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Adaptive Synchronization for a Class of Fractional Order Time-delay Uncertain Chaotic Systems via Fuzzy Fractional Order Neural Network

Xingpeng Zhang, Xiaohong Zhang, Dong Li*, and Dan Yang

Abstract: Uncertainty and delay are common phenomena in chaotic systems, but their existence will increase the difficulty of synchronization. For the sake of actualizing synchronization of fractional order time-delay uncertain chaotic systems, we propose an adaptive fractional order fuzzy neural network synchronization scheme based on the linear matrix inequalities. A fractional order radial basis functions neural network is applied to approximate uncertainties. According to the output of the neural network, we design a general adaptive controller for fractional order time-delay uncertain chaotic systems with different topological structure. Furthermore, we propose an adaptive fractional order fuzzy neural network by introducing fuzzy rules into the network. Then the fractional order extension of the Lyapunov direct method is utilized to demonstrate the stability of the error systems under the adaptive controller. Finally, numerical simulations are conducted to verify the effectiveness of the conclusions.

Keywords: Chaotic system, fractional order, neural network, time-delay, uncertain.

1. INTRODUCTION

Fractional calculus has recently become the hot research direction in the nonlinear control field owing to not only its widely and meaningful applications in science and engineering but also the incomplete mathematical theory. In fact, fractional order description is more consistent with the true nature of the system in reality, while integer-order systems are only an idealized and simple representation. Theoretical and experimental analysis has proved that the known chaotic systems will also possess chaotic phenomena although the order is a fractional order [1, 2].

The classical stability theory is not applicable to fractional systems for its distinctive definition. Therefore, researchers have proposed new methods to analyze the stability of fractional order systems. Laplace transformation [3, 4] is the first proposed solution. Then a derivation method of Lyapunov stability theory is proposed to analyze the stability of fractional order systems, which is the fractional order extension of Lyapunov direct methods [5–8]. Therefore, the synchronization methods of integer order chaotic systems, such as impulsive method [9], fuzzy method [10, 11], intermittent method [12], sliding mode [13] and so on, can also be applied to fractional-order ones by combining the fractional order stability theory. Accordingly, researchers have proposed a large number of synchronization methods for fractional order systems, such as impulsive synchronization [6], sliding mode synchronization [14], fuzzy synchronization [15], projective synchronization [16], adaptive synchronization [17] and so on.

Uncertainty and time-delay are two familiar phenomena in nonlinear systems, which increase the complexity and difficulty in realizing synchronization of the real systems. Even so, researches have designed a large number of synchronization methods for systems with uncertainties [18–20] and time-delay [21–23].

Based on the fractional order stability theory, many synchronization methods for fractional order chaotic systems with uncertainties are proposed. The methods widely used in practice are impulsive [24], sliding mode [25], adaptive [26], fuzzy [26], projective [27], adaptive neural network [28, 29], fuzzy neural network [30, 31] and so on. Time-delay emerges due to the finite speed of transmission and speeding as well as congestions [32]. It has been reported that the existence of time-delay may destroy synchronization [33]. Hence, the synchronization of chaotic systems with time-delay was studied in some literature [32–36].

And in fact, time-delay and uncertainty sometimes occur together, which makes the system more complex and

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Xingpeng Zhang, Xiaohong Zhang, and Dan Yang are with the School of Big Data and Software Engineering, Chongqing University, Chongqing 400030, China (e-mails: xpzhang@cqu.edu.cn, xhongz@cqu.edu.cn, dyang@cqu.edu.cn). Dong Li is with the College of Mathematics and Statistics, Chongqing University, Chongqing 400030, China (e-mail: donglicqu@gmail.com, Lid@cqu.edu.cn)
* Corresponding author.

difficult to control. It is inevitable to study the synchronization of fractional chaotic systems. Therefore, researches pay attention to the synchronization of fractional order time-delay uncertain chaotic systems [37–41]. Sliding mode method [37, 38], adaptive sliding robust mode control [39], adaptive fuzzy projective control [40], adaptive impulsive method [41] and integer order RBF neural network [42] and proposed.

Among these methods, some have unique characteristics and can be aggregated to achieve simpler and easier control [24, 28, 30, 38–41]. The radial basis function neural network has been theoretically proved to be able to approximate arbitrary continuous functions with arbitrary precisions, which is a good choice for solving uncertainties. And the adaptive method can let the network and system can easy learn the parameters. The fuzzy method is widely used because of its simple implementation and good effect on nonlinear functions. And the study on fractional order time-delay uncertain chaotic systems is worth doing further research to propose more methods for practical use.

All the above observations motivate us to carry out the present study: based on the fractional order extension of Lyapunov direction method and linear matrix inequality, we propose an adaptive fractional order neural network and adaptive fuzzy fractional order neural network to realize the synchronization of fractional order time-delay uncertain chaotic systems. Our proposed methods take full advantage of the fractional order neural network, adaptive method, and fuzzy method. We adopt refinement control, that is, each sub-controller has its own unique control function. This design not only makes the controller easy to understand but also facilitates the design in practice. Our study proposes the fractional order neural network for fractional order chaotic systems, which fill up the shortcomings of the fractional order neural network synchronization method.

Compared with other methods, our adaptive controller is subdivided into six sub-controllers corresponding to uncertain and time-delay terms. The first sub-controller is a gain controller whose primary role is to set the gain matrix and eliminate useless information. The second and third sub-controller are adaptive controllers corresponding to the unknown Lipschitz constants. The fourth sub-controller is a fractional order neural network controller, which is used to control the unknown nonlinear dynamics based on the neural network approximation. The last two sub-controller are used to control the effect of uncertainties corresponding to the nonlinear time-delay term.

In summary, our contributions mainly include the following three aspects. Firstly, our proposed adaptive fractional order neural network controller is applicable to all fractional order time-delay uncertain chaotic systems. Secondly, the structure of multiple sub-controllers makes the adaptive controller easier to understand and design. Thirdly, the design of fractional order neural network, adaptive law, and stability theory closer to the characteristics of the system itself.

The rest of the paper is organized as follows: In Section 2, basic definition, essential lemma and the model of fractional order time-delay uncertain chaotic systems are introduced. In Section 3, an adaptive controller is designed based on the output of the fractional order radial basis function neural network. In Section 4, we propose a new controller by introducing fuzzy logic rules into the fractional order radial basis function neural network. Then, the numerical simulations are given to illustrate the effectiveness of our theoretical results in Section 5.

2. PRELIMINARIES

In this section, we give some basic definitions, notations, and preliminary results.

2.1. Definition and lemma

Till now, there are many different definitions of fractional derivatives, that is, Riemann-Liouville, Grünwald-Letnikov and Caputo definitions. In this paper, we adopt the Caputo definition for the reason that the proof of Lemma 2 and Lemma 3 is based on this definition.

Definition 1 (The Fractional Derivative Caputo Right Hand Definition) [24, 32]**:** The Caputo fractional derivative of order q of a continuous function $f: R^+ \to R$ is defined as follows.

$$ {}_0^C D_t^q f(t) = \frac{1}{\Gamma(m)} \int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{q+1-m}} d\tau, \tag{1} $$

where $m-1 < q < m$, $\Gamma()$ is the gamma function, satisfying $\Gamma(z) = \int_o^\infty e^{-t} t^{z-1} dt$ and $\Gamma(z+1) = z\Gamma(z)$.

