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Finite-time Synchronization Control Relationship Analysis of Two Classes of Markovian Switched Complex Networks

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Abstract: In this paper, finite-time global synchronization control problem for a class of nonlinear coupling Markovian switched complex networks (NCMSCNs) is investigated. Furthermore, according to differentiability of nonlinear coupling function $g(x,y)$, $g(x,y)$ how to affect synchronization dynamics of the class of NCMSCNs is analyzed by two viewpoints. The first is that if $g(x,y)$ satisfies the Lipschitz condition and is derivable, the above question is discussed by taking $g(x,y) = L_1x+L_2y$, $g(x,y) = -L_1x+L_2y$, $g(x,y) = L_1x-L_2y$ and $g(x,y) = -L_1x-L_2y$, where $L_1 > 0$, $L_2 > 0$. The second is that if nonlinear coupling function $g(x,y)$ only satisfies the Lipschitz condition, by analyzing the differences of synchronization control rules for the class of NCMSCNs and a class of linear coupling Markovian switched complex networks (LCMSCNs), the problem is explored. Comparing the previous works [12,21,22,26,33,34], the main improvement of this paper is that the works of this paper extend the existed analyzing ideas of the finite-time global synchronization for nonlinear coupling complex networks, including NCMSCNs.

Keywords: Control rules, finite-time synchronization, linear coupling, nonlinear coupling, synchronization control.

1. INTRODUCTION

In the past few decades, because of the pioneering work of Watts and Strogatz [1], complex networks which consist of interacting dynamical states and interaction patterns have been extensively studied [1, 2]. In various fields, many applications of complex networks have been found. For example, communication networks, social networks, neural networks [3, 4]. Among the main research problems on complex networks, synchronization, as one of the most important collective dynamical behavior properties of the complex networks, has been aroused more and more concern by many researcher [5–7]. Up till now, there are a lot of different types of synchronization, for instance, pin cluster synchronization, finite-time synchronization, exponential synchronization and so on [8–10].

As is known to all, in a network environment, the dynamical behavior of each node may present randomly switching phenomenon due to environmental variance, component failures or repairs, and so on. If systems experience the above phenomena, they are usually called Markovian switched systems [11]. Recently, many results of synchronization for Markovian switched complex networks, which are one of Markovian switched systems, have been derived [12–20]. For instance, Dong *et al.* [12] investigated the exponential synchronization problem for a new array of nonlinearly and stochastically coupled networks with Markovian switching via events-triggered sampling. In [13], the authors proposed the issue of almost sure cluster synchronization in nonlinearly coupled complex networks with nonidentical nodes and time-varying delay. Besides these, due to the limited speed of signals traveling, processing speeds and the other environment elements, these cause to produce time delays in Markovian switched complex networks [21, 22].

In fact, the couplings which include the linear and nonlinear ones are important factors impacting the synchronization [23]. Until now, some works of synchronization for nonlinear coupling complex networks have been proposed [23–29]. Based on the norm-bounded conditions, an effective approach of solving nonlinear coupling one is to linearize nonlinear coupling function [23–28]. It is worth to mention that in [12, 23–25], although synchronization problems for the addressed complex networks with nonlinear coupling have been discussed, nonlinearity of nonlinear coupling one how to affect synchronization dynamics are still not analyzed. Until now, it is regrettable that extremely few publication [26] on the issue. In [26], Liu *et al.* fixed the intermittent control as the periodically intermittent control and chose three classes

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of nonlinear coupling functions to analyze dynamical behaviors of the synchronization for nonlinear coupled networks. Actually, from [23–25], it is not difficult to find that nonlinear coupling function $g(x)$ satisfies the Lipschitz condition, that is to say $\|g(x)-g(y)\| \leq L\|x-y\|$, where $L>0$ and $x,y \in R^n$. According to $\|g(x)-g(y)\| \leq L\|x-y\|$, we can get that $\lim_{x\to y}\frac{\|g(x)-g(y)\|}{|x-y\|} \leq L$. Let $\lim_{x\to y}\frac{\|g(x)-g(y)\|}{|x-y\|} = L$, then with increasing L, the nonlinearity of nonlinear coupling function $g(x)$ will become serious. That means L can be decided by $g(x)$. According to [26], the nonlinearity of nonlinear coupling function $g(x)$ is closely related to synchronization dynamics of the addressed complex network. Therefore, if $\lim_{x\to y}\frac{\|g(x)-g(y)\|}{|x-y\|} = L$, how to get L? The answer is that we have to choose many different classes of nonlinear coupling function $g(x)$ to obtain many L values by $\lim_{x\to y}\frac{\|g(x)-g(y)\|}{|x-y\|} = L$ condition. It is easily seen that this scheme is not practical and not flexible. If we directly adjust L to study the above problem, the question will become easy. If we use the viewpoint to solve the issue, how to do. This is very interesting.

In practice, especially in some engineering fields, it is desirable and more valuable that the convergence of a dynamical system is realized in finite-time rather than infinite time [30]. Therefore, recently, some results on the finite-time global synchronization problem for complex networks have been proposed [31–34]. In [31], the authors investigated the finite-time global synchronization of drive-response inertial memristive neural networks with time delay. Qiu *et al*. [32] proposed finite-time global synchronization of multi-weighted complex dynamical networks with and without coupling delay. Note that until now, about for finite-time global synchronization problem of Markovian switched complex networks, there are still a few publications [21, 22, 33, 34]. [21, 22] studied finite-time global synchronization for two classes of Markovian jump complex networks with partially unknown transition rates, respectively. In [33], the authors studied finite-time global synchronization and identification of drive complex network and response complex network with Markovian jumping parameters, stochastic perturbations and time delay. Liu *et al*. [34] investigated finite-time global synchronization for a class of neutral complex dynamical networks with Markovian switching, partly unknown transition rates and time-varying, mode- dependent delays. Besides these, Huang *et al*. [14] proposed finite-time H_∞ sampled-data synchronization for Markovian jump complex networks with time-varying delays. The [35] was concerned with finite-time cluster synchronization of Markovian switching complex networks with stochastic perturbations. Combining the above analysis of nonlinear coupling, Markovian switched complex networks and finite-time global synchronization, it is interesting and necessary to study finite-time global synchronization of nonlinear coupling Markovian switching complex networks, especially, to analyze nonlinear coupling how to impact synchronization dynamics for NCMSCNs in finite time. To the best of our knowledge, there are no results to be reported on the above topic.

Motivated by the above discussions, in this paper, we will focus on the following two problems for a class of NCMSCNs. Furthermore, in order to make nonlinear coupling one become more general than that of [23–25], nonlinear coupling one is $g(x,y)$, not $g(x)$. The reason is that $g(x,y)$ is more general than that of $g(x)$.

Problem 1 (The Effect of Nonlinear Coupling Function For Finite-time Global Synchronization Of NCMSCNs)**:** According to the differentiability of $g(x,y)$, we take two ideas to discuss the problem 1.

(i) The first idea. If $g(x,y)$ satisfies the Lipschitz condition and is derivable, there are $\|g(x_1,y_1)-g(x_2,y_2)\| \leq L_1\|x_1-x_2\|+L_2\|y_1-y_2\|$, $\|g_x^{'}(x,y)\| = \|\lim_{x\to x_1}\frac{g(x,y)-g(x_1,y)}{x-x_1}\| \leq L_1$, $\|g_y^{'}(x,y)\| = \|\lim_{y\to y_1}\frac{g(x,y)-g(x,y_1)}{y-y_1}\| \leq L_2$, x_1, x_2, y_1, $y_2 \in R^n$. Let $\|g_x^{'}(x,y)\| = L_1$ and $\|g_y^{'}(x,y)\| = L_2$, then one obtains $g(x,y) = L_1x+L_2y+C$, $g(x,y) = -L_1x+L_2y+C$, $g(x,y) = L_1x-L_2y+C$ and $g(x,y) = -L_1x-L_2y+C$, where C is constant. Thus, according to the 3rd paragraph analysis, problem 1 can be discussed by adjusting L_1 and L_2. Here, we make $C=0$. The reason is that problem 1 is only connected with L_1 and L_2. Problem 1 is analyzed by the following steps:

1) To get sufficient conditions of finite-time global synchronization for the NCMSCNs and the LCMSCNs.

2) To build the relationship conditions of finite-time global synchronization for the NCMSCNs and the LCMSCNs.

3) The effect of nonlinear coupling function for finite-time global synchronization of NCMSCNs is testified by adjusting L_1 and L_2.

(ii) The second idea. If nonlinear coupling function $g(x,y)$ only satisfies the Lipschitz condition, the first idea is not used. In this situation, how to study problem 1? In this paper, we adopt a new idea to investigate problem 1. The idea is to compare the differences of synchronization control rules for the NCMSCNs and the LCMSCNs. According to the differences, problem 1 can be explored. The steps are as follows: 1) Sufficient conditions of finite-time global synchronization for the NCMSCNs and the LCMSCNs are derived. 2) The difference relationship of synchronization control rules for the NCMSCNs and the LCMSCNs is built. 3) To analyze the relationship between the differences and problem 1. 4) To testify the results of 3).

Problem 2 (Finite-time Global Synchronization Conditions For LCMSCNs and NCMSCNs)**:** In order to solve problem 1, a class of LCMSCNs and a class of NCMSCNs are considered. By using feedback control technique, sufficient conditions of finite-time global synchronization for the two classes of Markovian switching complex networks are derived. The proposed schemes are testified by Chua's circuit networks.

Comparing with previous works [12, 21, 22, 26, 33, 34], combining the above analysis and problem 1, the main difficulties and the main improvements of this paper are as follows.

(i) The main difficulties of this paper are that how to solve problem 1 and how to analyze the conservatism of the proposed results based on problem 1. In [12], from the addressed model aspect, although the authors considered a class of nonlinearly and stochastically coupled networks with Markovian switching, they focused on sufficient conditions for the exponential synchronization of the addressed model via events-triggered sampling by self-adaptive learning. In [21, 22, 33, 34], the authors mainly concentrated on sufficient conditions for the exponential synchronization of finite-time global synchronization of the addressed LCMSCNs, instead of problems 1-2 of this paper. Despite Liu *et al*. [26] investigated the effect of nonlinear coupling function for global synchronization and synchronization conditions for nonlinear coupled complex network, there are three differences between [26] and this paper. The first is the addressed synchronization problem aspect. Global synchronization was considered in [26]. In this paper, we study finite-time global synchronization. The second is the addressed model aspect. In the model of this paper, there is Markovian switching. In the model of [26], there was not Markovian switching. The third is that viewpoints of analyzing problem 1 in the two papers are quite different. From the above analysis, it is observed that until now, there is no literatures of problem 1 based on the above two ideas. Therefore, to use the above viewpoints to study problem 1 is a great challenge.

