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A Multi-step Output Feedback Robust MPC Approach for LPV Systems with Bounded Parameter Changes and Disturbance

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Abstract: This paper considers a multi-step output feedback robust model predictive control (OFRMPC) approach for the linear parameter varying (LPV) systems with bounded changes of scheduling parameters and bounded disturbance. Less conservative bounds of future estimation error sets and system parametric uncertain sets are predicted by considering bounded changes of scheduling parameters in LPV systems. In the multi-step OFRMPC approach, an optimization problem is solved to obtain a sequence of controller gains, which considers predictions of future bounds of estimation error sets and system parametric uncertain sets. The optimized sequence of controller gains corresponding to a sequence of Lyaponov matrices have less constraint conditions and also introduce more degree of freedom for the optimization. The proposed multi-step OFRMPC guarantees robust uniform ultimately bounded of the estimation error and robust stability of the observer system. A numerical example is given to demonstrate the effectiveness of the approach.

Keywords: Model predictive control, multi-step control, output feedback, uncertain system.

1. INTRODUCTION

Robust model predictive control (RMPC) approaches that consider system uncertainty and guarantee robust stability of controlled systems have been extensively studied [1–6]. At each sampling time, the prediction of system future states and physical constraints are considered in the on-line RMPC optimization problem to obtain optimal control inputs. At the next sampling time, system measurements are updated and the prediction horizon is shifted one step forward, then the optimization problem is repeated. With the growing interest of RMPC in industry and academic community, results on MPC research mainly concentrate on issues such as robust stability against system uncertainty [7–9], reduction on the conservatism in controller design [10–16] and on-line computational burden [17, 18], and enlargement of feasible sets for constraint optimization problems [12, 19].

For real nonlinear dynamic systems, e.g., hydropower system [20, 21], modeling of nonlinear systems and ensuring stability of controlled systems are important. In RMPC approaches, model parametric uncertainty and system nonlinear dynamics can be described by linear parameter varying (LPV) systems. In the on-line RMPC optimization problem, in order to optimize the controller, at each sampling time, the min-max optimization that often considers all the possible realization of future model parametric uncertainty is solved to optimize an optimal controller. The quasi-min-max optimization problem is considered in [10, 11], which extends the approach in [7] and introduces a free control input for the on-line optimization problem. A periodic invariance method is proposed in [12], in which the state is allowed to leave a set temporarily but returns into it in finite steps. As a result, compared with [7], a larger feasible set for the optimization problem is obtained and also the control performance is improved. In [13–15, 19], the multi-step state feedback RMPC method optimizes a sequence of control inputs to steer the system state from one ellipsoidal set to another one and finally into a robust positively invariant (RPI) set. References [13, 14] and [15, 19] improve the control performance in [11] and [7, 12], respectively. Furthermore, bounded parameter changes are considered in [13, 14], which reduce the conservatism of future model parameter prediction in the multi-step state feedback RMPC optimization problem.

For real processes that true states are unmeasurable and disturbance exists, the output feedback RMPC (OFRMPC) is necessary for real applications. A common approach to OFRMPC problem is the combination

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of a state observer and the feedback controller based on estimated states. However, it is not trivial since the separate design of the observer system and the feedback controller based on the estimated state cannot guarantee the robust stability of the controlled system [22]. The current and future system states in [9, 16, 23, 24] are constrained in one RPI set, where all possible realizations of future system parametric uncertain sets and bounds of estimation error sets after the current sampling time are considered. A two-stage control mechanism for quasi-min-max OFRMPC is developed in [25], where the state is firstly driven into a prescribed neighborhood of the origin (terminal region), then the off-line controller gain corresponding to the terminal region makes the state converge to the origin.

The present paper considers a multi-step OFRMPC approach for LPV systems with bounded changes of the scheduling parameters and bounded disturbance. Different from the multi-step state feedback RMPC methods in [13–15, 19], true states are unmeasurable and therefore the uncertainty of the estimation error set and bounded disturbance should be considered in the optimization problem. Compared with the OFRMPC methods in [9, 16, 23, 24], by considering bounded changes of scheduling parameters in LPV systems, more less conservative future system parametric uncertain sets and bounds of future estimation error sets are predicted. The proposed multi-step OFRMPC optimizes a sequence of controller gains, where the prediction of bounds of future estimation error sets and parametric uncertain sets are considered. The optimized multi-step controller gains steer the estimated state from one ellipsoidal set to another one and finally into an RPI set. In the multi-step OFRMPC approach, the estimation error is robust uniform ultimately bounded (UUB) [30] with respect to the minimal RPI set, and the observer system is robust stability. The sequence of controller gains in the multi-step OFRMPC approach are associated with the sequence of Lyapunov matrices. Therefore, compared with the OFRMPC methods based on one common Lyaponov matrix in [23, 24], the multi-step controllers have less constraints and also introduce more degree of freedom for the optimization. The proposed multi-step OFRMPC method is advantageous for reducing the conservatism in the output feedback controller design and enlarging the region of attraction for robust optimization.

The rest of paper is organized as follows. In Section 2, the system model and the main goal of the paper are given. Furthermore, future system parameters are predicted by considering bounded changes of scheduling parameters in the LPV system. In Section 3, an off-line observer is designed to predict bounds of the future estimation error sets. The multi-step OFRMPC approach optimizing a sequence of controller gains to guarantee the robust stability of the observer system is given in Section 4. The overall algorithm with the proof of recursive feasibility and robust stability is summarized in Section 5. The computational burden of the proposed algorithm compared with other related methods is shown in Section 6. Simulation results are given in Section 7 to show the advantages of the method. Finally some conclusions and future work are summarized in Section 8.

Notations: Let $\mathbb{R}$, $\mathbb{R}_+$ $\mathbb{Z}$ and $\mathbb{Z}_+$ denote the set of real numbers, the set of non-negative real numbers, the set of integer numbers, and the set of non-negative integers, respectively. $\mathbb{Z}_{[s,k]}$ and $\mathbb{Z}_{[s,\infty)}$ denote the set of non-negative integers from s to k, and the set of non-negative integers that are greater than or equal to s, where $s,k \in \mathbb{Z}_+$. For any vector x and positive-definite matrix P, $\|x\|_P^2 \triangleq x^{\mathrm{T}}Px$. $x(i|k)$ is the value of x at time $k+i$, predicted at time k. $x(0|k)$ is the current value of $x(k)$. I is the identity matrix with appropriate dimension. Denote $i^+ \triangleq i+1$, and $\varepsilon_M \triangleq \{\xi|\xi^{\mathrm{T}}M\xi \le 1\}$ the ellipsoid associated with the symmetric positive-definite matrix M. All vector inequalities are interpreted in an element-wise sense. An element belonging to $\mathrm{Co}\{\cdot\}$ means that it is a convex combination of the elements in $\{\cdot\}$, with the scalar combing coefficients nonnegative and their sum equal to 1. The symbol "$\star$" induces a symmetric structure in the matrix inequalities. A value with superscript "$*$" means that it is the optimal solution of the optimization problem. The time-dependence of the MPC decision variable is often omitted for brevity.