Lemma 1 (Fractional-order extension of the Lyapunov direct method) [5]**:** Let $x = 0$ be an equilibrium point for the non-autonomous fractional order system ${}_0^C D_t^q x(t) = f(x,t)$. Assume that there exists a Lyapunov function $V(x,t)$ and class-K functions $\gamma_i, i = 1,2,3$, satisfying

$$ \begin{aligned} &\gamma_1 \|x(t)\| \le V(x,t) \le \gamma_2 \|x(t)\|, \\ &{}_0^C D_t^q V(x,t) \le -\gamma_3 \|x(t)\|, \end{aligned} \tag{2} $$

where $0 < q < 1$. Then fractional order system ${}_0^C D_t^q x(t) = f(x,t)$ is asymptotically stable.

Lemma 2 [7]**:** Let $x(t) \in R$ be a continuous and derivable function. Then, for any time instant $t > t_0$

$$ \frac{1}{2} {}_{t_0}^C D_t^q x^2(t) \le x(t) {}_0^C D_t^q x(t). \tag{3} $$

If $x(t) \in R^n$, the inequality satisfies

$$ \frac{1}{2} {}_{t_0}^C D_t^q (x^T(t) x(t)) \le x^T(t) {}_0^C D_t^q x(t). \tag{4} $$

Because we use linear matrix inequalities in the following proofs, we make little modifications to Lemma 2.

Lemma 3: Let $x(t) \in R^n$ be a continuous and derivable function vector. Then, for any time instant $t > t_0$,

$$ {}_{t_0}^{C}D_t^q(x^T(t)x(t)) \le x^T(t){}_0^CD_t^q x(t) + ({}_0^CD_t^q x(t))^T x(t). \tag{5}$$

The proof is straightforward, decomposing the expression (4) into two parts and applying Lemma 2.

2.2. Fractional order time-delay uncertain chaotic systems

In this paper, we consider a class of fractional order time-delay uncertain chaotic systems with different topological structure, where the drive and response systems are shown as follows:

$$\begin{aligned} {}_{t_0}^{C}D_t^q x(t) =& Ax(t) + f(x(t)) + Cx(t-\tau) \\ &+ F(x(t), x(t-\tau)), \end{aligned} \tag{6}$$

and

$$\begin{aligned} {}_{t_0}^{C}D_t^q y(t) =& By(t) + g(y(t)) + Dy(t-\tau) \\ &+ G(y(t), y(t-\tau)) + \triangle By(t) + \triangle g(y(t)) \\ &+ d(t) + E(y(t), y(t-\tau)) + u(t), \end{aligned} \tag{7}$$

where $x(t), y(t) \in R^n$ are the system state vectors, $A, B \in R^{n\times n}$ are the constant matrix, $C, B \in R^{n\times n}$ are the constant matrix corresponding to time-delay. $f(x(t)), g(y(t)) \in R^n$ are the nonlinear valued vector functions, $F(x(t), x(t-\tau)), G(y(t), y(t-\tau)) \in R^n$ are nonlinear time-delay vector functions. Besides, $\triangle B$ and $\triangle g(t)$ denotes the linear and nonlinear uncertain terms respectively. τ denotes time-delay. $E(y(t), y(t-\tau)) \in R^n$ denotes the uncertain terms of time delay. $u(t) \in R^n$ is the control input.

Let $e(t) = y(t) - x(t)$ be the error between the drive system (6) and response system (7). Therefore, we realize the synchronization between systems (6) and (7), if we can let $lim_{t\to\infty} e(t) = 0$. The dynamical expression of the error system is obtained as follow.

$$\begin{aligned} {}_{t_0}^{C}D_t^q e(t) =& By(t) + (B-A)x(t) + g(y(t)) \\ &- f(x(t))De(t-\tau) + (D-C)x(t-\tau) \\ &+ \Phi(y) + \Psi(x, y, t, t-\tau) \\ &+ E(y(t), y(t-\tau)) + u(t), \end{aligned} \tag{8}$$

where $\Psi(x, y, t, t-\tau) = G(y(t), y(t-\tau)) - F(x(t), x(t-\tau))$ and $\Phi(y) = \triangle By(t) + \triangle g(y(t)) + d(t)$.

2.3. Assumptions

In addition, we give three assumptions commonly used in nonlinear areas.

Assumption 1: The nonlinear vectors of the chaotic system satisfy the Lipschitz condition, i.e., $\forall x(t), y(t) \in R^n$, there exists positive constants l_f and L_g such

$$\|f(y(t)) - f(x(t))\| \le L_f \|y(t) - x(t)\|,$$

$$\|g(y(t)) - g(x(t))\| \le L_g \|y(t) - x(t)\|. \tag{9}$$

The most known chaotic systems satisfy Assumption 1, for instance, Lorenz system, Chen system, Rössler system, Qi system and so forth. Generally, the Lipschitz' constants L_f and L_g usually are unknown.

The chaotic systems will also show chaotic phenomena when time-delay to occur in nonlinear terms [43]. It should be noted that delay terms are usually closely related to the linear terms and nonlinear terms of the system. In many time-delay chaotic systems, it is a direct substitution in mathematical form. Hence, the time-delay term should also satisfy the assumption similar to the Lipschitz conditions.

In addition, the time-delay function vector is also bounded which have been introduced in the paper [32]. Hence, two assumptions about time-delay are introduced.

Assumption 2 [32]**:** There exist two constant matrix H_1, H_2 such the time-delay term $\Psi(x, y, t, t-\tau)$ also satisfies following inequality.

$$e^T(t)\Theta(x, y, t, t-\tau) \le e^T(t)H_1e(t) + e^T(t)H_2e(t-\tau), \tag{10}$$

where $\Theta(x, y, t, t-\tau) = \Psi(x, y, t, t-\tau) - G(x(t), x(t-\tau) + F(y(t), y(t-\tau))$.

Assumption 3 [23, 32]**:** There are two unknown constants λ_1 and λ_2 such the uncertainties corresponding to time-delay $E(y(t), y(t-\tau)) \in R^n$ satisfies following inequality.

$$\|E(y(t), y(t-\tau))\| \le \lambda_1\|y(t)\| + \lambda_2\|y(t-\tau)\|. \tag{11}$$

Remark 1: Let D^q be the simplified form of ${}_{t_0}^{C}D_t^q$.

3. ADAPTIVE FRACTIONAL ORDER NEURAL NETWORK CONTROLLER DESIGN FOR FRACTIONAL ORDER TIME-DELAY UNCERTAIN CHAOTIC SYSTEMS

In this section, we design an adaptive fractional order neural network controller, which has several sub-controller corresponding to the adaptive parameter uncertain and time-delay terms.

3.1. Applying neural network to approximate the uncertainties

The radial basis function (RBF) neural network is a three-layer neural network with fast local convergence property. It is also theoretically proved to be able to approximate arbitrary continuous functions with arbitrary precision. Therefore, we apply the RBF neural network to approximate uncertainties $\Phi(y) = \triangle By(t) + \triangle g(t) + d(t)$ in error system (8).

The second layer of the RBF neural network is customarily Gauss function.

$$\phi_j(y) = \exp[\frac{-(y-\xi_j)^2}{\delta_j^2}], \ \ (j = 1, 2, \ldots, m), \tag{12}$$

where $y \in R^n$ is the input of the neural network, which is also the intermediate output of the response system. ξ_j is the center of the neural cell lying in the i-th hidden layer, and δ_j are the width. $\phi(y) = [\phi_1(y), \phi_2(y), \ldots, \phi_m(y)]^T$.