(ii) The main improvement of this paper is that the works of this paper can extend the existed analyzing ideas of the finite-time global synchronization for nonlinear coupling complex networks, including NCMSCNs.

The main contributions of this paper is the above main improvement of this paper.

The rest of this paper is organized as follows: Model and preliminaries are presented in Section 2. In Section 3, the sufficient conditions of finite-time synchronization based on Problems 1 and 2 are given. Chua's circuit network simulations of the proposed results and the analysis of Problem 1 are provided in Section 4. The conclusions are made in Section 5.

2. PROBLEM FORMULATION AND PRELIMINARIES

In this section, problem formulation and preliminaries are briefly introduced.

Let $\{r(t), t \geq 0\}$ be a right-continuous Markovian chain on the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ taking values in a finite state space $S = \{1, 2, \cdots, s\}$ with a generator $\Pi = (\delta_{ij})_{s\times s}$ $(i, j \in S)$ given by

$$P\{r(t+\Delta t) = j|r(t) = i\} = \begin{cases} \delta_{ij}\Delta t + o(\Delta t), if(i \neq j), \\ 1+\delta_{ij}\Delta t + o(\Delta t), if(i = j), \end{cases}$$

where $\Delta t > 0$, $\lim_{\Delta t\to 0}(o(\Delta t)/\Delta t) = 0$, $\delta_{ij} > 0$ $(\forall i \neq j)$ is the transition rate from mode i to mode j and $\delta_{ii} = -\sum_{i\neq j}\delta_{ij} < 0$.

In this paper, we consider a class of NCMSCNs with time delay and a class of LCMSCNs with time delay, respectively. They are as follows:

$$\dot{x}_i(t) = c(r(t))\sum_{j=1}^{N} a_{ij}(r(t))\Gamma(r(t))g(x_j(t), x_j(t-\tau)) + B(r(t))f(x_i(t)) + u_i(t, r(t)), \quad (1)$$

$$\dot{x}_i(t) = c(r(t))\sum_{j=1}^{N} a_{ij}(r(t))\Gamma(r(t))(x_j(t) + x_j(t-\tau)) + B(r(t))f(x_i(t)) + u_i(t, r(t)), \quad (2)$$

where $i = 1, 2, \ldots, N$, $c(r(t))$ represents the coupling strength in mode $r(t)$ and $c(r(t)) > 0$, $\Gamma(r(t)) = (\gamma_{ij}) \in R^{n\times n}$ $(\gamma_{ii} > 0)$ is inner-coupling matrix in mode $r(t)$, $A(r(t)) = (a_{ij}(r(t)))_{N\times N}$ is outer-coupling matrix, $\tau > 0$ is coupling node time delay, $f(\cdot) : R^n \to R^n$ which stands for the activity of ith node is a vector-value function, $g(\cdot,\cdot)$ denotes nonlinear coupling function and $g(\cdot,\cdot) : R^n \to R^n$, $u_i(t, r(t))$ denotes the control input of ith node in mode $r(t)$.

Remark 1: Comparing the nonlinear coupling one, nonlinear coupling function in the network (1) of this paper is more general than that of [5, 23–28, 36]. The reason is that nonlinear coupling function was $g(x(t))$ in [5, 23–28, 36]. We can see that $g(x(t))$ is one special case of $g(x(t), x(t-\tau))$. Beside this, in [21, 22, 33, 34], the authors addressed LCMSCNs. Therefore, from coupling function aspect, the network (1) of this paper extends the models of [21, 22, 33, 34].

Let s(t) be the synchronization state of the networks (1)-(2), then s(t) satisfies

$$\dot{s}(t) = B(r(t))f(s(t)). \quad (3)$$

According to some literatures of finite-time synchronization for complex networks [21, 22, 31–34], and combining the networks (1)-(2), the controller is as follow:

$$u_i(t, r(t)) = -\varepsilon_i(r(t))\Gamma(r(t))e_i(t) - \frac{k_i(r(t))}{\rho(r(t))} sign(e_i(t))|e_i(t)|^{\beta}, \quad (4)$$

where $\varepsilon_i(r(t)) > 0$, $k_i(r(t)) > 0$, $\rho(r(t)) > 0$, $|e_i(t)|^{\beta} = (|e_{i1}(t)|^{\beta}, |e_{i2}(t)|^{\beta}, \ldots, |e_{in}(t)|^{\beta})^T$, $sign(\cdot)$ is the sign function and $sign(e_i(t)) = (sign(e_{i1}(t)), sign(e_{i2}(t)), \ldots, sign(e_{in}(t)))^T$, β satisfies $0 < \beta < 1$ and $\beta \in R$.

For notation simplicity, we denote $B(r(t))$, $\Gamma(r(t))$, $c(r(t))$, $a_{ij}(r(t))$, $\varepsilon_i(r(t))$, $k_i(r(t))$, and $\rho(r(t))$ as B_r, Γ_r, c_r, a^r_{ij}, ε_{ir}, $k_{ir}$and ρ_r, respectively.

Remark 2: Comparing the network models (1) and (2), there is no difference except coupling functions. Why we discuss them? The reason is closely related to problem 1 based on the second idea. According to the steps of the second idea, in order to analyze problem 1, firstly, finite-time synchronization conditions of the networks (1) and (2) need to be derived. Secondly, by using the obtained results, $\varepsilon^{(1)}_{ir}$ and $k^{(1)}_{ir}$ of the controller (4) for finite-time synchronization of the network (1) are designed. Similar to the above method, $\varepsilon^{(2)}_{ir}$ and $k^{(2)}_{ir}$ of the controller (4) for finite-time synchronization of the network (2) can also be designed. Thus, under the above two obtained controller (4), the networks (1) and (2) can achieve finite-time synchronization, respectively. Furthermore, it is not difficult to find the differences between $\varepsilon^{(1)}_{ir}$ and $\varepsilon^{(2)}_{ir}$, $k^{(1)}_{ir}$ and $k^{(1)}_{ir}$. Because the networks (1) and (2) are quite same except coupling functions $g(x_j(t),x_j(t-\tau))$ and $x_j(t)+x_j(t-\tau)$, the differences of the control rules of $\varepsilon^{(1)}_{ir}$, $\varepsilon^{(2)}_{ir}$, $k^{(1)}_{ir}$ and $k^{(1)}_{ir}$ are originated from the coupling functions of the networks (1) and (2). Therefore, by adopting the viewpoint, the problem 1 is discussed.

Subtracting (3) from (1) and (2), we can obtain the following error system of the networks (1) and (2), respectively.

$$\begin{aligned}\dot{e}_i(t) =& B_rF(e_i(t))+c_r\sum_{j=1}^{N}a^r_{ij}\Gamma_rG(e_j(t),e_j(t-\tau))\\ &-\varepsilon_{ir}\Gamma_re_i(t)-\frac{k_{ir}}{\rho_r}sign(e_i(t))|e_i(t)|^{\beta}, \qquad (5)\\ \dot{e}_i(t) =& B_rF(e_i(t))+c_r\sum_{j=1}^{N}a^r_{ij}(e_j(t)+e_j(t-\tau))\\ &-\varepsilon_{ir}\Gamma_re_i(t)-\frac{k_{ir}}{\rho_r}sign(e_i(t))|e_i(t)|^{\beta}, \qquad (6)\end{aligned}$$

where $i=1,2,\ldots,N$, $F(e_i(t))=f(x_i(t))-f(s(t))$, $G(e_j(t),e_j(t-\tau))=g(x_j(t),x_j(t-\tau))-g(s(t),s(t-\tau))$.

Remark 3: As is known to all, traditional finite-time control techniques are based on sliding mode controllers, which utilize *sign* function and give rise to the phenomenon of chattering. How to avoid this phenomenon in (4). From [28,37,39], chatting will occur when the control in the addressed system adopts switching function. Until now, some methods for attenuating chatting have been proposed [28, 37, 39]. In [37], although the *sign* function in the switching control term was used, the switching control term can be softened to be a smooth signal by using low-pass filter technique. Tang addressed that some "smooth" function must be used instead of the *sign* function in order to eliminate chatting of sliding mode control system [38]. Despite there is the *sign* function in the controller (4) of this paper, the phenomenon of chatting can not be happened because the switching control term $(\frac{\varepsilon_{ir}}{\rho(r(t))}|e_i(t)|^{\beta})sign(e_i(t))$ is smooth function when $e_i(t)>0$ and $e_i(t)<0$. The analysis is as follows. Assuming that $e_i(t)$ satisfies $\|\lim_{\triangle\to 0}e_i(t+\triangle)-e_i(t)\|=C_i$, where $C_i>0$, one obtains $\|\dot{e}_i(t)\|=\|\lim_{\triangle\to 0}\frac{e_i(t+\triangle)-e_i(t)}{\triangle}\|=\|\lim_{\triangle\to 0}\frac{C_i}{\triangle}\|\to+\infty$. Because the functions $F(e_i(t))$ and $G(e_j(t),e_j(t-\tau))$ satisfy assumption 1, the two functions are bounded. Thus, combining (5) and (6), we have $\|(-\varepsilon_{ir}\Gamma_re_i(t)-\frac{\varepsilon_{ir}}{\rho_r}sign(e_i(t))|e_i(t)|^{\beta})\|\to+\infty$. That means $\|e_i(t)\|\to+\infty$ and $\|\,|e_i(t)|^{\beta}\,\|\to+\infty$. It is clear that the result is wrong. In order to make $\|\dot{e}_i(t)\|$ bounded, we have $\|\dot{e}_i(t)\|=\|\lim_{\triangle\to 0}\frac{C_i}{\triangle}\|\le\chi<+\infty$. Thus, we get $\|\lim_{\triangle\to 0}(e_i(t+\triangle)-e_i(t))\|\to 0$. This shows that $e_i(t)$ is smooth function. Therefore, $e^{\beta}_i(t)$ $(e_i(t)>0, sign(e_i(t))=1)$ and $(-e_i(t))^{\beta}$ $(e_i(t)<0, sign(e_i(t))=-1)$ are smooth function. We can draw that $(\frac{\varepsilon_{ir}}{\rho(r(t))}|e_i(t)|^{\beta})sign(e_i(t))$ is smooth function when $e_i(t)>0$ and $e_i(t)<0$.