2. PROBLEM STATEMENT AND PREDICTION OF SYSTEM PARAMETRIC UNCERTAIN SETS

2.1. Problem statement

Consider the following discrete-time LPV system

$$x(k+1) = A(k)x(k) + B(k)u(k) + D(k)w(k), \quad (1)$$

$$y(k) = C(k)x(k) + E(k)w(k), \quad (2)$$

where $x \in \mathbb{R}^{n_x}$, $u \in \mathbb{R}^{n_u}$, $w \in \mathbb{R}^{n_w}$ and $y \in \mathbb{R}^{n_y}$ are the system state, input, disturbance and output, respectively. The disturbance is bounded and satisfies $w(k) \in \varepsilon_{P_w}$, where P_w is the shape matrix of the ellipsoidal set for the bounded disturbance. The control input and system state should satisfy

$$-\bar{u} \le u(k) \le \bar{u}, -\bar{\psi} \le \Psi x(k) \le \bar{\psi}, \quad (3)$$

where $\bar{u} = [\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_{n_u}]^{\mathrm{T}}$, $\bar{u}_s > 0$, $s \in \mathbb{Z}_{[1,n_u]}$; $\Psi \in \Re^{q \times n_x}$, $\bar{\psi} = [\bar{\psi}_1, \bar{\psi}_2, \ldots, \bar{\psi}_q]^{\mathrm{T}}$, $\bar{\psi}_j > 0$, $j \in \mathbb{Z}_{[1,q]}$. At each time k, model parameters $[A|B|C|D|E](k) \in \Omega_0 = \mathrm{Co}\{[A_l|B_l|C_l|D_l|E_l], \ l \in \mathbb{Z}_{[1,L]}$, and satisfying $[A|B|C|D|E](k) = \sum_{l=1}^{L} \lambda_l(k)[A_l|B_l|C_l|D_l|E_l]$, $\lambda_l(k) \ge 0$, $\sum_{l=1}^{L} \lambda_l(k) = 1$, where $\lambda(k) = [\lambda_1(k), \ldots, \lambda_L(k)]$ is the scheduling parameter of the LPV system at time k.

The present paper considers that the system state and disturbance are unmeasurable, the system output and the scheduling parameter are known at time k, while future scheduling parameters of the LPV system are uncertain and have bounded rate of their changes. The future system parametric uncertain sets are predicted by considering the bounded changes of the scheduling parameters in the LPV system. To stabilize the controlled system, an observer system is off-line designed to estimate the true state, and then a multi-step OFRMPC approach is proposed to guarantee robust stability of the observer system. As a result, a sequence of optimized controller gains are obtained such that the estimated state is steered to a neighborhood of the origin, and the true state will accordingly converge to a neighborhood of the origin.

2.2. Prediction of system parametric uncertain sets

Assume that $\lambda_l(k)$ are available at the current time k, and their rates of changes are bounded by

$$|\lambda_l(k+1)-\lambda_l(k)| \leq \Delta_l, l \in \mathbb{Z}_{[1,L]}, \tag{4}$$

where $\Delta_l \in \mathbb{R}_+$ are pre-specified scalars [13, 14]. The following Lemma 1 predicts system parametric uncertain sets for systems (1) and (2).

Lemma 1 [14]**:** Considering the uncertain parameters in Ω_0 and bounded changes of the scheduling parameters of the LPV system in (4), the future system uncertain parameters satisfy $[A|B|C|D|E](k+i) \in \Omega(k+i) = \text{Co}\{[A_l(k,i)|B_l(k,i)|C_l(k,i)|D_l(k,i)|E_l(k,i)], \quad i \in \mathbb{Z}_{[1,\infty)}$, where

$$\begin{aligned}&[A_l(k,i)|B_l(k,i)|C_l(k,i)|D_l(k,i)|E_l(k,i)]\\&=\sum_{t=1}^{L} b_t(i)[A_t|B_t|C_t|D_t|E_t]+(1-\sum_{t=1}^{L} b_t(i))\times\\&[A_l|B_l|C_l|D_l|E_l],\ t,l \in \mathbb{Z}_{[1,L]}.\end{aligned} \tag{5}$$

In (5), the scalars $b_t(1) = max\{\lambda_t(k)-\Delta_t, 0\}$; the scalars $\{b_t(i+1)\}$, $i \in \mathbb{Z}_{[1,\infty)}$, are calculated from the following iterations:

$$\begin{aligned}b_t(i+1) &= \max\{(1-\sum_{l=1}^{L} d_l(i))+d_t(i), b_t(i)\},\\ d_t(i+1) &= \min\{(1-\sum_{l=1}^{L} b_l(i))+b_t(i), d_t(i)\},\end{aligned} \tag{6}$$

where $d_t(1) = \min\{\lambda_t(k)+\Delta_t, 1\}$. Furthermore, the iteration terminates when

$$d_t(i)-b_t(i) \leq \min\{1-\sum_{t=1}^{L} b_t(i), \sum_{t=1}^{L} d_t(i)-1\}. \tag{7}$$

Based on Lemma 1 in [14], since $b_t(i)$ and $d_t(i)$, $i \in \mathbb{Z}_{[1,\infty)}$, are nondecreasing and non-increasing scalars, respectively, the convex sets satisfy $\Omega(k+i) \subseteq \Omega(k+i+h) \subseteq \Omega_0$, $i \in \mathbb{Z}_{[1,\infty)}$, $h \in \mathbb{Z}_{[1,\infty)}$ [14]. In the following Sections 3 and 4, the above predicted future system parametric uncertain sets are considered in the prediction of bounds of future estimation error sets and the multi-step OFRMPC optimization.