The third layer of RBF neural network is a linear output.

$$\widehat{\Phi}_i(y) = \sum_{j=1}^{m} w_{ij}^T \phi_j(y) = W_i^T \phi(y), \ \ (i = 1, 2, \ldots, n). \tag{13}$$

Then, the output is also the approximate value of $\Phi(y)$.

$$\widehat{\Phi}(y) = W^T \phi(y), \tag{14}$$

where $W = [W_1, W_2, \ldots, W_n] \in R^{m\times n}$, and $W_i = [W_{i1}, W_{i2}, \ldots, W_{im}]^T$ is the weight vector fo the network.
We use optional weight value to evaluate the weight values W, which is defined as:

$$\widetilde{W} = arg \min_{W\in\Omega}[\sup_{y\in S_y}|\widehat{\Phi}(y/W) - \Phi(y)], \tag{15}$$

where $\Omega = \{W :\| W \le M\}$ is a valid field of the parameter, and M is a designed parameter. $S_y \subset R^n$ is an allowable set of the state vector.
Then, the optimal estimate of uncertain terms as

$$\Phi(y) = \widetilde{W}^T \phi(y) + \varepsilon(y), \tag{16}$$

where $\|\varepsilon(y)\| \le \bar{\varepsilon}$, $\bar{\varepsilon}$ is the minimum approach error of RBF neural network, which is given and can be ignored when it is small enough.

3.2. Designing adaptive controller

Based on the output of the fractional order RBF neural network, we design an adaptive controller, which also handle the unknown constants corresponding to Assumptions 1 and 3.

$$\begin{aligned} u(t) =& u_c(t) + u_{L_f}(t) + u_{L_g}(t) + u_W(t) + u_{\lambda_1}(t) \\ &+ u_{\lambda_2}(t), \end{aligned} \tag{17}$$

where $u_c(t)$ is the main component of controller; $u_{L_f}(t)$ and $u_{L_g}(t)$ are designed to control the Lipschitz constants L_f and L_g respectively; $u_W(t)$ is applied to control the weight of fractional order RBF neural network; $u_{\lambda_1}(t)$ and $u_{\lambda_2}(t)$ are designed to control the unknown positive constants λ_1 and λ_2 corresponding to delay uncertain term.

And each sub-controller has the following form:

$$\begin{aligned} u_c =& -Ke(t) + f(y(t)) - g(x(t)) - (B - A)x(t) \\ &- (D - C)x(t-\tau) - G(x(t), x(t-\tau) \\ &+ F(y(t), y(t-\tau)), \\ u_{L_f} =& -\gamma_f \hat{L}_f e(t), \\ u_{L_g} =& -\gamma_g \hat{L}_g e(t), \\ u_W =& -W^T \phi(y), \\ u_{\lambda_1} =& \begin{cases} 0, & \|e^T(t)\| = 0, \\ \frac{-1}{\|e^T(t)\|}\rho_1 \hat{\lambda}_1 \|y(t)\|, & \text{otherwise}, \end{cases} \\ u_{\lambda_2} =& \begin{cases} 0, & \|e^T(t)\| = 0, \\ \frac{-1}{\|e^T(t)\|}\rho_2 \hat{\lambda}_2 \|y(t-\tau)\|, & \text{otherwise}, \end{cases} \end{aligned} \tag{18}$$

where K is the control gain matrix, $\hat{L}_f$ and $\hat{L}_g$ are the estimation value of unknown Lipschitz constants L_f and L_g respectively; $\hat{\lambda}_1$ and $\hat{\lambda}_2$ are the estimation value of unknown constants λ_1 and λ_2 respectively. γ_f, γ_g, ρ_1, ρ_2 are positive constants.

The adaptive law of unknown parameters satisfies following fractional derivative:

$$\begin{aligned} &D^q W_i = \mu_i e_i(t)\phi(y), \ \ (i = 1, 2, \ldots, n), \\ &D^q \hat{L}_f = \gamma_f e^T(t)e(t), \\ &D^q \hat{L}_g = \gamma_g e^T(t)e(t), \\ &D^q \hat{\lambda}_1 = \rho_1 e(t)\|y(t)\|, \\ &D^q \hat{\lambda}_2 = \rho_2 e(t)\|y(t-\tau)\|. \end{aligned} \tag{19}$$

3.3. Stability analysis

Theorem 1: Designing the controller $u(t)$ as (17) and each sub-controller as (18). The adaptive law of the weight and uncertain terms are designed as (19). Assume there exist two positive constants β_1 and β_2, and two semi-definite positive matrix P, Q, if the following conditions are satisfied.

(A1) The following linear matrix inequality holds.

$$\begin{pmatrix} (B-K)^T + B - K + H_1^T + H_1 + \beta_1 P & H_2 + D \\ H_2^T + D^T & \beta_2 Q \end{pmatrix} \le 0, \tag{20}$$

(A2) $\gamma_f > 1$, $\gamma_g > 1$, $\rho_1 > 1$, $\rho_2 > 1$.

Then the error system (8) is asymptotically stable.

Proof: Firstly, we construct a Lyapunov function based on Lemma 1 to analyze stability.

$$\begin{aligned} V(t) =& e^T(t)e(t) + \sum_{i=1}^{n} \frac{1}{\mu_i}(\widetilde{W}_i - W_i)^T(\widetilde{W}_i - W_i) \\ &+ (\hat{L}_f - L_f)^2 + (\hat{L}_g - L_g)^2 + (\hat{\lambda}_1 - \lambda_1)^2 \\ &+ (\hat{\lambda}_2 - \lambda_2)^2. \end{aligned} \tag{21}$$

Applying Lemma 3, the derivative of Lyapunov function satisfies

$$\begin{aligned} D^q V(t) =& e^T(t)D^q e(t) + (D^q e(t))^T e(t) \\ &- \sum_{i=1}^{n} \frac{1}{\mu_i}[(\widetilde{W}_i - W_i)^T D^q W_i + (D^q W_i)^T(\widetilde{W}_i - W_i)] \\ &+ 2(\hat{L}_f - L_f)D^q \hat{L}_f + 2(\hat{L}_g - L_g)D^q \hat{L}_g \\ &+ 2(\hat{\lambda}_1 - \lambda_1)D^q \hat{\lambda}_1 + (\hat{\lambda}_2 - \lambda_2)D^q \hat{\lambda}_2. \end{aligned} \tag{22}$$

Substituting controller (17) whose each sub-controller is (18) into the error system (8), we get the new presentation of the error system.

$$\begin{aligned}D^q e(t) =&(B-K)e(t)+g(y(t))-g(x(t))+De(t-\tau)\\&+f(y(t))-f(x(t))+\Theta(x,y,t,t-\tau)\\&+E(y(t),y(t-\tau))+(\widetilde{W}-W)\phi(y)\\&+\triangle\psi(t,t-\tau)-\gamma_f\hat{L}_f e(t)-\gamma_g\hat{L}_g e(t)\\&-\frac{1}{\|e^T\|}\hat{\lambda}_1\|x(t)\|-\frac{1}{\|e^T\|}\hat{\lambda}_2\|x(t-\tau)\|.\end{aligned} \tag{23}$$

