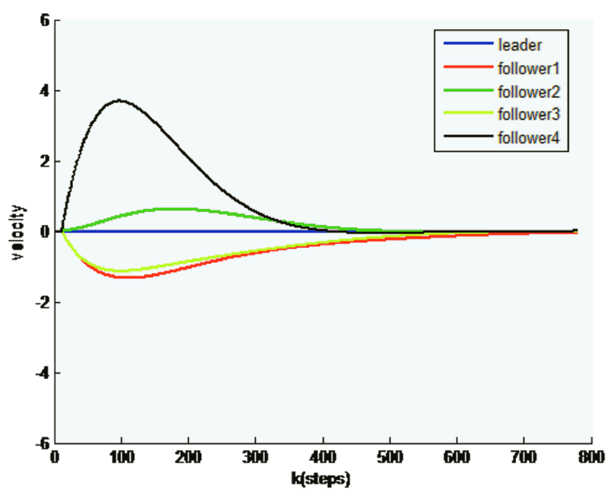


Fig. 7. Velocity trajectories of system (21) ($\tau = 10$).

topologies. Containment control of continuous-time and discrete-time multi-agent systems with communication time-delays are investigated, respectively. We showed sufficient conditions on the networks topology and time-delays to guarantee distributed containment control. There are still a number of related interesting problems deserving further investigation. Containment control problems for heterogeneous multi-agent systems or multi-agent systems with varying time-delays will be discussed in our future work. Meanwhile, it is desirable to study cooperative containment control of linear multi-agent with sampled data.

APPENDIX A

A.1. Proof of Lemma 4

Proof: Since $\bar{l}_{ij} = \Delta_i l_{ij}$from the definition of the matrix$\bar{\mathbf{L}}_x$. And from that the matrix $\mathbf{L}_1$ is invertible,

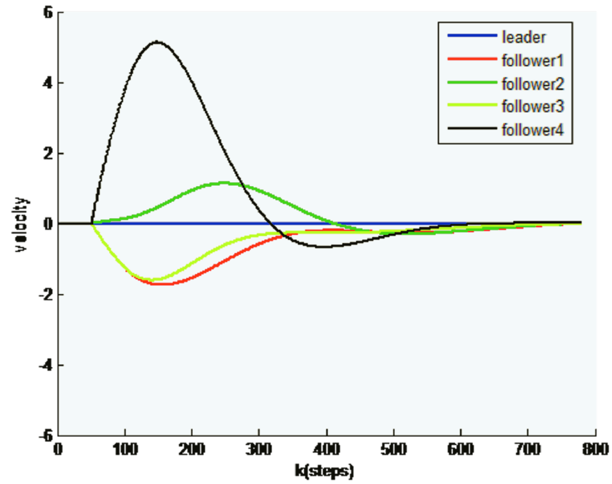


Fig. 8. Velocity trajectories of system (21) ($\tau = 50$).

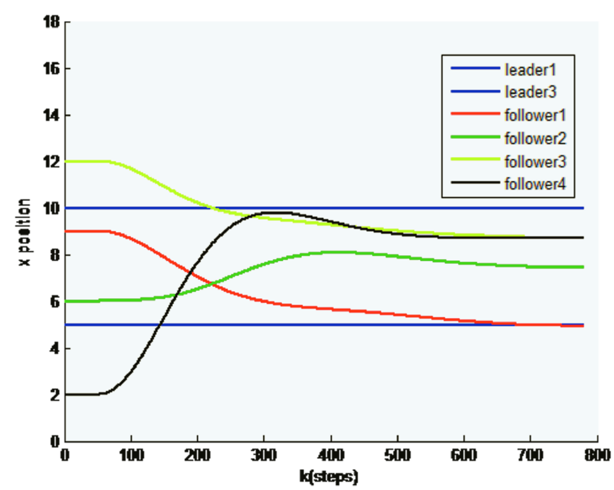


Fig. 9. Position trajectories of system (21) ($\tau = 50$).

we can know thatdet$(\mathbf{L}_1) > 0$ .And we have $\Delta_i^x > 0$, so we have $\det(\bar{\mathbf{L}}_{x1}) = \prod_{i=1}^{n-m} \Delta_i^x \det(\mathbf{L}_1) > 0$. The proof is completed. □