3. OFF-LINE OBSERVER SYSTEM AND BOUNDS OF FUTURE ESTIMATION ERROR SETS

3.1. Off-line observer system design

Since the true state is unmeasurable, the following observer system is designed:

$$\begin{aligned}\hat{x}(k+1) =& A(k)\hat{x}(k)+B(k)u(k)+L_p(y(k)\\&-C(k)\hat{x}(k)),\end{aligned} \tag{8}$$

where $\hat{x}(k) \in \mathfrak{R}^{n_x}$ is the estimated state, L_p is the off-line observer gain. The estimation error is defined as $e(k) \triangleq x(k)-\hat{x}(k)$. Based on the predictions of systems (1), (8), and the definition of the estimation error,

$$\begin{aligned}e(i+1|k) =& \sum_{l=1}^{L} \lambda_l(k+i)[(A_l(k,i)-L_pC_l(k,i))e(i|k)\\&+(D_l(k,i)-L_pE_l(k,i))w(k+i)],\end{aligned} \tag{9}$$

$$\sum_{l=1}^{L} \lambda_l(k+i) = 1, \ \ i \in \mathbb{Z}_+.$$

3.2. Preliminary definitions

Let $\mathcal{X}$ and $\mathcal{S}$ be two compact subsets of state space $\mathbb{R}^{n_x}$, which contains the origin as an interior point.

Definition 1 (minimal RPI set [26, 27])**:** At time k, if for any $e(k) \in \mathcal{S} \subseteq \mathbb{R}^{n_x}$ and any disturbance $w(k+i) \in \varepsilon_{P_w}, i \in \mathbb{Z}_+$, the condition $e(k+i) \in \mathcal{S}$ holds for $i \in \mathbb{Z}_{[1,\infty)}$, then the set $\mathcal{S}$ is said to be an RPI set for system (9). The set $\mathcal{S}$ is said to be the minimal RPI set for system (9) if $\mathcal{S}$ is included in all possible closed RPI sets.

Definition 2 (Robust uniformly ultimately bounded (UUB) [30])**:** Suppose that $\mathcal{X} \subset \mathcal{S}$, if for every initial condition $e(k) \in \mathcal{S}$ and any disturbance $w(k+i) \in \varepsilon_{P_w}, i \in \mathbb{Z}_+$, there exits an instant $K > 0$ such that $e(k+\tilde{i}) \in \mathcal{X}$ for $\forall \tilde{i} \geq K$, then system (9) is robust UUB.

Definition 3 (Quadratic boundedness [28,29])**:** System (9) is quadratically bounded with a common Lyapunov matrix P_e , if

$$\begin{aligned}&e^{\mathrm{T}}(i|k)P_e e(i|k) \geq 1\\&\Longrightarrow e^{\mathrm{T}}(i|k)P_e e(i|k) \geq e^{\mathrm{T}}(i+1|k)P_e e(i+1|k),\\&i \in \mathbb{Z}_+.\end{aligned} \tag{10}$$

For system (9), the following Lemma 2 optimizes an off-line observer gain and a minimal RPI set. The proof of Lemma 2 is given in Appendix A.1.

Lemma 2: If there exist matrices P_e, $Y_e \triangleq P_eL_p$ and $\theta \in (0,1)$, satisfying (12), then (10) holds. Further by maximizing the trace of matrix P_e subject to (12), equation (10) is guaranteed for $L_p = P_{e0}^{-1}Y_e^0$, where P_{e0} and Y_e^0 are the matrices optimized from problem (11)-(12). As a result, $\varepsilon_{P_{e0}}$ is the minimal RPI set with respect to the estimation error and bounded disturbance.

$$\max_{Y_e,P_e} \ (\text{trace}(P_e)), \tag{11}$$

$$\text{s.t.} \begin{bmatrix} (1-\theta)P_e & \star & \star \\ 0 & \theta P_w & \star \\ P_eA_l - Y_eC_l & P_eD_l - P_eE_l & P_e \end{bmatrix} \geq 0. \tag{12}$$

3.3. Prediction of bounds of future estimation error sets

Suppose that at time k, the bounds of the estimation error are known. The following Lemma 3 predicts the bounds of future estimation error sets at time $k+i+1$, $i \in \mathbb{Z}_+$. The proof of Lemma 3 is given in Appendix A.2.

Lemma 3: Suppose that at time k, the estimation error satisfies $e^{\mathrm{T}}(i|k)P_{e0}e(i|k) \leq \eta(k+i)$, $\eta(k+i) > 1$, $i \in \mathbb{Z}_+$. Considering the predicted system parametric uncertain sets in (5) and bounded disturbance, the bounds of future estimation error sets satisfies $e^{\mathrm{T}}(i+1|k)P_{e0}e(i+1|k) \leq \eta(k+i+1)$, where $\eta(k+i+1)$ are calculated by

$$\min_{\eta(k+i+1),\phi_1(i)\geq 0,\phi_2(i)\geq 0} \ \eta(k+i+1), \tag{13}$$

$$\text{s.t.} \sum_{l=1}^{L} \lambda_l(k+i) \times \begin{bmatrix} \Lambda_1 & \star & \star & \star \\ 0 & \phi_2(i)P_{e0} & \star & \star \\ 0 & 0 & \phi_1(i)P_w & \star \\ 0 & \Lambda_{2l} & \Lambda_{3l} & P_{e0}^{-1} \end{bmatrix} \geq 0, \tag{14}$$

$$\Lambda_1 = \eta(k+i+1) - \phi_1(i) - \phi_2(i)\eta(k+i),$$
$$\Lambda_{2l} = A_l(k,i) - L_pC_l(k,i), \Lambda_{3l} = D_l(k,i) - L_pE_l(k,i),$$

when $e^{\mathrm{T}}(i|k)P_{e0}e(i|k) \leq \eta(k+i) \leq 1$, $e^{\mathrm{T}}(i+1|k)P_{e0}e(i+1|k) \leq 1$ always holds and the estimation error stays in the set $\varepsilon_{P_{e0}}$ thereafter.

At the current time k, λ_l are available, and the estimation error satisfies $e^{\mathrm{T}}(0|k)P_{e0}e(0|k) \leq \eta(k)$. By iteratively solving problem (13)-(14), the scalars $\eta(k+i+1)$, $i \in \mathbb{Z}_+$, related to future bounds of estimation error sets can be obtained. Replace the matrix P_e in (A.2) by P_{e0}, and consider $||w(k+i)||_{P_w}^2 \leq 1$, $e^{\mathrm{T}}(i|k)P_{e0}e(i|k) \leq \eta(k+i)$, then for $i \in \mathbb{Z}_+$,

$$\begin{aligned} & e^{\mathrm{T}}(i+1|k)P_{e0}e(i+1|k) \\ & \leq (1-\theta)e^{\mathrm{T}}(i|k)P_{e0}e(i|k) + \theta||w(k+i)||_{P_w}^2 \\ & \leq 1 + (1-\theta)(\eta(k+i)-1). \end{aligned} \tag{15}$$

According to (15), since $\theta \in (0,1)$, when $\eta(k+i) > 1$, the bounds of future estimation error sets will decrease with the evolution of time; when $\eta(k+i) \leq 1$, it is easy to see that $e^{\mathrm{T}}(i+1|k)P_{e0}e(i+1|k) \leq 1$ always holds, i.e., the estimation error converges within the set $\varepsilon_{P_{e0}}$ and will stay in it thereafter. In this case, let $\eta(k+i+1) = 1$, $i \in \mathbb{Z}_{[1,\infty)}$.