Applying Assumptions 1, 2 and 3, then the sub-term $e^T(t)D^q e(t)+(D^q e(t))^T e(t)$ in (22) satisfies

$$\begin{aligned}&e^T(t)D^q e(t)+(D^q e(t))^T e(t)\\&\le e^T(t)[(B-K)^T+(B-K)+H_1^T+H_1]e(t)\\&\quad+e^T(t)(H_2+D)e(t-\tau)+e^T(t-\tau)(H_2+D)^T e(t)\\&\quad+2L_g e^T(t)e(t)+2L_f e^T(t)e(t)\\&\quad+e^T(t)(\widetilde{W}-W)^T\phi(y)+\phi^T(y)(\widetilde{W}-W)e(t)\\&\quad+e^T(t)P(\lambda_1\|x(t)\|+\lambda_2\|x(t-\tau)\|)\\&\quad-2\gamma_f\hat{L}_f e^T e(t)-2\gamma_g\hat{L}_g e^T e(t)\\&\quad-2\|e(t)\|\rho_1\hat{\lambda}_1\|x(t)\|-2\|e(t)\|\rho_2\hat{\lambda}_2\|x(t-\tau)\|.\end{aligned} \tag{24}$$

For other sub-items of (22), the fractional order adaptive law (19) is considered, then we get the following results.

$$\begin{aligned}&\sum_{i=1}^{n}\frac{1}{\mu_i}[(\widetilde{W}_i-W_i)^T D^q W_i+(D^q W_i)^T(\widetilde{W}_i-W_i)]\\&=-\sum_{i=1}^{n}\frac{1}{\mu_i}(\widetilde{W}_i-W_i)^T\mu_i e_i(t)\phi(y)\\&\quad-\sum_{i=1}^{n}\frac{1}{\mu_i}\mu_i\phi^T(y)e_i(t)(\widetilde{W}_i-W_i)\\&=-e^T(t)(\widetilde{W}-W)^T\phi(y)-\phi^T(y)(\widetilde{W}-W)e(t),\end{aligned} \tag{25}$$

and

$$\begin{aligned}e_{L_f}^T D^q e_{L_f}^T&=(\hat{L}_f-L_f)\gamma_f e^T(t)e(t),\\e_{L_g}^T D^q e_{L_g}^T&=(\hat{L}_g-L_g)\gamma_g e^T(t)e(t),\\e_{\lambda_1}^T D^q e_{\lambda_1}^T&=(\hat{\lambda}_1-\lambda_1)\rho_1 e^T(t)\|x(t)\|,\\e_{\lambda_2}^T D^q e_{\lambda_2}^T&=(\hat{\lambda}_2-\lambda_2)\rho_2 e^T(t)\|x(t-\tau)\|.\end{aligned} \tag{26}$$

Combining inequality (24), (25) and (26) to rewrite and simplify the fractional derivate of Lyapunov function (22) leads to

$$\begin{aligned}D^q V(t)\le&e^T(t)[(B-K)^T+(B-K)+H_1^T+H_1]e(t)\\&+e^T(t)H_2 e(t-\tau)+e^T(t-\tau)H_2^T e(t)\\&+2(1-\gamma_f)L_f\|e(t)\|^2+2(1-\gamma_g)L_g\|e(t)\|^2\\&+(1-\rho_1)\lambda_1\|e(t)x(t)\|\\&+(1-\rho_2)\lambda_2\|e(t)x(t-\tau)\|.\end{aligned} \tag{27}$$

Applying LMI method to solve the time-delay terms in inequality (27) i.e.,

$$\begin{aligned}&e^T(t)[(B-K)^T+(B-K)+H_1^T+H_1]e(t)\\&\quad+e^T(t)H_2 e(t-\tau)+e^T(t-\tau)H_2^T e(t)\\&=\begin{pmatrix}e(t)\\e(t-\tau)\end{pmatrix}^T\begin{pmatrix}\Xi & H_2+D\\H_2^T+D^T & 0\end{pmatrix}\begin{pmatrix}e(t)\\e(t-\tau)\end{pmatrix},\end{aligned} \tag{28}$$

where $\Xi=(B-K)^T+(B-K)+H_1^T+H_1$.
Then, we apply condition (A1) and (A2).

$$\begin{aligned}&D^q V(t)\\&\le\begin{pmatrix}e^T(t)\\e^T(t-\tau)\end{pmatrix}\begin{pmatrix}-\beta_1 P & 0\\0 & -\beta_2 Q\end{pmatrix}\begin{pmatrix}e(t)\\e(t-\tau)\end{pmatrix}\\&\quad+(1-\gamma_f)L_f P\|e(t)\|^2+(1-\gamma_g)L_g P\|e(t)\|^2\\&\quad+(1-\rho_1)\lambda_1 P\|e(t)x(t)\|\\&\quad+(1-\rho_2)\lambda_2 P\|e(t)x(t-\tau)\|\\&\le 0.\end{aligned} \tag{29}$$

Obviously, inequality (29) satisfies Lemma 1. Hence, error system (8) tends to be stable, which implies synchronization of fractional order uncertain chaotic system (6) and (7) is achieved. □

4. ADAPTIVE FRACTIONAL ORDER FUZZY NEURAL NETWORK CONTROLLER DESIGN FOR FRACTIONAL ORDER TIME-DELAY UNCERTAIN CHAOTIC SYSTEMS

In this section, we introduce fuzzy logic rules into RBF neural network to propose an adaptive controller. The designed fuzzy RBF neural network is also used to approximate the uncertainties.

4.1. Applying fuzzy neural network to approximate the uncertainties

Fuzzy radial basis function network (RRBF) can realize fortification by adding a layer based on the RBF network. The inputs of the FRBF are the elements in vector y.

The second layer is the membership layer, where each node in this layer corresponds to one linguistic label of one of the input variables in the input layer [18]. With the choice of the Gaussian membership function, the operation performed in this layer is

$$\phi_j(y_k)=exp[\frac{-(y_k-\xi_{kj})^2}{\delta_{kj}^2}], \tag{30}$$

where ξ_{kj} and δ_{kj} ($j=1,2,\ldots,m$, $k=1,2,\ldots,r$) are the mean and standard deviation of the Gaussian function of the k-th partition for the j-th input variable y.

And the third layer is the rule layer, where each node in this layer represents one fuzzy logic rule and performs precondition matching of a rule [18]. The output of a rule node in this layer is calculated by the product operation as follows:

$$r_j(\phi_j(y_k)) = \prod_{k=1}^{r} \phi_j(y_k) = \prod_{k=1}^{r} exp[\frac{-(y_k - \xi_{kj})^2}{\delta_{kj}^2}], \tag{31}$$

where r_j represents the j-th output of the rule layer, which also represents the firing strength of the corresponding fuzzy rule.
Hence, the final output of FRBF as

$$R_i(\phi_j(y_k)) = \sum_{j=1}^{m} w_{ij} r_j(\phi_j(y_k)), \tag{32}$$

where $i = 1, 2, \ldots, n$.
The output of the network is also the approximate value of $\Phi(y)$. Therefore,

$$\widehat{\Phi}(y) = R_i(\phi_j(y_k)) = W^T r(y), \tag{33}$$

where $W = [W_1, W_2, \ldots, W_n] \in R^{m\times n}$, and $W_i = [W_{i1}, W_{i2}, \ldots, W_{im}]^T$ is the weight vector for network.
We use optional weight value to evaluate the weight values W, which is defined as

$$\widetilde{W} = arg \min_{W\in\Omega} [\sup_{y\in S_y} |\widehat{\Phi}(y/W) - \Phi(y)],$$

where $\Omega = \{W :\| W \leq M\}$ is a valid field of the parameter, and M is a designed parameter. $S_y \subset R^n$ is an allowable set of the state vector.
Then, the optimal estimate of uncertain terms as

$$\Phi(y) = \widetilde{W}^T r(y) + \varepsilon(y),$$

where $\|\varepsilon(y)\| \leq \bar{\varepsilon}$, $\bar{\varepsilon}$ is the minimum approach error of RBF neural network, which is given and can be ignored when it is small enough.