In order to obtain the main results, the following definition, assumptions and lemmas are needed.

Definition 1: The networks (1) and (2) are said to achieve global synchronization in finite-time t^*, if there exists a constant $t^*>0$ depends on the initial state vector value $x(t)$, for any $t\ge t^*$, such that

$$E\|x_i(t)-s(t)\|=0, as\ \ t\to t^*$$

holds for any $i\in\{1,2,\ldots,N\}$, where $x(t)=(x^T_1(0),\ldots,x^T_N(0))^T$, $s(t)=(s_1(t),\ldots,s_n(t))^T\in R^n$ is the synchronization state of the the networks (1) and (2).

Assumption 1: The functions $f(\cdot)$ and $g(\cdot,\cdot)$ satisfy the Lipschitz condition. That means there exist constants $L>0$, $L_1>0$ and $L_2>0$ such that

$$\begin{aligned}&\|f(x)-f(y)\|\le L\|x-y\|, \forall x,y\in R^n.\\ &\|g(x_1,y_1)-g(x_2,y_2)\|\le L_1\|x_1-x_2\|+L_2\|y_1-y_2\|,\\ &\forall x_1,x_2,y_1,y_2\in R^n.\end{aligned}$$

Remark 4: To deal with nonlinear coupling, there are some methods existed. According to the existed literatures [5, 23–28, 36], which were about to investigate synchronization problems for nonlinear coupling complex networks, some methods of solving nonlinear coupling include:

1) Nonlinear coupling function $g(x)$ is processed by Lipschitz condition. This can be seen in [23, 24].

2) $g(x)-\alpha x$ is solved by Lipschitz condition, where $\alpha>0$. The [5, 25, 36] showed the method.

3) Nonlinear coupling function is bounded. The scheme was used in [27].

4) Nonlinear coupling function satisfies Lipschitz condition and is bounded. This method was adopted in [28].

5) Nonlinear coupling function satisfies the local Lipschitz condition and cooperative property of the nonlinear protocol. This was shown in [26]. Actually, from the above existed methods, in order to obtain proposed

results, nonlinear coupling functions need to satisfy norm-bounded condition. Furthermore, comparing nonlinear coupling methods of [5, 23–28, 36] and this paper, one obtains that the following results:

1) In [23, 24], $g(x)$ satisfies $\|g(x)-g(y)\| \leq L\|x-y\|$, where $L>0$. It is clear that $\|g(x)-g(y)\| \leq L\|x-y\|$ is one special case of $\|g(x_1,y_1)-g(x_2,y_2)\| \leq L_1\|x_1-x_2\| + L_2\|y_1-y_2\|$, where $L_1>0$ and $L_2>0$.

2) In [5, 25, 36], $g(x)$ satisfies $|(g(x)-g(y))-\alpha(x-y)| \leq \beta|x-y|$, where $\alpha>0$, $\beta>0$, $|\cdot|$ means the absolute value. Let $\|(g(x)-g(y))-\alpha(x-y)\| \leq \beta\|x-y\|$. If $g(x)$ satisfies Lipschitz condition, one obtains that $\|(g(x)-g(y))-\alpha(x-y)\| \leq \|(g(x)-g(y))\|+\alpha\|(x-y)\| \leq \gamma\|x-y\|+\alpha\|(x-y)\| = (\gamma+\alpha)\|(x-y)\| = L\|(x-y)\|$, where $L=\gamma+\alpha$, $\gamma>0$ and $\alpha>0$. That means under the above condition, the method of solving $g(x)$ in [5, 25, 36] is one special case of assumption 1 in this paper.

3) In [27], $g(x)$ satisfies $\|g(x)\| \leq \zeta$, where $\zeta>0$. If $\zeta = L\|x-y\|$, $\|g(x)\| \leq \zeta$ is one special case of $\|g(x_1,y_1)-g(x_2,y_2)\| \leq L_1\|x_1-x_2\|+L_2\|y_1-y_2\|$.

4) In [28], $g(x,y)$ satisfies $\|(g(x,y)-g(x,y))\| \leq \vartheta\|x-y\|+\theta(y)$, where $\vartheta>0$ and $\theta(y)\geq 0$. Let $\theta(y)=L_2\|y_2-y_1\|$, then the method of [28] and this paper is same.

5) In [26], $g(x,y)$ satisfies three conditions: (i) $g(x,y)$ is a continuous mapping and satisfies the local Lipschitz condition. (ii) $g(x,y)$ satisfies $(x-y)g(y,x) \leq -\alpha(x-y)^2$, where $\forall x \neq y$, $\alpha>0$. (iii) $g(x,y)=-g(y,x)$. In Assumption 1 of this paper, if $g(x,y)$ satisfies the local Lipschitz condition, the conservatism of the method which dealt to nonlinear coupling function $g(x,y)$ in [26] is higher than that of Assumption 1 in this paper.

Assumption 2: Time delay in the networks (1)-(2) satisfies $0 \leq \tau \leq \tau_M$.

Assumption 3 (Yin *et al*. [40])**:** Let $0<\beta<1$ and $\lambda>0$, there exists a continuous function $g:[0,\infty)\to[0,\infty)$ with $g(0)>0$, for any $0\leq u\leq t$, such that

$$g(t)-g(u) \leq -\lambda\int_u^t (g(s))^{\beta}ds.$$

Assumption 4: Suppose that the initial condition of the networks (1) and (2) are given by $x_i(z)=\varphi_i(z)\in C([-\tau,0],R^n), i=1,2,\ldots,N$, where $C([-\tau,0],R^n)$ denotes the set of continuous functions mapping the interval $[-\tau,0]$ into R^n.

Lemma 1 (Bhat *et al*. [41])**:** Suppose that function $V(t):[0,\infty)\to[0,\infty)$ is differentiable (the derivative of V(t) at 0 is in fact its right derivative) and

$$\frac{dV(t)}{dt} \leq -\eta V^{\alpha}(t),$$

where $\eta>0$ and $0<\alpha<1$. Then V(t) will reach zero at finite time $t^* \leq V^{1-\alpha}(t)/(\eta(1-\alpha))$ and V(t)=0 for all $t\geq t^*$.

Lemma 2 (Boyd *et al*. [42])**:** For any vector $x,y\in R^n$ and one positive definite matrix $Q>0$, the following inequality holds

$$2x^Ty \leq x^TQ^{-1}x+y^TQy.$$

Lemma 3 (Mei *et al*. [43])**:** Let $x_1,x_2,\ldots,x_n\in R^n$ are any vectors and $0<q<2$ is a real number satisfying

$$\|x_1\|^q+\cdots+\|x_n\|^q \geq (\|x_1\|^2+\cdots+\|x_n\|^2)^{q/2}.$$

Lemma 4 (Wang and Xiao [44])**:** If $a_1,a_2,\cdots,a_n\geq 0$ and $0<p\leq 1$, then

$$(\sum_{i=1}^n a_i)^p \leq \sum_{i=1}^n a_i^p.$$

Lemma 5: If one positive definite matrix $Q=(q_{ij})_{n\times n}\in R^{n\times n}$ and $Q>0$, then the following statements are equivalent:
1) $q_{ii}>0$.
2) Every real eigenvalue of Q is positive.

Lemma 6: If $g(x,y)$ is derivable and $g(x,y)$ satisfies the Lipschitz condition, there exist constants $L_1>0$ and $L_2>0$ such that $\|g_x^{'}(x,y)\| \leq L_1$, $\|g_y^{'}(x,y)\| \leq L_2$, where $g_x^{'}(x,y)$ and $g_y^{'}(x,y)$ are the partial derivatives of $g(x,y)$ with respect to independent variables x and y, respectively.

Proof: Because $g(x,y)$ is derivable and $g(x,y)$ satisfies the Lipschitz condition, that means there are

$$\|g_x^{'}(x,y)\| = \|\lim_{x\to x_1}\frac{g(x,y)-g(x_1,y)}{x-x_1}\| \leq L_1.$$

$$\|g_y^{'}(x,y)\| = \|\lim_{y\to y_1}\frac{g(x,y)-g(x,y_1)}{y-y_1}\| \leq L_2.$$

Remark 5: In this paper, according to assumption 1, lemma 6, and combining the first idea, one obtains $g(x,y)=L_1x+L_2y$, $g(x,y)=L_1x-L_2y$, $g(x,y)=-L_1x+L_2y$ and $g(x,y)=-L_1x-L_2y$, where $L_1>0$, $L_2>0$. It is observed that $g(x,y)=L_1x-L_2y$ and $g(x,y)=-L_1x+L_2y$ are not strictly increasing. In [26], in order to avoid nonlinear coupling function $g(x,y)$ to increase strictly, and make $g(x,y)$ become more general, $g(x,y)$ satisfies the local Lipschitz condition and cooperative property of the nonlinear protocol. From strict increase aspect, the techniques of this paper and [26], which are to process $g(x,y)$, are very similar. But for problem 1, the method of this paper is more practical and more flexible than that of [26]. The reason is that the scheme of this paper is based on adjusting L_1 and L_2. The way of [26] was built on choosing different classes of nonlinear functions $g(x,y)$ according to $(x-y)g(y,x)\leq-\alpha(x-y)^2$.

Remark 6: Although $g(x,y)=L_1x-L_2y$ and $g(x,y)=-L_1x+L_2y$ are not strictly increasing, nonlinearity of $g(x,y)$ will become more and more serious if L_1 and L_2 are increased. The reason is that in the first idea of problem 1, there are $\|g_x^{'}(x,y)\| = \|\lim_{x\to x_1}\frac{g(x,y)-g(x_1,y)}{x-x_1}\| = L_1$ and $\|g_y^{'}(x,y)\| = \|\lim_{y\to y_1}\frac{g(x,y)-g(x,y_1)}{y-y_1}\| = L_2$.

3. MAIN RESULTS

In this section, finite-time synchronization of the networks (1)-(2) are studied. Furthermore, in order to analyze problem 1, Theorems 2-3 and Corollaries 1-3 are derived, respectively.