4. MULTI-STEP OFRMPC APPROACH

In this section, the multi-step OFRMPC approach optimizes the following sequence of controller gains:

$$u(i|k) = \begin{cases} F(k+i)\hat{x}(i|k), & i \in \mathbb{Z}_{[0,N-1]}, \\ F(k+N)\hat{x}(i|k), & i \in \mathbb{Z}_{[N,\infty)}, \end{cases} \tag{16}$$

where $F(k+i)$, $i \in \mathbb{Z}_{[0,N]}$, is the controller gain at time $k+i$. After time $k+N$, the controller gain is always $F(k+N)$.

For $i \in \mathbb{Z}_+$, the prediction models for systems (1) and (8) based on (9) and (16) are (17) and (18), respectively.

$$\begin{aligned} \hat{x}(i+1|k) = & \sum_{l=1}^{L} \lambda_l(k+i)[\Phi_l(k,i)\hat{x}(i|k) + L_pC_l(k,i) \\ & \times e(i|k) + L_pE_l(k,i)w(k+i)], \\ \hat{x}(k+1) = & \hat{x}(1|k), \end{aligned} \tag{17}$$

$$\begin{aligned} x(i+1|k) = & \sum_{l=1}^{L} \lambda_l(k+i)[\Phi_l(k,i)\hat{x}(i|k) + A_l(k,i) \\ & \times e(i|k) + D_l(k,i)w(k+i)], \\ \Phi_l(k,i) = & A_l(k,i) + B_l(k,i)F(k+i). \end{aligned} \tag{18}$$

Define the ellipsoidal sets $\varepsilon_{Q_c^{-1}(k,i)}, i \in \mathbb{Z}_{[0,N]}$, as the sequence of ellipsoidal sets that contain the current and predicted future estimated states. At the current time k, if (19) holds, the estimated state satisfies $\hat{x}(0|k) \in \varepsilon_{Q_c^{-1}(k,0)}$.

$$\begin{bmatrix} 1 & \star \\ \hat{x}(0|k) & Q_c(k,0) \end{bmatrix} \geq 0. \tag{19}$$

Theorem 1 guarantees that the estimated state is steered from one ellipsoidal set to another one and finally into an RPI set, which considers the prediction of bounds of future estimation error sets and future parametric uncertain sets.

Theorem 1: For the uncertain system (17), suppose that at time k, $\hat{x}(i|k) \in \varepsilon_{Q_c^{-1}(k,i)}$, $i \in \mathbb{Z}_{[0,N]}$, $||w(k+i)||_{P_w}^2 \leq 1$ and $e^{\mathrm{T}}(i|k)\frac{P_{e0}}{\eta(k+i)}e(i|k) \leq 1$, where $\eta(k)$ is known at time k, $\eta(k+i)$, $i \in \mathbb{Z}_{[1,N]}$, are obtained from Lemma 3. Considering the exactly known scheduling parameter at the current time k, and the predicted system parametric uncertain sets in (5), suppose that there exist positive scalars $\{\alpha_1, \alpha_2\} \in (0,1)$, $\gamma > 0$, and $Y(k+i) = F(k+i)Q_c(k,i)$, such that (19) and the following conditions (20)-(25) are satisfied,

$$\Upsilon^{\mathrm{c}}(k,i) = \sum_{l=1}^{L} \lambda_l(k+i)\widetilde{\Upsilon}_l^{\mathrm{C}}(k,i) \geq 0, \ i \in \mathbb{Z}_{[0,N-1]}, \tag{20}$$

$$\widetilde{\Upsilon}_l^{\mathrm{c}}(k,i) = \begin{bmatrix} \Delta_{1i} & \star & \star \\ 0 & \alpha_2 P_w & 0 \\ \Delta_{2li} & \Delta_{3li} & \Delta_{4i} \end{bmatrix} \geq 0, \tag{21}$$

$$\Delta_{1i} = \begin{bmatrix} (1-\alpha_1)Q_c(k,i) & \star \\ 0 & (\alpha_1-\alpha_2)\frac{P_{e0}}{\eta(k+i)} \end{bmatrix},$$

$$\Delta_{2li} = \begin{bmatrix} \Delta_{21li} & L_p C_l(k,i) \\ \mathscr{Q}^{1/2} C_l Q_c(k,i) & \mathscr{Q}^{1/2} C_l(k,i) \\ \mathscr{Q}^{1/2} Y(k+i) & 0 \end{bmatrix},$$

$$\Delta_{21li} = A_l(k,i)Q_c(k,i) + B_l(k,i)Y(k+i),$$

$$\Delta_{3li} = \begin{bmatrix} L_p E_l(k,i) \\ \mathscr{Q}^{1/2} E_l(k,i) \\ 0 \end{bmatrix},$$

$$\Delta_{4i} = \begin{bmatrix} Q_c(k,i+1) & \star & \star \\ 0 & \gamma I & \star \\ 0 & 0 & \gamma I \end{bmatrix}.$$

$$\Upsilon^{\mathrm{I}}(k,N) = \sum_{l=1}^{L} \lambda_l(k+N)\widetilde{\Upsilon}^{\mathrm{I}}_l(k,N) \geq 0, \tag{22}$$

$$\widetilde{\Upsilon}^{\mathrm{I}}_l(k,N) = \begin{bmatrix} \Delta_{5l} & \star & \star \\ 0 & \alpha_2 P_w & 0 \\ \Delta_{6l} & \Delta_{7l} & \Delta_8 \end{bmatrix} \geq 0, \tag{23}$$

$$\Delta_{5l} = \begin{bmatrix} (1-\alpha_1)Q_c(k,N) & \star \\ 0 & (\alpha_1-\alpha_2)\frac{P_{e0}}{\eta(k+N)} \end{bmatrix},$$