In addition, we also apply this FRBF network to approximate the uncertainties corresponding to the time-delay instead of Lemma 3. In order to achieve this, the input of the network should be $y(t)$ and $y(t-\tau)$. Therefore, the approximate value of uncertainties $\Phi(y(t)) + E(y(t), y(t-\tau))$ is as follows:

$$\Phi(y(t)) + E(y(t), y(t-\tau)) = \widetilde{W}^T r(y, t-\tau) + \varepsilon(y). \tag{34}$$

Remark 2: This network implements the method of fuzzy reasoning. The number of nodes in each layer represents the number of inputs, partitions, rules, and dimensions respectively.

4.2. Designing fuzzy adaptive controller

Considering a fuzzy rule basis with R rules which have the following $If - Then$ forms.

Plant rule i: if z_1 is M_{i1}, z_2 is M_{i2}, z_3 is M_{i3}. Then

$$\begin{aligned} D^q x(t) &= A_i x(t) + C_i x(t-\tau), \qquad (35)\\ D^q y(t) &= B_i y(t) + D_i y(t-\tau) + \widetilde{W}^T r(y, t-\tau) + u_i(t). \end{aligned}$$

Also let $e(t) = y(t) - x(t)$ be the error between drive and response system. Then we get the error system under the $i-th$ fuzzy rule as

$$\begin{aligned} D^q e(t) =& B_i y(t) - A_i x(t) + D_i y(t-\tau) - C(i)x(t-\tau)\\ &+ \widetilde{W}^T r(y, t-\tau) + u_i(t). \end{aligned} \tag{36}$$

Based on the output of the fractional order FRBF neural network, we also design a fuzzy adaptive controller, which also satisfies the above fuzzy rule, i.e. plant rule i: if z_1 is M_{i1}, z_2 is M_{i2}, z_3 is M_{i3}. Then

$$\begin{aligned} u_i(t) =& -K_i e(t) - (B_i - A_i)x(t) - (D_i - C_i)x(t-\tau)\\ &- W^T r(y, t-\tau). \end{aligned} \tag{37}$$

The updating law of the fractional order FRBF neural network satisfies the fractional order differential.

$$D^q W_i = \mu_i e_i(t) r(y, t-\tau), \quad (i = 1, 2, \ldots, n). \tag{38}$$

Therefore, combining the controller (37), the error system (36) can be rewritten as

$$\begin{aligned} D^q e(t) =& (B_i - K_i)e(t) + D_i e(t-\tau)\\ &+ (\widetilde{W} - W)^T r(y, t-\tau). \end{aligned} \tag{39}$$

Next, using singleton fuzzifier, product inference, and weighted average anti-fuzzification, the system (13) can be rewritten as

$$\begin{aligned} D^q e(t) =& \sum_{i=1}^{r} h_i(e(t))((B_i - K_i)e(t) + D_i e(t-\tau))\\ &+ (\widetilde{W} - W)^T r(y, t-\tau), \end{aligned} \tag{40}$$

where $h_i(e(t)) = \frac{\omega_i(e(i))}{\sum_{i=1}^{r} \omega_i(e(i))}$, $\omega_i(e(t)) = \prod_{j=1}^{3} M_{ij}(e(t))$, and $\sum_{i=1}^{r} \omega_i(e(i)) > 0$, $\sum_{i=1}^{r} h_i(e(t)) = 1$.

4.3. Stability analysis

A new theorem is proposed based on the adaptive fuzzy fractional order FRBF neural network controller.

Theorem 2: Designing the fuzzy controller $u(t)$ as (37). The adaptive law of the weight satisfies (38). Assume there exist two positive constants α_i and β_i, and two semi-definite positive matrix P and Q. If the following LMI condition satisfies.

$$\begin{pmatrix} (B_i - K_i)^T + (B_i - K_i) + \alpha_i P & D_i \\ D_i^T & \beta_i Q \end{pmatrix} \leq 0, \tag{41}$$

Then, the fuzzy error system (40) is asymptotically stable.

Proof: Constructing a Lyapunov function which satisfies Lemma 1.

$$V(t) = e^T(t)e(t) + \sum_{i=1}^{n} \frac{1}{\mu_i} (\widetilde{W}_i - W_i)^T (\widetilde{W}_i - W_i). \quad (42)$$

According to Lemma 3, the derivate of Lyapunov function (42) satisfies

$$\begin{aligned} D^q V(t) =& e^T(t) D^q e(t) + (D^q e(t))^T e(t) \quad (43) \\ & - \sum_{i=1}^{n} \frac{1}{\mu_i} [(\widetilde{W}_i - W_i)^T D^q W_i + (D^q W_i)^T (\widetilde{W}_i - W_i)]. \end{aligned}$$

Therefore, applying (40), the $e^T(t)D^q e(t) + (D^q e(t))^T e(t)$ in (43) satisfies

$$\begin{aligned} & e^T(t) D^q e(t) + (D^q e(t))^T e(t) \\ &= \sum_{i=1}^{r} h_i(e(t)) e^T(t) ((B_i - K_i)^T + B_i - K_i) e(t) \\ &\quad + \sum_{i=1}^{r} h_i(e(t)) e^T(t) D_i e(t-\tau) \\ &\quad + \sum_{i=1}^{r} h_i(e(t)) e^T(t-\tau) D_i^T e(t) \quad (44) \\ &\quad + e^T(t)(\widetilde{W} - W)^T r(y,t,t-\tau) \\ &\quad + r^T(y,t,t-\tau)(\widetilde{W} - W) e(t). \end{aligned}$$

For other items of (37), the fractional order adaptive law (38) is considered, then we get the following results.

$$\begin{aligned} & \sum_{i=1}^{n} \frac{1}{\mu_i} [(\widetilde{W}_i - W_i)^T D^q W_i + (D^q W_i)^T (\widetilde{W}_i - W_i)] \\ &= - \sum_{i=1}^{n} \frac{1}{\mu_i} (\widetilde{W}_i - W_i)^T \mu_i e_i(t) r(y,t,t-\tau) \\ &\quad - \sum_{i=1}^{n} \frac{1}{\mu_i} \mu_i r^T(y,t,t-\tau) e_i(t) (\widetilde{W}_i - W_i) \quad (45) \\ &= -e^T(t)(\widetilde{W} - W)^T r(y,t,t-\tau) \\ &\quad - r^T(y,t,t-\tau)(\widetilde{W} - W) e(t). \end{aligned}$$

Combining (44) and (45), the derivate of Lyapunov function (43) can be simplified as

$$\begin{aligned} & D^q V(t) \\ &= \sum_{i=1}^{r} h_i(e(t)) e^T(t) ((B_i - K_i)^T + B_i - K_i) e(t) \\ &\quad + \sum_{i=1}^{r} h_i(e(t)) e^T(t) D_i e(t-\tau) \\ &\quad + \sum_{i=1}^{r} h_i(e(t)) e^T(t-\tau) D_i^T e(t) \quad (46) \\ &= \sum_{i=1}^{r} h_i(e(t)) \begin{pmatrix} e(t) \\ e(t-\tau) \end{pmatrix}^T \begin{pmatrix} \Xi & D_i \\ D_i^T & 0 \end{pmatrix} \begin{pmatrix} e(t) \\ e(t-\tau) \end{pmatrix}. \end{aligned}$$

Therefore, based on the LMI condition (41), the derivate of Lyapunov function (43) satisfies

$$\begin{aligned} & D^q V(t) \\ & \leq -\sum_{i=1}^{r} h_i(e(t)) [\alpha_i e^T(t) P e(t) + \beta_i e^T(t-\tau) Q e(t-\tau)] \\ & \leq -[\alpha e^T(t) P e(t) + \beta e^T(t-\tau) Q e(t-\tau)] \sum_{i=1}^{r} h_i(e(t)) \\ & \leq -[\alpha e^T(t) P e(t) + \beta e^T(t-\tau) Q e(t-\tau)] \\ & \leq 0. \quad (47) \end{aligned}$$

Obviously, inequality (47) satisfies Lemma 1. Hence, error system (40) is asymptotically stable. □

5. NUMERICAL SIMULATION

We provide the following numerical examples in this section to verify our main results developed in this paper.