3.1. Finite-time Global Synchronization Condition For The Network (1).

Theorem 1: Let Assumptions 1-4 hold, then the network (1) is global synchronization under the set of controller (4) in finite time t^* if the following conditions are satisfied:

1) If $p \neq r$, $q_p\rho_p - a_r \leq 0$, otherwise, if $p = r$, $q_p\rho_p - a_r \geq 0$, where $r, p \in S$.

2) The following LMI holds:

$$\Phi_r \leq 0, \tag{7}$$

where

$$\Phi_r = \begin{bmatrix} \Phi_r^{(1)} & 0 \\ \star & \Phi_r^{(2)} \end{bmatrix} \leq 0,$$

$$\begin{aligned}\Phi_r^{(1)} = {} & \theta_r I_N \otimes I_n + I_N \otimes B_r Q_r^{-1} B_r^T \\ & + 2L_1^2 c_r \|Q_{r(1)}\| I_N \otimes I_n \\ & + c_r (A_r \otimes \Gamma_r) Q_{r(1)}^{-1} (A_r \otimes \Gamma_r)^T - 2\Xi_r \otimes \Gamma_r,\end{aligned}$$

$$\Phi_r^{(2)} = (2L_2^2 c_r \|Q_{r(1)}\| - \rho_r + \upsilon) I_N \otimes I_n,$$

$$\theta_r = \rho_r + L^2\|Q_r\| + \sum_{p=1}^{s} \frac{\delta_{rp} q_p}{q_r} - \upsilon,$$

$$\Xi_r = diag\{\varepsilon_{1r}, \dots, \varepsilon_{Nr}\}.$$

3) t^* is estimated by $t^* \leq \tau + \frac{\rho^{\frac{1+\beta}{2}} V(0,r(0))^{1-\frac{1+\beta}{2}}}{\nu(1-\frac{1+\beta}{2})}$, $k = \min_{i\in\{1,2,\dots,N\}}^{r\in S}\{k_{ir}\}$, $\lambda > 0$, $0 < \beta < 1$, $\upsilon > 0$, $\nu = min\{\lambda\upsilon, 2k\}$, $\rho = \max_{r\in S}\{\rho_r\}$, $V(0,r(0)) = q_{r(0)} \sum_{i=1}^{N} e_i^T(0) e_i(0)$, $e_i(0) (i = 1,2,\dots,N)$ *is the initial condition.*

Proof: Construct a Lyapunov-Krasovskii functional candidate as

$$\begin{aligned}V(e(t),t,r(t)) = {} & q_r[\sum_{i=1}^{N} e_i^T(t) e_i(t) \\ & + \rho_r \sum_{i=1}^{N} \int_{t-\tau}^{t} e_i^T(s) e_i(s) ds],\end{aligned} \tag{8}$$

where $q_r \geq \rho_r > 0, r \in S$.

Computing $\mathcal{L}V(e(t),t,r)$ along the trajectory of error system (5), one can obtain that

$$\begin{aligned}& \mathcal{L}V(e(t),t,r) \\ = {} & V_t(e(t),t,r) + V_e(e(t),t,r) \\ & \times [B_r F(e_i(t)) + c_r \sum_{j=1}^{N} a_{ij}^r \Gamma_r G(e_j(t), e_j(t-\tau)) \\ & - \varepsilon_{ir} \Gamma_r e_i(t) - \frac{k_{ir}}{\rho_r} sign(e_i(t)) |e_i(t)|^\beta] \\ & + \sum_{p=1}^{s} \delta_{rp} V(e(t),t,p) \\ = {} & q_r \rho_r \sum_{i=1}^{N} [e_i^T(t) e_i(t) - e_i^T(t-\tau) e_i(t-\tau)] \\ & + 2q_r \sum_{i=1}^{N} e_i^T(t) [B_r F(e_i(t)) \\ & + c_r \sum_{j=1}^{N} a_{ij}^r \Gamma_r G(e_j(t), e_j(t-\tau)) \\ & - \varepsilon_{ir} \Gamma_r e_i(t) - \frac{k_{ir}}{\rho_r} sign(e_i(t)) |e_i(t)|^\beta] \\ & + \sum_{p=1}^{s} \delta_{rp} V(e(t),t,p).\end{aligned} \tag{9}$$

From Assumption 1 and Lemma 2, we have

$$\begin{aligned}& 2q_r \sum_{i=1}^{N} e_i^T(t) B_r F(e_i(t)) \\ & \leq q_r \sum_{i=1}^{N} [e_i^T(t) B_r Q_r^{-1} B_r^T e_i(t) + L^2 \|Q_r\| e_i^T(t) e_i(t)],\end{aligned} \tag{10}$$

$$\begin{aligned}& 2q_r \sum_{i=1}^{N} e_i^T(t) c_r \sum_{j=1}^{N} a_{ij}^r \Gamma_r G(e_j(t), e_j(t-\tau)) \\ & = 2q_r c_r e^T(t) (A_r \otimes \Gamma_r) G(e(t), e(t-\tau)) \\ & \leq q_r c_r [e^T(t) (A_r \otimes \Gamma_r) Q_{r(1)}^{-1} (A_r \otimes \Gamma_r)^T e(t) \\ & \quad + G^T(e(t), e(t-\tau)) Q_{r(1)} G(e(t), e(t-\tau))] \\ & \leq q_r c_r [e^T(t) (A_r \otimes \Gamma_r) Q_{r(1)}^{-1} (A_r \otimes \Gamma_r)^T e(t) \\ & \quad + 2L_1^2 \|Q_{r(1)}\| e^T(t) e(t) \\ & \quad + 2L_2^2 \|Q_{r(1)}\| e^T(t-\tau) e(t-\tau)].\end{aligned} \tag{11}$$

Because of $\sum_{r,p\in S} \delta_{rp} = 0$, for $\forall a_r > 0 (r \in S)$, we can get

$$\sum_{r,p\in S} \delta_{rp} a_r = 0. \tag{12}$$

Thus,

$$\begin{aligned}& \sum_{p=1}^{s} \delta_{rp} V(e(t),t,p) \\ & = \sum_{p=1}^{s} \delta_{rp} q_p \left[\sum_{i=1}^{N} e_i^T(t) e_i(t) c + \rho_p \sum_{i=1}^{N} \int_{t-\tau}^{t} e_i^T(s) e_i(s) ds\right] \\ & = \sum_{p=1}^{s} \delta_{rp} q_p \sum_{i=1}^{N} e_i^T(t) e_i(t) \\ & \quad + \sum_{r,p\in S} \delta_{rp} (q_p \rho_p - a_r) \sum_{i=1}^{N} \int_{t-\tau}^{t} e_i^T(s) e_i(s) ds,\end{aligned} \tag{13}$$

Denoting $\Xi_r = diag\{\varepsilon_{1r}, \dots, \varepsilon_{Nr}\}$, $k = \min_{i\in\{1,2,\dots,N\}}^{r\in S}\{k_{ir}\}$, and substituting (10)-(13) into (9), then taking the expectation on both sides of (9), according to the condition (1) in Theorem 1, we have

$$\begin{aligned}
&E[\mathcal{L}V(e(t),t,r)]\\
&\leq E\{q_r[e^T(t)\Phi_r^{(1)}e(t)+e^T(t-\tau)\Phi_r^{(2)}e(t-\tau)]\\
&\quad +q_r\upsilon[\sum_{i=1}^{N}e_i^T(t)e_i(t)-\sum_{i=1}^{N}e_i^T(t-\tau)e_i(t-\tau)]\\
&\quad -\frac{2q_rk}{\rho_r}\sum_{i=1}^{N}\sum_{l=1}^{n}|e_{il}(t)|^{1+\beta},
\end{aligned} \tag{14}$$

where

$$\begin{aligned}
\Phi_r^{(1)}&=\theta_r I_N\otimes I_n+I_N\otimes B_rQ^{-1}B_r^T+2L_1^2c_r\|Q_{r1}\|I_N\otimes I_n\\
&\quad +c_r(A_r\otimes\Gamma_r)Q_{r(1)}^{-1}(A_r\otimes\Gamma_r)^T-2\Xi_r\otimes\Gamma_r,\\
\Phi_r^{(2)}&=(2L_2^2c_r\|Q_{r1}\|-\rho_r+\upsilon)I_N\otimes I_n,\\
\theta_r&=\rho_r+L^2\|Q_r\|+\sum_{p=1}^{s}\frac{\delta_{rp}q_p}{q_r}-\upsilon.
\end{aligned}$$

By Lemma 4, we get

$$\begin{aligned}
&-\frac{2q_rk}{\rho_r}\sum_{i=1}^{N}\sum_{l=1}^{n}|e_{il}(t)|^{1+\beta}=-\frac{2q_rk}{\rho_r}\sum_{i=1}^{N}(e_i^T(t)e_i(t))^{\frac{1+\beta}{2}}\\
&\leq -2k(\frac{q_r}{\rho_r}\sum_{i=1}^{N}e_i^T(t)e_i(t))^{\frac{1+\beta}{2}},
\end{aligned} \tag{15}$$

where $q_r\geq\rho_r>0$.
Let $\lambda>0$ and $\upsilon>0$, then combining Assumption 3 and Lemma 4, we obtain

$$\begin{aligned}
&q_r\upsilon[\sum_{i=1}^{N}e_i^T(t)e_i(t)-\sum_{i=1}^{N}e_i^T(t-\tau)e_i(t-\tau)]\\
&\leq -\lambda\upsilon\sum_{i=1}^{N}\int_{t-\tau}^{t}(q_re_i^T(s)e_i(s))^{\frac{1+\beta}{2}}ds\\
&\leq -\lambda\upsilon(q_r\sum_{i=1}^{N}\int_{t-\tau}^{t}e_i^T(s)e_i(s)ds)^{\frac{1+\beta}{2}}.
\end{aligned} \tag{16}$$

Thus, substituting (15)-(16) into (14), we have

$$\begin{aligned}
&E[\mathcal{L}V(e(t),t,r(t))]\\
&\leq E\{q_r[e^T(t)\Phi_r^{(1)}e(t)e^T(t-\tau)\Phi_r^{(2)}e(t-\tau)]\\
&\quad -\Big[2k(\frac{q_r}{\rho_r}\sum_{i=1}^{N}e_i^T(t)e_i(t))^{\frac{1+\beta}{2}}\\
&\quad +\lambda\upsilon(q_r\sum_{i=1}^{N}\int_{t-\tau}^{t}e_i^T(s)e_i(s)ds)^{\frac{1+\beta}{2}}\Big]\}.
\end{aligned} \tag{17}$$