$$\Delta_{6l} = \begin{bmatrix} \Delta_{61l} & L_p C_l(k,N) \\ \mathscr{Q}^{1/2} C_l Q_c(k,N) & \mathscr{Q}^{1/2} C_l(k,N) \\ \mathscr{Q}^{1/2} Y(k+N) & 0 \end{bmatrix},$$

$$\Delta_{61l} = A_l(k,N)Q_c(k,N) + B_l(k,N)Y(k+N),$$

$$\Delta_{7l} = \begin{bmatrix} L_p E_l(k,N) \\ \mathscr{Q}^{1/2} E_l(k,N) \\ 0 \end{bmatrix},$$

$$\Delta_8 = \begin{bmatrix} Q_c(k,N) & \star & \star \\ 0 & \gamma I & \star \\ 0 & 0 & \gamma I \end{bmatrix}.$$

$$Q_c(k,i) \geq Q_c(k,i+1),\ i \in \mathbb{Z}_{[0,N-1]}, \tag{24}$$

$$1 - \alpha_1 - \alpha_2 \geq 0. \tag{25}$$

As a result, the controller gains are $F(k+i) = Y(k+i)Q_c^{-1}(k,i)$, $i \in \mathbb{Z}_{[0,N]}$, which steer the estimated state $\hat{x}(0|k)$ from $\varepsilon_{Q_c^{-1}(k,0)} \to \varepsilon_{Q_c^{-1}(k,1)} \to \cdots \to \varepsilon_{Q_c^{-1}(k,N)}$, and finally the future estimated states stay in the set $\varepsilon_{Q_c^{-1}(k,N)}$ thereafter under the controller gain $F(k+N)$.

Proof: When $\hat{x}(i|k) \in \varepsilon_{Q_c^{-1}(k,i)}$, $e^{\mathrm{T}}(i|k)\frac{P_{e0}}{\eta(k+i)}e(i|k) \leq 1$, and $||w(k+i)||^2_{P_w} \leq 1$, $i \in \mathbb{Z}_{[0,N-1]}$, the convergence of the future estimated state is guaranteed if there exist positive scalars α_1, α_2 and γ such that

$$\begin{aligned} &\hat{x}^{\mathrm{T}}(i|k)Q_c^{-1}(k,i)\hat{x}(i|k) - \hat{x}^{\mathrm{T}}(i^+|k)Q_c^{-1}(k,i^+)\hat{x}(i^+|k) \\ &\quad - \alpha_1[\hat{x}^{\mathrm{T}}(i|k)Q_c^{-1}(k,i)\hat{x}(i|k) - e^{\mathrm{T}}(i|k)\frac{P_{e0}}{\eta(k+i)}e(i|k)] \\ &\quad - \alpha_2[e^{\mathrm{T}}(i|k)\frac{P_{e0}}{\eta(k+i)}e(i|k) - w^{\mathrm{T}}(k+i)P_w w(k+i)] \\ &\geq \frac{1}{\gamma}\left[\|y(i|k)\|^2_{\mathscr{Q}} + \|u(i|k)\|^2_{\mathscr{R}}\right], i \in \mathbb{Z}_{[0,N-1]}. \end{aligned} \tag{26}$$

Rearrange (26), it can be obtained that

$$\begin{aligned} &(1-\alpha_1)x_c^{\mathrm{T}}(i|k)Q_c^{-1}(k,i)x_c(i|k) + (\alpha_1-\alpha_2)e^{\mathrm{T}}(i|k) \\ &\quad \times \frac{P_{e0}}{\eta(k+i)}e(i|k) + \alpha_2 w^{\mathrm{T}}(k+i)P_w w(k+i) \\ &\quad - x_c^{\mathrm{T}}(i^+|k)Q_c^{-1}(k,i^+)x_c(i^+|k) \\ &\geq \frac{1}{\gamma}\left[\|y(i|k)\|^2_{\mathscr{Q}} + \|u(i|k)\|^2_{\mathscr{R}}\right],\ i \in \mathbb{Z}_{[0,N-1]}. \end{aligned} \tag{27}$$

The sufficient and necessary condition for (27) is

$$\begin{aligned} &\Delta_{9i} - \Delta_{10li}^{\mathrm{T}}Q_c^{-1}(k,i^+)\Delta_{10li} \\ &\geq \frac{1}{\gamma}[\Delta_{11li}^{\mathrm{T}}\mathscr{Q}\Delta_{11li} + \Delta_{12i}^{\mathrm{T}}\mathscr{R}\Delta_{12i}], \end{aligned} \tag{28}$$

$$\Delta_{9i} = \begin{bmatrix} (1-\alpha_1)Q_c^{-1}(k,i) & \star & \star \\ 0 & (\alpha_1-\alpha_2)\frac{P_{e0}}{\eta(k+i)} & \star \\ 0 & 0 & \alpha_2 P_w \end{bmatrix},$$

$$\Delta_{10li} = \begin{bmatrix} \Phi_l(k,i), L_p C_l(k,i), L_p E_l(k,i) \end{bmatrix},$$

$$\Delta_{11li} = \begin{bmatrix} I, C_l(k,i), E_l(k,i) \end{bmatrix},$$

$$\Delta_{12i} = \begin{bmatrix} F(k,i), 0, 0 \end{bmatrix}, i \in \mathbb{Z}_{[0,N-1]}.$$

By applying the Schur complement, and the convergence transformation via $\mathrm{diag}\{Q_c(k,i), I\}$, then letting $Y(k+i) = F(k+i)Q_c(k,i)$, one can obtain (20). The controller gains corresponding to the sets $\varepsilon_{Q_c^{-1}(k,i)}$ are $F(k+i) = Y(k+i)Q_c^{-1}(k,i)$, $i \in \mathbb{Z}_{[0,N-1]}$. By simultaneously satisfying (19), (20), (24) and (25), it can be inferred that $\hat{x}(i|k) \in \varepsilon_{Q_c^{-1}(k,i)}$, $i \in \mathbb{Z}_{[0,N]}$. The controller gains $F(k+i)$, $i \in \mathbb{Z}_{[0,N-1]}$, steer the estimated state $\hat{x}(0|k)$ from $\varepsilon_{Q_c^{-1}(k,0)} \to \varepsilon_{Q_c^{-1}(k,1)} \to \cdots \to \varepsilon_{Q_c^{-1}(k,N)}$, where $\varepsilon_{Q_c^{-1}(k,0)} \supseteq \varepsilon_{Q_c^{-1}(k,1)} \supseteq \cdots \supseteq \varepsilon_{Q_c^{-1}(k,N)}$ due to the consideration of (24).