A.2. Proof of Lemma 5

Proof: That had been done in [26], and we omit it due to space limitation. □

A.3. Proof of Lemma 6

Proof: Let z be any eigenvalue of the matrix $\mathbf{H}$, note that

$$\begin{aligned}
&\det(z\mathbf{I}_{2(n-m)} - \mathbf{H}) \\
&= \det(\begin{bmatrix} (z-1)\mathbf{I}_{n-m} & -\varepsilon\mathbf{I}_{n-m} \\ \chi\varepsilon\mathbf{I}_{n-m} & (z-1+\varepsilon)\mathbf{I}_{n-m} \end{bmatrix}) \\
&= \det(z^2 + (\varepsilon-2)z + 1 - \varepsilon + \varepsilon^2\chi).
\end{aligned}$$

One can have

$$\vec{z}_{i1} = \frac{(2-\varepsilon)+\varepsilon\sqrt{1-\chi}}{2},$$
$$\vec{z}_{i2} = \frac{(2-\varepsilon)-\varepsilon\sqrt{1-\chi}}{2}, \quad (i=1,2,\ldots,n-m).$$

One can easily verify that $|z_i| < 1 \Leftrightarrow \rho(\mathbf{H}) < 1$ when $0 < \chi < 1$ holds. The proof is completed. □

A.4. Proof of Lemma 8

Proof: Let $p(y) = y^{\tau+1} - \gamma y^{\tau} - l$, then

$$p'(y) = (\tau+1)y^{\tau} - \gamma\tau y^{\tau-1} = y^{\tau-1}((\tau+1)y - \gamma\tau).$$

Let $q(y) = (\tau+1)y - \gamma\tau$, then one can have $q'(y) = \tau + 1 > 0$, so we know that

$$q(y) > q(\gamma) = (\tau+1)\gamma - \gamma\tau = \gamma > 0, \quad y \in (\gamma, 1).$$

Therefore, $p'(y) = y^{\tau-1}q(y) > 0$, $y \in (\gamma,1)$. It means that $p(y)$ is monotonically increasing for $y \in (\gamma,1)$. Since $p(1) = 1-\gamma-l > 0$, so there exists at least a $y \in (\gamma,1)$ such that $y^{\tau+1} - \gamma y^{\tau} - l > 0$. The proof is completed. □

A.5. Proof of Lemma 9

Proof: It suffices to verify that$(\mathbf{I}_{2(n-m)} - \mathbf{H} - \mathbf{B})\mathbf{E}(k)$ =0, that is

$$\mathbf{I}_{2(n-m)} - \mathbf{H} - \mathbf{B} = \begin{bmatrix} \mathbf{0}_{n-m} & -\varepsilon\mathbf{I}_{n-m} \\ \chi\varepsilon\bar{\mathbf{L}}_{x1} & \varepsilon\mathbf{I}_{n-m} + \chi\varepsilon\bar{\mathbf{L}}_{v1} \end{bmatrix}. \quad (A.1)$$

And

$$\begin{bmatrix} \mathbf{0}_{n-m} & -\varepsilon\mathbf{I}_{n-m} \\ \chi\varepsilon\bar{\mathbf{L}}_{x1} & \varepsilon\mathbf{I}_{n-m} + \chi\varepsilon\bar{\mathbf{L}}_{v1} \end{bmatrix} \begin{pmatrix} E_x(k) \\ E_v(k) \end{pmatrix} = 0. \quad (A.2)$$

From (A.2), it is clear that both $E_v = 0$ and $\chi\varepsilon\bar{\mathbf{L}}_{x1}e_x = 0$ hold. And from Lemma 4, $\det(\bar{\mathbf{L}}_{x1}) > 0 \Rightarrow E_x = 0$. So the result as follow holds true.

$$E = \begin{pmatrix} E_x \\ E_v \end{pmatrix} = 0.$$

It means that (29) has a unique solution 0 if Assumption 1 is satisfied ($\chi \neq 0, \varepsilon \neq 0$). □

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