Let $\nu=min\{\lambda\upsilon,2k\}$, then by Lemmas 3-4, and the condition (2) in Theorem 1, we get

$$\begin{aligned}
&E[\mathcal{L}V(e(t),t,r(t))]\\
&\leq -2\varepsilon E\{((\frac{q_r}{\rho_r}\sum_{i=1}^{N}e_i^T(t)e_i(t))^{\frac{1+\beta}{2}}\\
&\quad +(q_r\sum_{i=1}^{N}\int_{t-\tau}^{t}e_i^T(s)e_i(s)ds)^{\frac{1+\beta}{2}})\}\\
&\leq -2\varepsilon E\{((\frac{1}{\rho_r}(q_r\sum_{i=1}^{N}e_i^T(t)e_i(t)\\
&\quad +q_r\rho_r\sum_{i=1}^{N}\int_{t-\tau}^{t}e_i^T(s)e_i(s)ds)^{\frac{1+\beta}{2}})\}\\
&\leq -\frac{2\varepsilon}{\rho^{\frac{1+\beta}{2}}}E[V^{\frac{1+\beta}{2}}(e(t),t,r(t))],
\end{aligned} \tag{18}$$

where $\rho=\max_{r\in S}\{\rho_r\}$.
For any $t_0\geq\tau>0$, we have $E[V^{\frac{1+\beta}{2}}(t_0)]=(E[V(t_0)])^{\frac{1+\beta}{2}}$. Therefore, we can obtain

$$E[\mathcal{L}V(e(t),t,r)]\leq -\frac{\nu}{\rho^{\frac{1+\beta}{2}}}(E[V(t)])^{\frac{1+\beta}{2}}. \tag{19}$$

According to Lemma 1, $E[V(t)])$ converges to zero in finite time and finite time is estimated by

$$t^*\leq\tau+\frac{\rho^{\frac{1+\beta}{2}}V(0,r(0))^{1-\frac{1+\beta}{2}}}{\nu(1-\frac{1+\beta}{2})}. \tag{20}$$

This shows that $V(e(t),t,r)=0$ if $t\geq t^*$. That means $e_i(t)=0$ if $t\geq t^*$. By Definition 1, if $t\geq t^*$, we have $E\|x_i(t)-s(t)\|=0$. Hence, the network (1) will achieve finite-time global synchronization under the controller (4) within finite time t^*. This completes the proof. □

Remark 7: Under Theorem 1, the network (1) with the controller (4) can achieve synchronization with finite-time t^*. If problem 1 is analyzed by Theorem 1, the following steps are needed: Step 1: Nonlinear coupling function $g^{(1)}(x(t),x(t-\tau))$ is chosen. Step 2: According to Assumption 1, $L_1^{(1)}$ and $L_2^{(1)}$ are decided. Step 3: By Theorem 1, $\varepsilon_{ir}^{(1)}$ and $k_{ir}^{(1)}$ for finite-time synchronization of the network (1) are designed, where $i=1,2,...,N$ and $r=1,2,...,s$. Step 4: By simulation, problem 1 is testified. Step 5: The $\varepsilon_{ir}^{(1)}$ and $k_{ir}^{(1)}$ are fixed. Step 6: A new nonlinear coupling function $g^{(2)}(x(t),x(t-\tau))$ is chosen. Step 7: A new $L_1^{(2)}$ and $L_2^{(2)}$ are decided by Assumption 1. Step 8: Substituting $L_1^{(2)}$, $L_2^{(2)}$, $\varepsilon_{ir}^{(1)}$ into the condition (2) of Theorem 1, if the condition (2) of Theorem 1 is held, return step 4, otherwise, continue step 9. Step 9: By Theorem 1, $\varepsilon_{ir}^{(2)}$ and $k_{ir}^{(2)}$ for finite-time synchronization of the network (1) are designed. Step 10: Let $\varepsilon_{ir}^1:=\varepsilon_{ir}^2$ and $k_{ir}^{(1)}:=k_{ir}^{(2)}$, then return step 4. From steps 1-2 and 6-7, it is observed that in order to testify problem 1, it is necessary to get L_1 and L_2 by choosing different classes of $g(x(t),x(t-\tau))$. It is clear that this method is not practical. In [26], the steps of the problem 1 is very similar to that of the above steps.

3.2. The Effect of Nonlinear Coupling Function For Finite-time Global Synchronization Of The Network (1)

The proof of Theorems 2-4 is similar to that of Theorem 1, we only give the results.

3.2.1 The first idea

Case 1: In the network (1), let $g(x(t),x(t-\tau)) = \hat{L}x(t)+L_2x(t-\tau)$, where $\hat{L}=L_1$ or $\hat{L}=-L_1$, $L_1>0, L_2>0$.

1) To get sufficient conditions of finite-time global synchronization for the network (1) in case 1.

Theorem 2: Let Assumptions 1-4 hold, then the network (1) in case 1 is global synchronization under the set of controller (4) in finite time t^* if the following conditions are satisfied:

$$\begin{aligned}\tilde{\Phi}_r^{(1)} =&\theta_r I_N\otimes I_n + I_N\otimes B_rQ_r^{-1}B_r^T+2c_r\hat{L}(A_r\otimes\Gamma_r)\\&+c_rL_2(A_r\otimes\Gamma_r)\tilde{Q}_{r(1)}^{-1}(A_r\otimes\Gamma_r)^T-2\Xi_r\otimes\Gamma_r,\\ \tilde{\Phi}_r^{(2)} =&(c_rL_2\|\tilde{Q}_{r(1)}\|-\rho_r+\upsilon)I_N\otimes I_n.\end{aligned}$$

The other parameters of Theorems 1 and 2 are quite same.

2) To build the relationship conditions of Theorems 1 and 2.

Corollary 1: Let $M_r^{(1)} = \tilde{\Phi}_r^{(1)}-\Phi_r^{(1)} = 2c_r(\hat{L}(A_r\otimes\Gamma_r)-L_1^2\|Q_{r1}\|I_N\otimes I_n)+c_r(L_2(A_r\otimes\Gamma_r)\tilde{Q}_{r(1)}^{-1}(A_r\otimes\Gamma_r)^T-(A_r\otimes\Gamma_r)Q_{r(1)}^{-1}(A_r\otimes\Gamma_r)^T)$, and $M_r^{(2)} = \tilde{\Phi}_r^{(2)}-\Phi_r^{(2)} = c_rL_2(\|\tilde{Q}_{r(1)}\|-2L_2\|Q_{r(1)}\|)I_N\otimes I_n$. If $M_r^{(1)}\leq 0$ and $M_r^{(2)}\leq 0$, under Theorem 2, the network (1) of case 1 with the controller (4) must satisfy Theorem 1.

Proof: According to Theorems 1-2, we have

$$\begin{aligned}M_r^{(1)} =&\tilde{\Phi}_r^{(1)}-\Phi_r^{(1)} = 2c_r(\hat{L}(A_r\otimes\Gamma_r)-L_1^2\|Q_{r1}\|I_N\otimes I_n)\\&+c_r(L_2(A_r\otimes\Gamma_r)\tilde{Q}_{r(1)}^{-1}(A_r\otimes\Gamma_r)^T\\&-(A_r\otimes\Gamma_r)Q_{r(1)}^{-1}(A_r\otimes\Gamma_r)^T), \quad (21)\\ M_r^{(2)} =&\tilde{\Phi}_r^{(2)}-\Phi_r^{(2)}\\ =&c_rL_2(\|\tilde{Q}_{r(1)}\|-2L_2\|Q_{r(1)}\|)I_N\otimes I_n. \quad (22)\end{aligned}$$

Under Theorem 1, we have $\Phi_r^{(1)}\leq 0$, $\Phi_r^{(2)}\leq 0$. Thus, if $M_r^{(1)}\leq 0$ and $M_r^{(2)}\leq 0$, one obtains that $\tilde{\Phi}_r^{(1)}\leq 0$, $\tilde{\Phi}_r^{(2)}\leq 0$. This completes the proof. □

Case 2: In the network (1), let $g(x(t),x(t-\tau)) = \hat{L}x(t)-L_2x(t-\tau)$, where $\hat{L}=L_1$ or $\hat{L}=-L_1$, $L_1>0, L_2>0$.

1) To get sufficient conditions of finite-time global synchronization for the network (1) in case 2.

Theorem 3: Let Assumptions 1-4 hold, then the network (1) in case 2 is global synchronization under the set of controller (4) in finite time t^* if the following conditions are satisfied:

$$\begin{aligned}\hat{\Phi}_r =&\begin{bmatrix}\hat{\Phi}_r^{(1)} & -c_rL_2(A_r\otimes\Gamma_r)\\ \star & \hat{\Phi}_r^{(2)}\end{bmatrix}\leq 0,\\ \hat{\Phi}_r^{(1)} =&\theta_r I_N\otimes I_n + I_N\otimes B_rQ_r^{-1}B_r^T+2c_r\hat{L}(A_r\otimes\Gamma_r)\\&-2\Xi_r\otimes\Gamma_r,\\ \hat{\Phi}_r^{(2)} =&(-\rho_r+\upsilon)I_N\otimes I_n.\end{aligned}$$

The other parameters of Theorems 1 and 3 are quite same.

2) To build the relationship conditions of Theorems 1 and 3.

Corollary 2: If

$$\hat{\Phi}_r = \begin{bmatrix}\hat{M}_r^{(1)} & -c_rL_2(A_r\otimes\Gamma_r)\\ \star & \hat{M}_r^{(2)}\end{bmatrix}\leq 0$$

holds, under Theorem 3, the network (1) of case 2 with the controller (4) must satisfy Theorem 1, where $\hat{M}_r^{(1)} = c_r(2\hat{L}(A_r\otimes\Gamma_r)-2L_1^2\|Q_{r1}\|I_N\otimes I_n-A_r\otimes\Gamma_r)Q_{r(1)}^{-1}(A_r\otimes\Gamma_r)^T)$, and $\hat{M}_r^{(2)} = -2L_2^2c_r\|Q_{r(1)}\|I_N\otimes I_n$.