Once the estimated state is steered into the set $\varepsilon_{Q_c^{-1}(k,N)}$, the future estimated state will be RPI in the set $\varepsilon_{Q_c^{-1}(k,N)}$ if

$$\begin{aligned} &\hat{x}^{\mathrm{T}}(i|k)Q_c^{-1}(k,N)\hat{x}(i|k) - \hat{x}^{\mathrm{T}}(i^+|k)Q_c^{-1}(k,N)\hat{x}(i^+|k) \\ &\quad - \alpha_1[\hat{x}^{\mathrm{T}}(i|k)Q_c^{-1}(k,N)\hat{x}(i|k) - e^{\mathrm{T}}(i|k)\frac{P_{e0}}{\eta(k+N)}e(i|k)] \\ &\quad - \alpha_2[e^{\mathrm{T}}(i|k)\frac{P_{e0}}{\eta(k+N)}e(i|k) - w^{\mathrm{T}}(k+i)P_w w(k+i)] \\ &\geq \frac{1}{\gamma}\left[\|y(i|k)\|^2_{\mathscr{Q}} + \|u(i|k)\|^2_{\mathscr{R}}\right],\ i \in \mathbb{Z}_{[N,\infty)}. \end{aligned} \tag{29}$$

Rearrange (29), then it can be obtained that

$$\begin{aligned} &(1-\alpha_1)x_c^{\mathrm{T}}(i|k)Q_c^{-1}(k,N)x_c(i|k) + (\alpha_1-\alpha_2)e^{\mathrm{T}}(i|k) \\ &\quad \times \frac{P_{e0}}{\eta(k+N)}e(i|k) + \alpha_2 w^{\mathrm{T}}(k+i)P_w w(k+i) - x_c^{\mathrm{T}}(i^+|k) \\ &\quad \times Q_c^{-1}(k,N)x_c(i^+|k) \geq \frac{1}{\gamma}\left[\|y(i|k)\|^2_{\mathscr{Q}} + \|u(i|k)\|^2_{\mathscr{R}}\right], \\ &i \in \mathbb{Z}_{[N,\infty)}. \end{aligned} \tag{30}$$

Similar to the procedure for obtaining (20), equation (30) is guaranteed by (22). □

Lemma 4 deals with the input and state constraints in (3). The proof is given in Appendix A.3.

Lemma 4: Considering system (18) and the controller gains in (16), the constraints on the control input and true state in (3) are satisfied if there exist symmetric positive-definite matrices $Q_c(k,i)$ and $Y(k+i) = F(k+i)Q_c(k,i)$, $i \in \mathbb{Z}_{[0,N]}$, such that (19), (20), (22), (24), (25), and the following conditions are satisfied:

$$\widetilde{\Upsilon}^{\mathrm{U}}(k,i) = \begin{bmatrix} Q_c(k,i) & \star \\ \xi_s Y(k+i) & \bar{u}_s^2 \end{bmatrix} \geq 0, \tag{31}$$

$$s \in \mathbb{Z}_{[0,n_u]},\ i \in \mathbb{Z}_{[0,N]},$$

$$\Upsilon^{\mathrm{S}}(k,i) = \sum_{l=1}^{L} \lambda_l(k+i)\widetilde{\Upsilon}_l^{\mathrm{S}}(k,i) \geq 0, \widetilde{\Upsilon}_l^{\mathrm{S}}(k,i) = \tag{32}$$

$$\begin{bmatrix} Q_c(k,i) & \star & \star & \star \\ 0 & \frac{P_{e0}}{\eta(k+i)} & \star & \star \\ 0 & 0 & P_w & \star \\ \Delta & \sqrt{3}\Psi_t A_l(k,i) & \sqrt{3}\Psi_t D_l(k,i) & \bar{\psi}_t^2 \end{bmatrix},$$

$$\Delta = \sqrt{3}\Psi_t[A_l(k,i)Q_c(k,i) + A_l(k,i)Y(k+i)],$$

$$i \in \mathbb{Z}_{[0,N]}, t \in \mathbb{Z}_{[0,q]},$$

where ξ_s is the s-th row of the n_u-order identity matrix, and Ψ_t is the t-th row of the matrix Ψ.

From the above derivations, at each time k, the multi-step OFRMPC optimization problem is solved by

$$\min_{\alpha_1,\ \alpha_2,\ \gamma,\ Y(k+i), Q_c(k,i)} \gamma,$$

$$\text{s.t. } (19), (20), (22), (24), (25), (31), (32). \tag{33}$$

In problem (33), the scalar γ is the optimized objective function. In problem (33), model parameters in $\Upsilon^{c}(k,0)$ and $\Upsilon^{\mathrm{S}}(k,0)$ (see (20) and (32)) are exactly known since $\lambda_l(k)$, $l \in \mathbb{Z}_{[0,L]}$, are exactly available at the current time k. After time k, by considering convexity of the optimization, vertices of the predicted future system model parametric uncertain sets in (5) are substituted into problem (33).

Remark 1: Compared with the OFRMPC methods in [23, 24], at the current and future time, all the possible realizations of model parametric uncertainty and bounds of estimation error sets are considered, which amount to $N = 0$, $[A_l(k,i)|B_l(k,i)|C_l(k,i)|D_l(k,i)|E_l(k,i)] \in \Omega_0, i \in \mathbb{Z}_{[0,\infty)}$, and $e^{\mathrm{T}}(i|k)P_{e0}e(i|k) \leq \eta(k), i \in \mathbb{Z}_{[0,\infty)}$, in the present paper. When problem (33) is solved, the multi-step controller gains in (16) are calculated as $F(k+i) = Y(k+i)Q_c^{-1}(k,i)$. The sequence of controller gains $F(k+i)$, $i \in \mathbb{Z}_{[0,N]}$, in problem (33) are associated with the Lyapunov matrices $Q_c^{-1}(k,i)$, $i \in \mathbb{Z}_{[0,N]}$, which introduce more degree of freedom for the optimization. In [23, 24], the controller gains are related to one common Lyapunov matrix. The larger N is selected, the less conservative of the system parametric uncertainty and bounds of estimation error sets will be involved. However, the computational burden will increase (see Section 6). The selection of N should consider the trade-off between control performance and computational burden.

5. OVERALL ALGORITHM, RECURSIVE FEASIBILITY AND ROBUST STABILITY

5.1. The overall algorithm

The overall algorithm includes the off-line stage to obtain observer gain and a minimal RPI set for the estimation error, the on-line prediction of future parametric uncertain sets and bounds of estimation error sets, and the multi-step OFRMPC optimization to obtain the sequence of controller gains.