5.1. Adaptive fractional order neural network synchronization for fractional order time-delay uncertain Lorenz system

The model of the fractional order time-delay Lorenz system is shown as follows:

$$\begin{aligned} & \begin{pmatrix} D^q x_1(t) \\ D^q x_2(t) \\ D^q x_3(t) \end{pmatrix} \\ &= \begin{pmatrix} -\sigma & \sigma & 0 \\ \gamma & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} + \begin{pmatrix} 0 \\ -x_1(t)x_3(t) \\ x_1(t)x_2(t) \end{pmatrix} \\ &+ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\beta \end{pmatrix} \begin{pmatrix} x_1(t-\tau) \\ x_2(t-\tau) \\ x_3(t-\tau) \end{pmatrix}. \quad (48) \end{aligned}$$

When the parameters are appropriately chosen, i.e., $\sigma = 10$, $\beta = 8/3$, $\gamma = 28$, time-delay $\tau = 0.4$ and order $q = 0.95$, the fractional order time-delay Lorenz system also show chaos phenomenon (Fig. 1 and paper [35]).

When fractional order time-delay Lorenz system has uncertainty, the mathematic model is below.

$$\begin{aligned} & \begin{pmatrix} D^q y_1(t) \\ D^q y_2(t) \\ D^q y_3(t) \end{pmatrix} \\ &= \begin{pmatrix} -\sigma & \sigma & 0 \\ \gamma & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{pmatrix} + \begin{pmatrix} 0 \\ -y_1(t)y_3(t) \\ y_1(t)y_2(t) \end{pmatrix} \\ &\quad + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\beta \end{pmatrix} \begin{pmatrix} y_1(t-\tau) \\ y_2(t-\tau) \\ y_3(t-\tau) \end{pmatrix} + \begin{pmatrix} 0.1(y_2(t)-y_1(t)) \\ 0.1(y_1(t)-y_2(t)) \\ 0 \end{pmatrix} \\ &\quad + \begin{pmatrix} 0 \\ 0.1y_1(t)y_3(t) \\ 0.1y_1(t)y_2(t) \end{pmatrix} + \begin{pmatrix} 0.01\sin(\pi t) y_2(t) y_3(t) \\ 0.01\sin(\pi t) y_1(t) y_3(t) \\ 0.01\sin(\pi t) y_1(t) y_2(t) \end{pmatrix} \end{aligned}$$

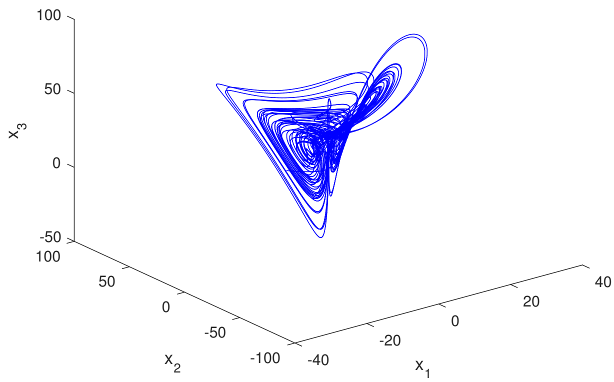

Fig. 1. The fractional order time-delay Lorenz chaotic attractors with $\sigma = 10$, $\beta = 8/3$, $\gamma = 28$, $q = 0.95$, and $\tau = 0.4$.

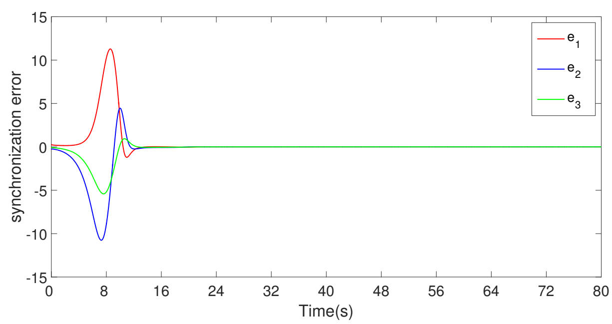

Fig. 2. The chaos synchronization error between system (48) and (49) under controller (50) with $k = -10$.

$$+\begin{pmatrix} 0 \\ 0 \\ -0.1y_3(t-\tau) \end{pmatrix}. \tag{49}$$

Because this is the self-synchronization of fractional order Lorenz system, hence the adaptive parameters are only γ_f and ρ_2. Therefore, design the controller based on equation the adaptive controller (17) and (18) satisfying the adaptive law (19).

$$\begin{aligned} u_1(t) &= -k_1 e_1(t) - W_1^T\phi(y) - \gamma_f \hat{L}_f e_1(t), \\ u_2(t) &= -k_2 e_2(t) - W_2^T\phi(y) - \gamma_f \hat{L}_f e_2(t), \\ u_3(t) &= -k_3 e_3(t) - W_3^T\phi(y) - \gamma_f \hat{L}_f e_3(t) \\ &\quad - \rho_2 \hat{\lambda}_2 e_3(t)\|y_3(t-\tau)\|. \end{aligned} \tag{50}$$

In controller (50), the gain matrix is $K = diag(-10, -10, -10)$, and the parameters are $\mu = 1$, $\gamma_f = 1.2$, $\rho_2 = 1.2$. Then the result of synchronization under controller (50) is shown in Fig. 2 and Fig. 3. The numerical results in Fig. 3 also illustrate that the adaptive parameters eventually tends to a stable value.

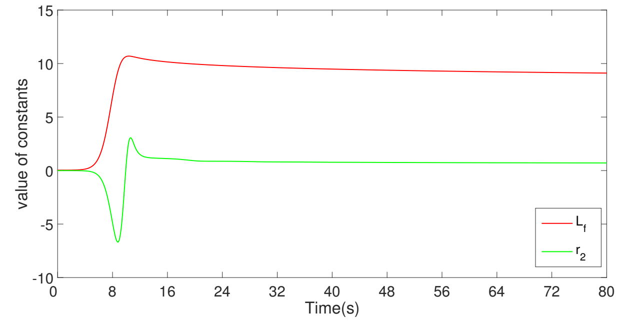

Fig. 3. State curve of uncertain parameters (γ_f and ρ_2) between system (48) and (49) under controller (50).