Proof: The proof is similar to that of Corollary 1. □

Remark 8: Under Corollaries 1-2, the network (1) in cases 1-2 must satisfy Theorem 1. Thus, combining the first idea and Corollaries 1-2, problem 1 can be analyzed. The steps is as follows:

Step 1: To choose $L_1^{(1)}$ and $L_2^{(1)}$.

Step 2: According to Corollaries 1-2, $\varepsilon_{ir}^{(1)}$ and $k_{ir}^{(1)}$ for finite-time synchronization of the network (1) in Cases 1-2 are designed, where $i=1, 2, ..., N$ and $r=1, 2, ..., s$.

Step 3: By simulation, problem 1 is testified.

Step 4: The $\varepsilon_{ir}^{(1)}$ and $k_{ir}^{(1)}$ are fixed.

Step 5: A new $L_1^{(2)}$ and $L_2^{(2)}$ are chosen.

Step 6: To substitute $L_1^{(2)}$, $L_2^{(2)}$, $\varepsilon_{ir}^{(1)}$ and $k_{ir}^{(1)}$ into Corollaries 1-2. If Corollaries 1-2 are held, return step 3, otherwise, continue Step 7.

Step 7: Under Corollaries 1-2, $\varepsilon_{ir}^{(2)}$ and $k_{ir}^{(2)}$ for finite-time synchronization of the network (1) in Cases 1-2 are designed.

Step 8: Let $\varepsilon_{ir}^{(1)} := \varepsilon_{ir}^{(2)}$ and $k_{ir}^{(1)} := k_{ir}^{(2)}$ then return Step 3. Comparing steps of Remarks 7 and 8, it is seen that the scheme of Remark 8 for Problem 1 are more practical and more simple than that of Remark 7. Comparing [26], the merits in Remark 8 are more useful than that of [26]. It is pity that if nonlinear coupling function $g(x,y)$ is not derivable, the above scheme for Problem 1 in Remark 8 can not be applied.

Remark 9: Theorem 1 and Corollaries 1-2 are based on Assumption 1 without Lemma 6 and Assumption 1 or with Lemma 6, respectively. The disadvantages of Corollaries 1-2 is that the conservatism of Corollaries 1-2 is higher than that of Theorem 1. This can be seen from the process of proving Theorem 1 and Corollaries 1-2. The advantages of Corollaries 1-2 is that for problem 1, the method based on Corollaries 1-2 is more practical than that of Theorem 1. This can be observed from Remarks 7 and 8.

3.2.2 The second idea

Theorem 4: Let Assumptions 1-4 hold, then the network (2) is global synchronization under the set of controller (4) in finite time t^* if the following conditions are satisfied:

$$\check{\Phi}_r^{(1)} =\theta_r I_N\otimes I_n + I_N\otimes B_rQ_r^{-1}B_r^T+2c_r(A_r\otimes\Gamma_r)$$

$$+c_r(A_r\otimes\Gamma_r)\tilde{Q}^{-1}_{r(1)}(A_r\otimes\Gamma_r)^T - 2\Xi_r\otimes\Gamma_r,$$
$$\check{\Phi}^{(2)}_r = (c_r\|\tilde{Q}_{r(1)}\| - \rho_r + \upsilon)I_N\otimes I_n.$$

The other parameters of Theorems 1 and 4 are quite same.

Corollary 3: Let $\check{M}^{(1)}_r = \check{\Phi}^{(1)}_r - \Phi^{(1)}_r = 2c_r((A_r\otimes\Gamma_r) - L_1^2\|Q_{r(1)}\|I_N\otimes I_n) + c_r((A_r\otimes\Gamma_r)\tilde{Q}^{-1}_{r(1)}(A_r\otimes\Gamma_r)^T - (A_r\otimes\Gamma_r)Q^{-1}_{r(1)}(A_r\otimes\Gamma_r)^T)$, and $\check{M}^{(2)}_r = \check{\Phi}^{(2)}_r - \Phi^{(2)}_r = c_r(\|\tilde{Q}_{r(1)}\| - 2L_2^2\|Q_{r(1)}\|)I_N\otimes I_n$. If $\check{M}^{(1)}_r \le 0$ and $\check{M}^{(2)}_r \le 0$, under Theorem 1, the network (1) with the controller (4) must satisfy Theorem 4.

Proof: The proof is similar to that of Corollary 1.

Next, in order to make variables in Theorem 4 and Corollary 3 understand easily, here, $\check{\Phi}^{(1)T4}_r$, $\check{\Phi}^{(1)C3}_r$, ρ^{T4}, ρ^{C3}, Ξ^{T4}_r, Ξ^{C3}_r, ε^{T4}_{ir}, ε^{C3}_{ir}, k_{T4}, k_{C3}, t^*_{T4} and t^*_{C3} stand for $\check{\Phi}^{(1)}_r$, ρ, Ξ_r, ε_{ir}, k and t^* of Theorem 4 and Corollary 3, respectively.

Corollary 4: In Corollary 3, if $\check{\Phi}^{(1)C3}_r \le \check{M}^{(1)}_r < \check{\Phi}^{(1)T4}_r \le 0$, $\rho^{T4} = \rho^{C3}$ and $k_{T4} = k_{C3}$, for finite-time global synchronization of the network (2), the control rule under Corollary 3 is better than that of Theorem 4.

Proof: According to Corollary 3, one obtains that $\check{M}^{(1)}_r = \check{\Phi}^{(1)C3}_r - \Phi^{(1)}_r \le 0$, $\Phi^{(1)}_r \le 0$. Thus, we have $\check{\Phi}^{(1)C3}_r \le \check{M}^{(1)}_r \le 0$. From Theorem 4, we get $\check{\Phi}^{(1)T4}_r \le 0$. If $\check{M}^{(1)}_r < \check{\Phi}^{(1)T4}_r \le 0$, there is $\check{\Phi}^{(1)C3}_r \le \check{M}^{(1)}_r < \check{\Phi}^{(1)T4}_r \le 0$. Thus, one has $\Xi^{C3}_r > \Xi^{T4}_r > 0$. That means $\varepsilon^{C3}_{ir} > \varepsilon^{T4}_{ir} > 0$. From the inequality (14) of Theorem 1, one obtains that $\triangle E[\mathcal{L}V(e(t),t,r(t))] = \{E[\mathcal{L}V^{C3}(e(t),t,r(t))] - E[\mathcal{L}V^{T4}(e(t),t,r(t))]\} \le E\{q_r[e^T(t)(\Xi^{C3}_r - \Xi^{T4}_r)e(t)] < 0$. Under Theorem 4, one has $E[\mathcal{L}V^{T4}(e(t),t,r(t))] \le 0$. Thus, there is $E[\mathcal{L}V^{C3}(e(t),t,r(t))] < E[\mathcal{L}V^{T4}(e(t),t,r(t))] \le 0$. That means if $\check{\Phi}^{(1)C3}_r \le \check{M}^{(1)}_r < \check{\Phi}^{(1)T4}_r \le 0$, $\rho^{T4} = \rho^{C3}$ and $k_{T4} = k_{C3}$, global synchronization dynamics of the network (2) under Corollary 3 is better than that of the network (2) under Theorem 4. The proof is completed. □

Remark 10: According to Corollary 4, two synchronization control rules can be designed. The first is based on $\check{M}^{(1)}_r < \check{\Phi}^{(1)T4}_r \le 0$. The second is built on $\check{\Phi}^{(1)C3}_r \le \check{M}^{(1)}_r < \check{\Phi}^{(1)T4}_r \le 0$. Therefore, there is $\varepsilon^{C3}_{ir} > \varepsilon^{T4}_{ir}$. Furthermore, if $L^{(2)}_1 > L^{(1)}_1 > 0$, combining Corollary 4, there is $\check{\Phi}^{(1)C3(2)}_r < \check{\Phi}^{(1)C3(1)}_r \le \check{M}^{(1)}_r < \check{\Phi}^{(1)T4}_r \le 0$. This shows that $\varepsilon^{C3(2)}_{ir} > \varepsilon^{C3(1)}_{ir} > \varepsilon^{T4}_{ir} > 0$. Therefore, under Corollary 4, with increasing L_1, the difference of the two synchronization control rules for the network (2) becomes more and more significant. The above analysis shows that the synchronization control rules based on Corollary 4 is only related to L_1. From assumption 1, $g(x(t),x(t-\tau))$ is closely connected with L_1 and L_2. Therefore, if nonlinearity of $g(x(t),x(t-\tau))$ is caused by $x(t-\tau)$, the above technique built on Corollary 4 for problem 1 is invalid. In the future, the issue will be considered.

Remark 11: From Corollary 3, two conclusions can be obtained as follows. (i) If $\check{M}^{(1)}_r \le 0$ and $L_2 \ge \frac{\sqrt{2}}{2}$, under Theorem 1, the network (1) with the controller (4) must satisfy Theorem 4. In Corollary 3, let $\tilde{Q}_{r(1)} = Q_{r(1)}$, then $\check{M}^{(1)}_r = \check{\Phi}^{(1)}_r - \Phi^{(1)}_r = 2c_r((A_r\otimes\Gamma_r) - L_1^2\|Q_{r(1)}\|I_N\otimes I_n)$, and $\check{M}^{(2)}_r = \check{\Phi}^{(2)}_r - \Phi^{(2)}_r = c_r\|\tilde{Q}_{r(1)}\|(1-2L_2^2)I_N\otimes I_n$. If $L_2 \ge \frac{\sqrt{2}}{2}$, there is $\check{M}^{(2)}_r \le 0$. (ii) Let $\check{M}^{(1)}_r = (m^r_{ij})_{(N*n)*(N*n)}$. If $\check{M}^{(1)}_r > 0$ and $0 < L_2 < \frac{\sqrt{2}}{2}$, under Theorem 4, the network (2) with the controller (4) must satisfy Theorem 1. Actually, the result is not held. The reason is that in $\check{M}^{(1)}_r$, there is $L_1^2\|Q_{r(1)}\| \ge 0$ and $A_r\otimes\Gamma_r < 0$. This shows $m^r_{ij} < 0$. By Lemma 5, if $\check{M}^{(1)}_r > 0$, there is $m^r_{ij} > 0$.