Algorithm 1:

Off-line stage: solve problem (11)-(12) to obtain P_{e0} and $L_p = P_{e0}^{-1}Y_e^0$.

On-line stage: Choose the initial estimated state $\hat{x}(0|k)$. Let $\eta(0) = e^{\mathrm{T}}(0|k)P_{e0}e(0|k)$. At each time $k \geq 0$, perform the following steps:

1) Predict system parametric uncertain sets by (5) and bounds of estimation error sets by Lemma 3.
2) Solve problem (33) to obtain the optimal solution $\{\alpha_1, \alpha_2, \gamma, Y(k+i), Q_c(k,i)\}$, $i \in \mathbb{Z}_{[0,N]}$.
3) Calculate the control input by $u(0|k) = Y(k)Q_c^{-1}(k,0)\hat{x}(0|k)$. Implement $u(0|k)$ to system (1), and let $\hat{x}(k+1) = A(k)\hat{x}(0|k) + B(k)u(0|k) + L_p(y(k) - C(k)\hat{x}(0|k))$.

In Algorithm 1, problems (11)-(12) and (33) can be solved via the linear matrix inequality (LMI) toolbox, where the scalars θ and α_1 are linear searched over the interval $(0,1)$.

5.2. Recursive feasibility and robust stability

Theorem 2: For system (1), Algorithm 1 is performed. If problem (33) is feasible at time $k = 0$, then the recursive feasibility of problem (33) and the robust stability of system (1)-(2) is guaranteed. With the evolution of time, the scalar γ converges to a constant value, and the system output and control input converge to a neighborhood of origin, respectively. The input and state constraints in (3) are satisfied for all $k \geq 0$.

Proof: Suppose that problem (33) is solved at time $k \geq 0$, the optimal solution $\Sigma^*(k) = \{\alpha_1^*,\ \alpha_2^*, \gamma^*, Y^*(k), \ldots, Y^*(k+N), Q_c^*(k,0), \ldots, Q_c^*(k,N)\}$ is obtained. At time k, the control inputs $u(i|k) = Y^*(k+i)[Q_c^*]^{-1}(k,i)\hat{x}(i|k)$, $i \in \mathbb{Z}_{[0,N]}$, steer the estimated state $\hat{x}(0|k)$ from $\varepsilon_{[Q_c^*]^{-1}(k,0)}$ into the sequence of sets $\varepsilon_{[Q_c^*]^{-1}(k,i)}$, and finally the estimated state is RPI in the set $\varepsilon_{[Q_c^*]^{-1}(k,N)}$. At time $k+1$, construct the solution $\Sigma(k+1) = \{\alpha_1^*,\ \alpha_2^*,\ \gamma^*, Y^*(k+1), \ldots, Y^*(k+N), Y^*(k+N), Q_c^*(k,1), \ldots, Q_c^*(k,N), Q_c^*(k,N)\}$. Considering the non-increasing of scalars $\eta(k+i)$ in Lemma 3 and (24), it can be seen that $\Sigma(k+1)$ is a feasible solution for (20), (22), (25), (31) and (32). By choosing $Q_c(k+1,0) = Q_c^*(k,1)$, and considering $\hat{x}(1|k) \in \varepsilon_{[Q_c^*]^{-1}(k,1)}$ and $\varepsilon_{[Q_c^*]^{-1}(k,0)} \supseteq \varepsilon_{[Q_c^*]^{-1}(k,1)}$, (19) is satisfied at time $k+1$. Therefore, the optimal solution of problem (33) at time k is a feasible solution for problem (33) at time $k+1$, i.e, the recursive feasibility of the optimization problem is guaranteed. By solving problem (33) at time $k+1$, it can be obtained that $\gamma^*(k+1) \le \gamma^*(k)$. With the evolution of time, $\gamma^*(k)$ will tend to a constant value. By summing (27) from $i=0$ to $N-1$, and (30) from $i=N$ to ∞, respectively, then adding the summations together and considering and $||\hat{x}(0|k)||^2_{Q_c(k,0)} \le 1$, $||w(k+i)||^2_{P_w} \le 1$ and $e^{\mathrm{T}}(i|k)\frac{P_{e0}}{\eta(k+t)}e(i|k) \le 1$, $i \in \mathbb{Z}_+$, then

$$J_\infty(k) = \sum_{i=0}^{\infty}\left[\|y(i|k)\|^2_{\mathscr{Q}} + \|u(i|k)\|^2_{\mathscr{R}}\right] < \gamma^*(k). \quad (34)$$

Therefore, $\gamma^*(k)$ is an upper bound of $J_\infty(k)$ at each time k. Consider the following nominal systems (35)-(37) (i.e., systems (2), (16) and (17) without the consideration of the uncertainties of estimation error and bounded disturbance).

$$\hat{x}_u(i+1|k) = \sum_{l=1}^{L}\lambda_l(k+i)\Phi_l(k,i)\hat{x}_u(i|k), \quad (35)$$

$$u_u(i|k) = F(k+i)\hat{x}_u(i|k), \hat{x}_u(0|k) = \hat{x}(0|k), \quad (36)$$

$$y_u(0|k) = C\hat{x}_u(0|k), i \in \mathbb{Z}_+, \quad (37)$$

By applying the convergence transformation and the Schur complement, (20) and (22) also imply (38) and (39), respectively.

$$\begin{aligned}&(1-\alpha_1)Q_c^{-1}(k,i) - \Phi_l^{\mathrm{T}}(k,i)Q_c^{-1}(k,i+1)\Phi_l(k,i)\\ &\ge \frac{1}{\gamma^*(k)}\left[C_l^{\mathrm{T}}(k,i)\mathscr{Q}C_l^{\mathrm{T}}(k,i) + F^{\mathrm{T}}(k+i)\mathscr{R}F(k+i)\right],\\ &\quad i \in \mathbb{Z}_{[0,N-1]}, \quad (38)\end{aligned}$$

$$\begin{aligned}&(1-\alpha_1)Q_c^{-1}(k,N) - \Phi_l^{\mathrm{T}}(k,N)Q_c^{-1}(k,N)\Phi_l(k,N)\\ &\ge \frac{1}{\gamma^*(k)}\left[C_l^{\mathrm{T}}(k,i)\mathscr{Q}C_l^{\mathrm{T}}(k,i) + F^{\mathrm{T}}(k+N)\mathscr{R}F(k+N)\right]. \quad (39)\end{aligned}$$