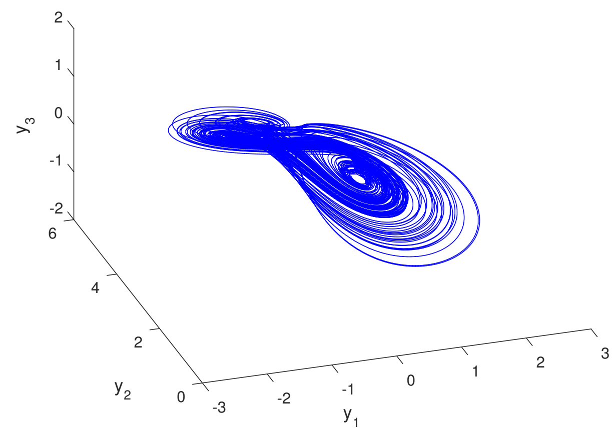

Fig. 4. The fractional order time-delay financial chaotic attractors with $a = 3$, $b = 0.1$, $c = 1$, $q = 0.95$, and $\tau = 0.5$.

5.2. Adaptive fractional order neural network synchronization for fractional order time-delay uncertain Lorenz system and financial system

Then, we do the simulation of synchronization of the chaotic system with the different structure. Considering fractional order time-delay financial system [43] as the the response system.

$$\begin{aligned} &\begin{pmatrix} D^q y_1(t) \\ D^q y_2(t) \\ D^q y_3(t) \end{pmatrix} \\ &= \begin{pmatrix} -a & 0 & 1 \\ 0 & -b & 0 \\ 0 & 0 & -c \end{pmatrix}\begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}\begin{pmatrix} y_1(t-\tau) \\ y_2(t-\tau) \\ y_3(t-\tau) \end{pmatrix} \\ &\quad + \begin{pmatrix} y_1(t)y_2(t-\tau) \\ 1 - y_1^2(t-\tau) \\ 0 \end{pmatrix}. \end{aligned} \tag{51}$$

When the parameters are approximately chosen, i.e. $(a,b,c) = (3, 0.1, 1)$, $\tau = 0.5$, $q = 0.95$, the fractional order time-delay financial system will also has chaotic phenomena [43] shown in Fig. 4.

When fractional order time-delay financial system has uncertainties

$$\begin{pmatrix} D^q y_1(t) \\ D^q y_2(t) \\ D^q y_3(t) \end{pmatrix} = \begin{pmatrix} -a & 0 & 1 \\ 0 & -b & 0 \\ 0 & 0 & -c \end{pmatrix}\begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}\begin{pmatrix} y_1(t-\tau) \\ y_2(t-\tau) \\ y_3(t-\tau) \end{pmatrix} + \begin{pmatrix} y_1(t)y_2(t-\tau) \\ 1-y_1^2(t-\tau) \\ 0 \end{pmatrix} + \begin{pmatrix} 0.1(y_3(t)-y_1(t)) \\ -0.1y_2(t) \\ -0.1y_3(t) \end{pmatrix} + \begin{pmatrix} 0.1y_1(t)y_2(t-\tau) \\ 1-0.1y_1^2(t-\tau) \\ 0 \end{pmatrix} + \begin{pmatrix} 0.01\sin(\pi t)y_2(t)y_3(t) \\ 0.01\sin(\pi t)y_1(t)y_3(t) \\ 0.01\sin(\pi t)y_1(t)y_2(t) \end{pmatrix}. \tag{52}$$

Because chaos is bounded, there exist a positive constant M such that $\|y(t)\| \le M$. Then Assumption 2 satisfies

$$\begin{aligned} &\begin{pmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \end{pmatrix}^T \begin{pmatrix} y_1(t)y_2(t-\tau) - x_1(t)x_2(t-\tau) \\ y_1^2(t-\tau) - x_1^2(t-\tau) \\ -e_1(t-\tau) - \beta e_3(t) \end{pmatrix} \\ &= e_1(t)(y_1(t)y_2(t-\tau) - x_1(t)y_2(t-\tau)) \\ &\quad + e_1(t)(x_1(t)y_2(t-\tau) - x_1(t)x_2(t-\tau)) \\ &\quad + e_2(y_1(t-\tau) - x_1(t-\tau))(y_1(t-\tau) + x_1(t-\tau)) \\ &\quad - e_3(t)e_1(t-\tau) - \beta e_3(t)e_3(t-\tau) \\ &= y_2(t-\tau)e_1(t)e_1(t) + x_1(t)e_1(t)e_2(t-\tau) \\ &\quad + (y_1(t-\tau) + x_1(t-\tau))e_2(t)e_1(t-\tau) \\ &\quad - e_3(t)e_1(t-\tau) - \beta e_3(t)e_3(t-\tau) \\ &= e^T(t)H_1e(t) + e^T(t)H_2e(t-\tau), \end{aligned} \tag{53}$$

where

$$H_1 = \begin{pmatrix} y_2(t-\tau) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$H_2 = \begin{pmatrix} 0 & x_1(t) & 0 \\ y_1(t-\tau)+x_1(t-\tau) & 0 & 0 \\ -1 & 0 & -\beta \end{pmatrix}.$$

Therefore, the adaptive controller is design based on (17), (18) and (19).

$$\begin{aligned} u_1(t) &= -k_1e_1(t) + (a-\sigma)x_1(t) + \sigma x_2(t) - x_3(t) \\ &\quad - W_1^T\phi(y) - x_1(t)x_2(t-\tau) - (\gamma_f\hat{L}_f + \gamma_g\hat{L}_g)e_1(t) \\ &\quad - \rho_1\hat{\lambda}_2e_1(t)\|y_1(t)\| - \rho_2\hat{\lambda}_2e_1(t)\|y_1(t-\tau)\|, \\ u_2(t) &= -k_2e_2(t) + (b-1)x_2(t) - y_1(t)y_3(t) \\ &\quad - x_1^2(t-\tau) - W_2^T\phi(y) - \gamma_f\hat{L}_fe_2(t) - \gamma_g\hat{L}_ge_1(t) \\ &\quad - \rho_1\hat{\lambda}_2e_2(t)\|y_2(t)\| - \rho_2\hat{\lambda}_2e_2(t)\|y_2(t-\tau)\|, \\ u_3(t) &= -k_3e_3(t) + x_3(t) + x_1(t-\tau) - \beta x_3(t) \\ &\quad + y_1(t)y_2(t) - W_3^T\phi(y) - \gamma_f\hat{L}_fe_3(t) - \gamma_g\hat{L}_ge_3(t) \\ &\quad - \rho_1\hat{\lambda}_1e_3(t)\|y_3(t) - \rho_2\hat{\lambda}_2e_3(t)\|y_3(t-\tau)\|. \end{aligned} \tag{54}$$

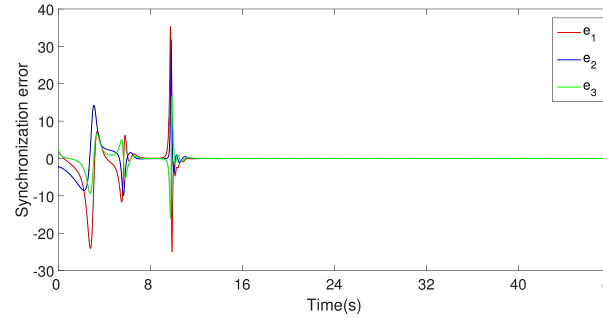

Fig. 5. The chaos synchronization error between system (48) and (52) under controller (54) with $k = -10$.

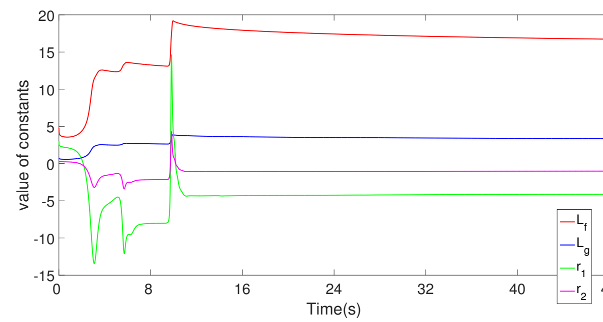

Fig. 6. The state curve of uncertain parameters (γ_f, γ_g, ρ_1, and ρ_2) between system (48) and (52) under controller (54).