4. SIMULATIONS

In this section, three examples are given to illustrate the effectiveness of the derived results. The initial conditions of the numerical simulations are taken as: $x_1(0) = (1,2,3)^T$, $x_2(0) = (3,1,1)^T$, $x_3(0) = (1,2,3)^T$. The total error and the synchronization total error of the network are defined as $e(t) = \sum_{i=1}^3\sum_{l=1}^3|e_{il}(t)|$. For given rate transition matrix, a Markov chain can be generated. We consider the following rate transition matrix:

$$\Pi = \begin{bmatrix} -2 & 2 \\ 5 & -5 \end{bmatrix}. \qquad (23)$$

A network composed of three Chua's circuits [45–47] is considered. A single Chua's circuit is illustrated in Fig. 1.

In the circuit, there are two linear capacitors $C1$ and $C2$, a nonlinear resistor NR, a linear resistor R and a linear inductor L. Let $C1 = C_1$, $C2 = C_2$, $i1 = i_L$, $v1 = v_1$ and $v2 = v_2$, then the circuit equations are as follows:

$$C_1\dot{v}_1 = \frac{1}{R}(v_2 - v_1) - \tilde{f}(v_1),$$
$$C_2\dot{v}_2 = \frac{1}{R}(v_1 - v_2) + i_L,$$
$$L\dot{i}_L = -v_2,$$

where $\tilde{f}(v_1)$ is that

$$\tilde{f}(v_1) = G_{b1}v_1 + 0.5(G_{a1} - G_{b1})(|v_1 + 1| - |v_1 - 1|).$$

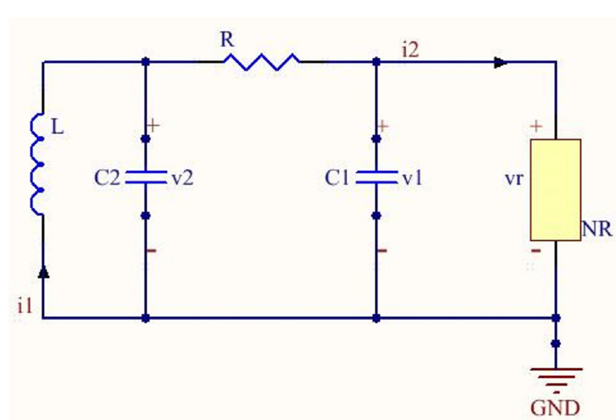

Fig. 1. A single Chua's circuit model.

Let $\dot{v}_1 = \dot{x}_{i1}$, $\dot{v}_2 = \dot{x}_{i2}$, $\dot{i}_L = \dot{x}_{i3}$, $h_c = \frac{1}{RC_1}$, $p_c = \frac{1}{C_1}$, $q_c = \frac{1}{RC_2}$, $r_c = \frac{1}{C_2}$ and $z_c = \frac{1}{L}$, then the above Chua's circuit equations can be expressed by

$$\begin{bmatrix}\dot{x}_{i1}\\ \dot{x}_{i2}\\ \dot{x}_{i3}\end{bmatrix} = \begin{bmatrix}-h_c & h_c & 0\\ q_c & -q_c & r_c\\ 0 & -z_c & 0\end{bmatrix}\begin{bmatrix}x_{i1}\\ x_{i2}\\ x_{i3}\end{bmatrix} + \begin{bmatrix}-p_c\tilde{f}(x_{i1})\\ 0\\ 0\end{bmatrix}. \tag{24}$$

Then, combining (24), the networks (1) and (2) with controller (4) can be described by

$$\begin{aligned}\dot{x}_i(t) =& c_r \sum_{j=1}^{3} a_{ij}^r \Gamma_r g(x_j(t), x_j(t-\tau))\\ &+ B_r f(x_i(t)) + u_i(t,r),\end{aligned} \tag{25}$$

$$\begin{aligned}\dot{x}_i(t) =& c_r \sum_{j=1}^{3} a_{ij}^r \Gamma_r (x_j(t) + x_j(t-\tau))\\ &+ B_r f(x_i(t)) + u_i(t,r),\end{aligned} \tag{26}$$

and $f(x_i(t))$ is that

$$\begin{bmatrix}f_1(x_{i1}(t))\\ f_2(x_{i2}(t))\\ f_3(x_{i3}(t))\end{bmatrix} = \begin{bmatrix}-h_c & h_c & 0\\ q_c & -q_c & r_c\\ 0 & -z_c & 0\end{bmatrix}\begin{bmatrix}x_{i1}\\ x_{i2}\\ x_{i3}\end{bmatrix} + \begin{bmatrix}-p_c\tilde{f}(x_{i1})\\ 0\\ 0\end{bmatrix},$$

where $r = 1,2$, $i = 1,2,3$, $L_1 > 0$ and $L_2 > 0$, $x_j(t) = [x_{j1}(t), x_{j2}(t), x_{j3}(t)]^T$, $x_j(t-\tau) = [x_{j1}(t-\tau), x_{j2}(t-\tau), x_{j3}(t-\tau)]^T$.

In the networks (25) and (26), let $c_1 = 1$, $c_2 = 1$, $\tau = 0.3$, $h_c = p_c = 0.2$, $q_c = r_r = 0.3$, $z_c = 0.5$, $G_{b1} = -0.0714$ and $G_{a1} = -0.219$, $\Gamma_1 = \Gamma_2 = diag\{1,1,1\}$, $B_1 = B_2 = diag\{1,1,1\}$, and the other parameters are as follows:

$$A_1 = \begin{bmatrix}2.2 & 1 & 1\\ 1 & 1.1 & 0\\ 1 & 0 & 1.2\end{bmatrix}, \quad A_2 = \begin{bmatrix}1.2 & 0 & 1\\ 0 & 1.2 & 1\\ 0.5 & 1 & 1.5\end{bmatrix}.$$

Example 1: In case 1, according to $\hat{L} = L_1$ or $\hat{L} = -L_1$, one obtains $g(x_j(t), x_j(t-\tau)) = L_1 x_j(t) + L_2 x_j(t-\tau)$ and $g(x_j(t), x_j(t-\tau)) = -L_1 x_j(t) + L_2 x_j(t-\tau)$.

Firstly, let $g(x_j(t), x_j(t-\tau)) = L_1 x_j(t) + L_2 x_j(t-\tau)$, $L_1 = L_2 = 0.2$ and $L = 0.6$. Under Corollary 1, one obtains that $a_1 = a_2 = 4$, $\lambda = 4$, $\upsilon = 1$, $\beta = 0.5$, $q_1 = q_2 = 1$, $\rho_1 = \rho_2 = 4$, $V(0) = 39$, $\Xi_r = diag\{3.5, 2.5, 3.5\}$, $k_{ir} = 2.5$, $t^* \leq 7.37$. Then, we fix Ξ_r, k_{ir} and let $L_1 = L_2 = 0.9, 1.3$. Simulation results are shown in Fig. 2.

Secondly, let $g(x_j(t), x_j(t-\tau)) = -L_1 x_j(t) + L_2 x_j(t-\tau)$. Similar to the above process, under Corollary 1, we get $\Xi_r = diag\{3.6, 2.7, 3.6\}$, $k_{ir} = 2.5$, $t^* \leq 7.37$. Let $L_1 = L_2 = 0.2, 0.9, 1.3$, then $\Xi_r = diag\{3.6, 2.7, 3.6\}$ and $k_{ir} = 2.5$ are fixed. Fig. 3 gives the simulation results.

Example 2: In case 2, there is $g(x_j(t), x_j(t-\tau)) = L_1 x_j(t) - L_2 x_j(t-\tau)$ and $g(x_j(t), x_j(t-\tau)) = -L_1 x_j(t) - L_2 x_j(t-\tau)$.

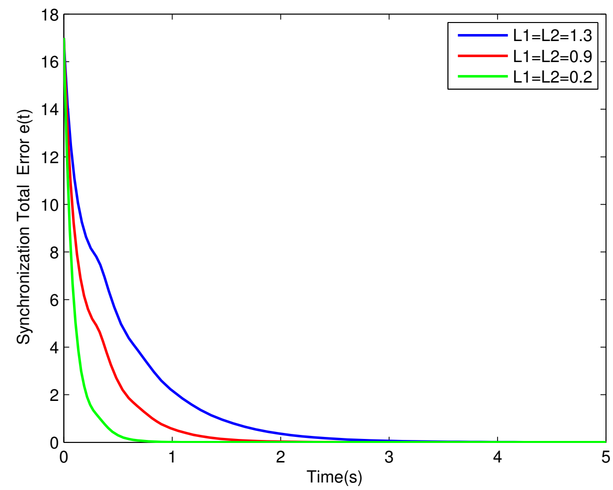

Fig. 2. Synchronization total error trajectories for the first case of Example 1.

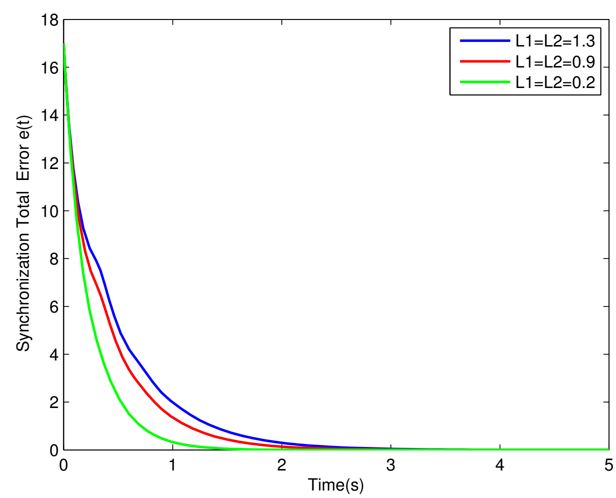

Fig. 3. Synchronization total error trajectories for the second case of Example 1.

Similar to example 1, let $g(x_j(t), x_j(t-\tau)) = L_1 x_j(t) - L_2 x_j(t-\tau)$, $L_1 = L_2 = 0.2$ and $L = 0.6$, respectively. Under Corollary 2, we have $a_1 = a_2 = 4$, $\lambda = 4$, $\upsilon = 1$, $\beta = 0.5$, $q_1 = q_2 = 1$, $\rho_1 = \rho_2 = 4$, $V(0) = 39$, $\Xi_r = diag\{3.7, 2.6, 3.7\}$, $k_{ir} = 2.5$, $t^* \leq 7.37$. Let $L_1 = L_2 = 0.8$ and 1.3, then Ξ_r and k_{ir} are fixed. The simulation results are in Fig. 4.

If $g(x_j(t), x_j(t-\tau)) = -L_1 x_j(t) - L_2 x_j(t-\tau)$, $L_1 = L_2 = 0.2$ and $L = 0.6$, under Corollary 2, one has $\Xi_r = diag\{3.8, 2.8, 3.8\}$, $k_{ir} = 2.5$, $t^* \leq 7.37$. Choosing $L_1 = L_2 = 0.6, 1.3$, and fixing Ξ_r and k_{ir}, the simulation results of Fig. 5 of are obtained.

Example 3: This example shows the results of Corollary 4.

Firstly, let

$$g(x_i(t), x_i(t-0.3))$$

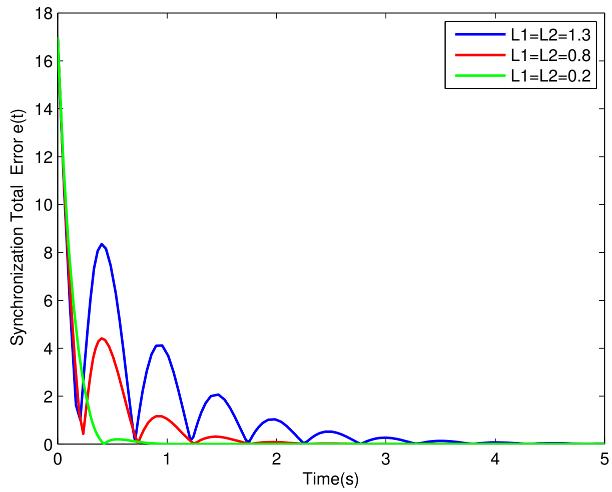

Fig. 4. Synchronization total error trajectories for the first case of Example 2.

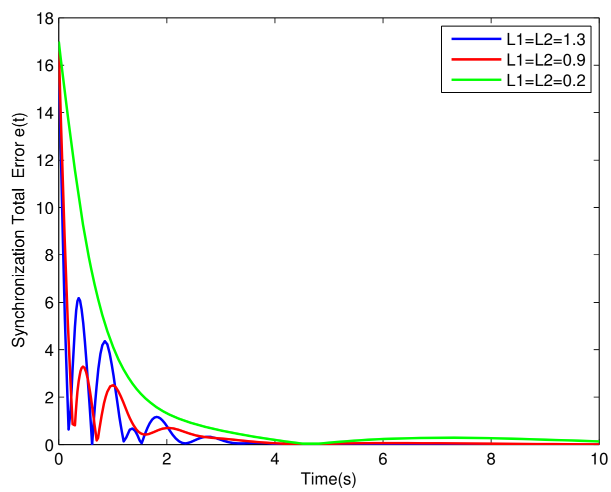

Fig. 5. Synchronization total error trajectories for the second case of Example 2.

$$=\begin{bmatrix}0.5\tanh(x_{i1}(t))+0.5\tanh(x_{i1}(t-0.3))\\0.5\tanh(x_{i2}(t))+0.5\tanh(x_{i2}(t-0.3))\\0.5\tanh(x_{i2}(t))+0.5\tanh(x_{i3}(t-0.3))\end{bmatrix}. \tag{27}$$

From $g(x_i(t),x_i(t-0.3))$, we have $L_1=0.5$, $L_2=0.5$. Let $a_1=a_2=4$, $\lambda=4$, $\upsilon=1$, $\beta=0.5$, $q_1=q_2=1$, $\rho_1=\rho_2=4$, $L=1$, we obtain $V(0)=39$, $k_{ir}=2.5$, $r=1,2$, $i=1,2,3$. According to Corollary 4, there are $\check{\Phi}_r^{(1)C3}\leq\check{M}_r^{(1)}<\check{\Phi}_r^{(1)T4}\leq 0$, $\rho^{T4}=\rho^{C3}$ and $k_{T4}=k_{C3}$. Thus, under Corollary 4, one obtains $\Xi_r^{T4}=diag\{2.8,2.5,2.5\}$, $\Xi_{r1}^{C3}=diag\{3.8,3,3.8\}$, $t^*\leq 7.37$.

Secondly, let

$$g(x_i(t),x_i(t-0.3))$$

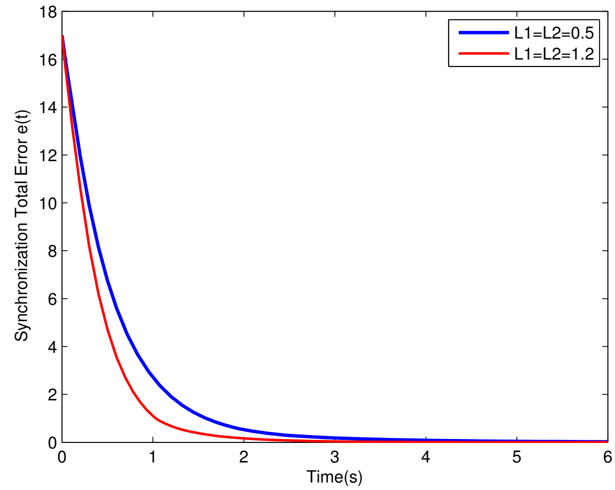

Fig. 6. Synchronization total error trajectories for Example 3.

$$=\begin{bmatrix}1.2\tanh(x_{i1}(t))+1.2\tanh(x_{i1}(t-0.3))\\1.2\tanh(x_{i2}(t))+1.2\tanh(x_{i2}(t-0.3))\\1.2\tanh(x_{i2}(t))+1.2\tanh(x_{i3}(t-0.3))\end{bmatrix}. \tag{28}$$

Similar to the above process, we get $L_1=1.2$, $L_2=1.2$, $k_{ir}=2.5$, $\Xi_r^{T4}=diag\{2.8,2.5,2.5\}$, $\Xi_r^{C3}=diag\{5.2,4.3,5.2\}$, $t^*\leq 7.37$. Fig. 6 shows the simulation results.

Remark 12: Examples 1-2 show that under Corollaries 1-2, the network (25) with controller (4), in which $g(x_j(t),x_j(t-0.3))=\hat{L}_1x_j(t)+\hat{L}_2x_j(t-\tau)$, $\hat{L}_1=L_1$, $\hat{L}_1=-L_1$, $\hat{L}_2=-L_2$, $\hat{L}_2=L_2$, $L_1>0$ and $L_2>0$, must achieve synchronization within finite-time t^*. Furthermore, from simulation results of Examples 1-2, it is seen that dynamics of the above network (25) from the initial state to synchronization state is closely related to L_1 and L_2. From example 3, it is observed that under Corollary 4, the linear coupling network (26) can realize finite-time synchronization. Meanwhile, when L_1 is increased, the synchronization dynamics of the network (26) is improved. This reflects that the analysis of remark 10 is reasonable.

Remark 13: From Examples 1-3, it can be obtained that the smaller nonlinearity of nonlinear coupling $f(x(t),x(t-\tau))$ is, the better its synchronization effect is. In fact, for problem 1, the conclusions of this paper and [26] are not conflictive. That is to say, the results of this paper and [26] are harmonious. In [26], in order to analyze problem 1, the authors chose nonlinear coupling functions $g_1(x,y)$, $g_2(x,y)$, $g_3(x,y)$ and made $(x-y)g_1(y,x)<(x-y)g_2(y,x)<(x-y)g_3(y,x)<0$ according to $(x-y)g(y,x)\leq-\alpha(x-y)^2$ and $\alpha>0$. Thus, the simulation results showed that the smaller $(x-y)g(y,x)$ was, the better the synchronization dynamics of the addressed networks was. The reason is analyzed as follows. From [26], we can see that in order to obtain the proposed results, nonlinear term $\sum_{j=1}^{N}\sum_{k=1}^{N} a_{jk}g(x_k(t),x_j(t))$ needed to be processed. For example, in the process of proving Theorem 1, a Lyapunov function $V_1(t)$ was defined and $\dot{V}_1(t) = T_1 + T_2 + T_3$ was computed. In $\dot{V}_1(t)$, T_2 was closely related to $(x-y)g(y,x)$ and $2T_2 = 2\sum_{j=1}^{N}\sum_{k=1}^{N} a_{jk}q_j[x_j(t)-x_k(t)]g(x_k(t),x_j(t))$, where $q_j > 0$. Combining $(x-y)g(y,x) \leq -\alpha(x-y)^2$, one obtained $2T_2 \leq -\alpha\sum_{j=1}^{N}\sum_{k=1}^{N} a_{jk}q_j[x_j(t)-x_k(t)]^2$. It was clear that $-\alpha[x_j(t)-x_k(t)]^2 \leq 0$. Therefore, if $\alpha[x_j(t)-x_k(t)]^2$ was increased, T_2 must be decreased. Thus, under $\dot{V}_1(t) \leq 0$, $\dot{V}_1(t)$ became smaller. This showed that under $(x-y)g(y,x) \leq -\alpha(x-y)^2$, if $(x-y)g(y,x)$ was smaller, the dynamics of the addressed network was better. In this paper, combining inequality (14) and $E[\mathcal{L}V(e(t),t,r)] \leq 0$, $E[\mathcal{L}V(e(t),t,r)]$ will decrease if L_1 and L_2 is decreased. This shows that synchronization dynamics of the network (1) is closely related to L_1 and L_2.

5. CONCLUSIONS

In this paper, two problems are explored. The first problem is that nonlinear coupling one how to affect the synchronization dynamics of the NCMSCNs is discussed. The second problem is that finite-time synchronization control problem with feedback control and nonlinear coupling one is investigated. Firstly, sufficient conditions of finite-time global synchronization of the NCMSCNs and the LCMSCNs are given. Secondly, the relationship conditions of finite-time global synchronization for the NCMSCNs and the LCMSCNs are built. Thirdly, the relationship of synchronization control rules for the NCMSCNs and the LCMSCNs is analyzed. At last, by Chua's circuit network simulations, the above questions are further testified. The conclusions are as follows. (i) Under the proposed Theorems and Corollaries, the addressed networks with controller can achieve synchronization within finite-time t^*. (ii) The nonlinearity of nonlinear coupling $g(x(t),x(t-\tau))$ is closely related to the network (1) with nonlinear coupling $g(x(t),x(t-\tau))$.

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