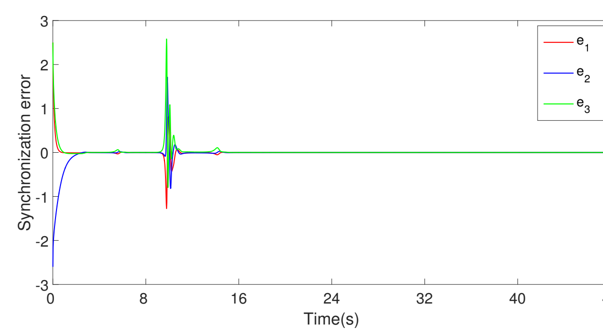

Fig. 7. The chaos synchronization error with the gain matrix $K = [-12, -9, 0; -30, -20, 14; -12, 0, -7]$.

In controller (54) the gain matrix is $K = diag(-10, -10, -10)$. Designing the update law of parameters as (19), where $\mu = 1$, $\gamma_f = 1.25$, $\gamma_g = 1.5$, $\rho_1 = \rho_2 = 1.75$. Then the result of synchronization under controller (54) is shown in Fig. 5 and Fig. 6. When the gain matrix is $K = \begin{pmatrix} -12 & -9 & 0 \\ -30 & -20 & 14 \\ -12 & 0 & -7 \end{pmatrix}$ shown in Fig. 7.

5.3. Adaptive fractional order fuzzy neural network synchronization for fractional order time-delay uncertain financial system

Applying the $IF-THEN$ rule proposed in section 4 on the drive and controlled response systems (48) and (51).

Plant rule i: if z_1 is M_{i1}, z_2 is M_{i2}, z_3 is M_{i3}, ($i = 1, 2, \ldots, r$). Then

$$\begin{aligned} D^q x(t) &= A_i x(t) + C_i x(t-\tau), \\ D^q y(t) &= B_i y(t) + D_i y(t-\tau) + \widetilde{W}^T r(y, t-\tau) + u_i(t). \end{aligned}$$

Consider the case where there are only two fuzzy rules, i.e., $r = 1, 2$. Hence, for $x_1(t) \in [d_1, d_2] = [2, 6]$.

$$A_1 = \begin{pmatrix} -\sigma & \sigma & 0 \\ \lambda & -1 & -d_1 \\ 0 & d_1 & 0 \end{pmatrix}, \ A_2 = \begin{pmatrix} -\sigma & \sigma & 0 \\ \lambda & -1 & -d_2 \\ 0 & d_2 & 0 \end{pmatrix},$$
$$B_1 = \begin{pmatrix} -a & -d_1 & 1 \\ d_1 & -b & 0 \\ 0 & 0 & -1 \end{pmatrix}, \ B_2 = \begin{pmatrix} -a & -d_2 & 1 \\ d_2 & -b & 0 \\ 0 & 0 & -1 \end{pmatrix},$$
$$C_1 = \begin{pmatrix} 0 & -d_1 & 0 \\ d_1 & 0 & 0 \\ 0 & 0 & -\beta \end{pmatrix}, \ C_2 = \begin{pmatrix} 0 & -d_2 & 0 \\ d_2 & 0 & 0 \\ 0 & 0 & -\beta \end{pmatrix},$$
$$D_1 = \begin{pmatrix} 0 & -d_1 & 0 \\ 0 & 0 & d_1 \\ -1 & 0 & 0 \end{pmatrix}, \ D_2 = \begin{pmatrix} 0 & -d_2 & 0 \\ 0 & 0 & d_2 \\ -1 & 0 & 0 \end{pmatrix}.$$

And the membership function M_1 and M_2 are

$$M_1 = \frac{-x_1 + d_2}{d_2 + d_1}, \ M_2 = \frac{x_1 - d_1}{d_2 + d_1}.$$

Then the gain matrix is designed as $K = diag(-5, -5, -5)$. The initial value of drive and response systems are $x0 = [-2, 4, -2]$ and $y0 = [2, -3, 1]$. Then the controller $u_i(t)$ is designed as (37). Therefore, the synchronization error is shown in Fig. 8.

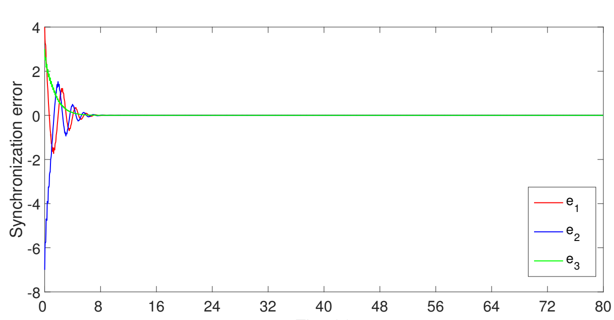

Fig. 8. The chaos synchronization error between fractional order time-delay uncertain Lorenz and financial system under the fuzzy neural network.

6. DISCUSSION AND CONCLUSION

6.1. Discussion

Our proposed adaptive controller can well implement the synchronization of fractional order time-delay uncertain chaotic system, But there are still limitations. In the process of using fractional order neural network to approximate uncertainties, we only update the weights W of the last layer but neglect the center ξ and width δ of Gaussian function.

The cause of this problem is chain rule of derivatives does not apply to the three common definitions of fractional calculus. Therefore, this is the challenge we are facing and will be the focus of our future research. In the following research, we will try to use the local fractional derivatives, which is a new definition, to study the synchronization method of fractional order neural network.

6.2. Conclusion

In this paper, two adaptive controllers based on fractional order radial basis function neural network and fuzzy fractional order radial basis function neural network are proposed for the synchronization of the fractional order time-delay uncertain controller. The scheme is based on the linear matrix inequality, the properties of the fractional order neural network, the fractional order differential and fuzzy method. The design of the controller as a combination of several sub-controllers is also very helpful for the practical use of this method.

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Xingpeng Zhang received his B.S. degree in Jinan University, Shandong, China, in 2012 and his M.S. degree in applied mathematics from Chongqing University, Chongqing, China, in 2015. He is currently pursuing a Ph.D. degree in software engineering from Chongqing University, China. His current research interests include dynamics of fractional order chaotic systems and deep learning.

Xiaohong Zhang received his M.S. degree in applied mathematics from Chongqing University, Chongqing, China, where he also received his Ph.D. degree in computer software and theory, in 2006. He is currently a professor in School of Software Engineering at Chongqing University. His current research interests include data mining of software engineering, topic modeling, image semantic analysis, video analysis and deep learning.

Dong Li received his B.S. and M.S. degrees in applied mathematics from Chongqing University, Chongqing, China, in 2000 and 2003, and his Ph.D. degree in engineering Chongqing University, Chongqing, China, in 2008. He is currently an Associate Professor with Chongqing University. His current research interests include nonlinear and complex system, financial mathematics and application of dynamical system, dynamical system and its application.

Dan Yang received his B.S., M.S., and Ph.D. degrees from Chongqing University, Chongqing, China, in 1982, 1985, and 1997, respectively. He is currently a professor in School of Big Data and Software Engineering at Chongqing University. His current research interests include software engineering, topic modeling, image semantic analysis and video analysis.