Since $\alpha_1 \in (0,1)$, (38) and (39) guarantee that

$$\begin{aligned}&Q_c^{-1}(k,i) - \Phi_l^{\mathrm{T}}(k,i)Q_c^{-1}(k,i+1)\Phi_l(k,i)\\ &> \frac{1}{\gamma^*(k)}\left[C_l^{\mathrm{T}}(k,i)\mathscr{Q}C_l^{\mathrm{T}}(k,i) + F^{\mathrm{T}}(k+i)\mathscr{R}F(k+i)\right],\\ &\quad i \in \mathbb{Z}_{[0,N-1]}, \quad (40)\end{aligned}$$

$$\begin{aligned}&Q_c^{-1}(k,N) - \Phi_l^{\mathrm{T}}(k,N)Q_c^{-1}(k,N)\Phi_l(k,N)\\ &> \frac{1}{\gamma^*(k)}\left[C_l^{\mathrm{T}}(k,N)\mathscr{Q}C_l^{\mathrm{T}}(k,N) + F^{\mathrm{T}}(k+N)\mathscr{R}F(k+N)\right]. \quad (41)\end{aligned}$$

For system (35)-(37), the above conditions (40) and (41) guarantee that

$$\begin{aligned}&\hat{x}_u^{\mathrm{T}}(i|k)Q_c^{-1}(k,i)\hat{x}_u(i|k) - \hat{x}_u^{\mathrm{T}}(i^+|k)Q_c^{-1}(k,i^+)\hat{x}_u(i^+|k)\\ &> \frac{1}{\gamma^*(k)}\left[\|y_u(i|k)\|^2_{\mathscr{Q}} + \|u_u(i|k)\|^2_{\mathscr{R}}\right], \quad i \in \mathbb{Z}_{[0,N-1]}, \quad (42)\end{aligned}$$

$$\begin{aligned}&\hat{x}_u^{\mathrm{T}}(i|k)Q_c^{-1}(k,N)\hat{x}_u(i|k) - \hat{x}_u^{\mathrm{T}}(i^+|k)Q_c^{-1}(k,N)\hat{x}_u(i^+|k)\\ &> \frac{1}{\gamma^*(k)}\left[\|y_u(i|k)\|^2_{\mathscr{Q}} + \|u_u(i|k)\|^2_{\mathscr{R}}\right], \quad i \in \mathbb{Z}_{[N,\infty)}. \quad (43)\end{aligned}$$

By summing (42) from $i=0$ to $N-1$, and (43) from $i=N$ to ∞, respectively, and adding the summations together and applying (19),

$$\sum_{i=0}^{\infty}\left[\|y_u(i|k)\|^2_{\mathscr{Q}} + \|u_u(i|k)\|^2_{\mathscr{R}}\right] < \gamma^*(k). \quad (44)$$

With the evolution of time, $\gamma^*(k)$ will tend to a constant value implies that the system output $y_u(k)$ and input $u_u(k)$ will converge to the origin. Since the estimation error is robust UUB with respect to the minimal RPI set and bounded disturbance exists, the estimated state and control input will converge to a neighborhood of origin. Therefore, the true state will converge to a neighborhood of origin due to the convergence of estimated states and the estimation error. The input and state constraints are satisfied due to (31) and (32), respectively. □

6. THE COMPARISON OF COMPUTATIONAL BURDEN

We compare Algorithm 1 with Algorithm 1 in [23] and Algorithm 3 in [24]. The complexity analysis for the compared algorithms is listed in Table 1. The complexity analysis for the optimization problem solved by an LMI tool in the compared algorithms is polynomial-time, which (regarding the fastest interior-point algorithms) is proportional to $\mathfrak{K}^3\mathfrak{L}$, where $\mathfrak{K}$ is the number of scalar LMI variables and $\mathfrak{L}$ is the number of scalar LMI rows [32]. Compared with Algorithm 1 in [23], the larger N leads to the increase of $\mathfrak{K}$ and $\mathfrak{L}$, which will result in the increase of the on-line computational burden.

7. NUMERICAL EXAMPLE

Consider the following LPV model, where the system model parameters are

$$A_1 = \begin{bmatrix} 0.8227 & -0.00168 \\ 6.1233 & 0.9367 \end{bmatrix}, B_1 = \begin{bmatrix} -0.000092 \\ 0.1014 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0.9654 & -0.00182 \\ -0.6759 & 0.9433 \end{bmatrix}, B_2 = \begin{bmatrix} -0.000097 \\ 0.1016 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 0.8895 & -0.00294 \\ 2.9447 & 0.9968 \end{bmatrix}, B_3 = \begin{bmatrix} -0.000157 \\ 0.1045 \end{bmatrix},$$

$$A_4 = \begin{bmatrix} 0.8930 & -0.00062 \\ 2.7738 & 0.8864 \end{bmatrix}, B_4 = \begin{bmatrix} -0.000034 \\ 0.0986 \end{bmatrix},$$

Table 1. The comparison of complexity analysis.

The compared algorithms	Optimization problem	The prediction of bounds of estimation error sets
Algorithm 1 in [23]	Problem (22) in [23]: $\mathfrak{K}=Ln_un_x+n_x(n_x+1)+\frac{1}{2}n_u(n_u+1)+\frac{1}{2}q(q+1)+2,$ $\mathfrak{L}=4n_x+L(2n_x+n_w+n_y+n_u)+2n_x+2n_u+L(2n_x+n_w+q)+q$	Problem (23)-(24) in [23]: $\mathfrak{K}=3,\ \mathfrak{L}=1+n_w++n_x$
Algorithm 1 in [24]	Problem (42) in [24]: $\mathfrak{K}_M=\frac{1}{2}L(L+1)n_x^2+Ln_x(n_y+n_u)+n_un_y+2n_x(n_x+1)+\frac{1}{2}n_u(n_u+1)+\frac{1}{2}q(q+1)+2,$ $\mathfrak{L}_M=\frac{(L+n-1)!}{2(L-1)!}(4n_x+n_w+n_y+n_u)+L\frac{(L+n-1)!}{2(L-1)!}(2n_x+2n_w+n_y)+L(2n_x+n_w+n_u)+5n_x+n_u+q+1$	Problem (50) in [24]: $\mathfrak{K}=\frac{1}{2}n_x(n_x+1),\ \mathfrak{L}=\hat{p}n_p^w+n_x;$ Problem (51) in [24]: $\mathfrak{K}=\bar{p},\ \mathfrak{L}=2\bar{p}\hat{p}n_p^w$
Algorithm 1	Problem (33): $\mathfrak{K}=2+(N+1)(n_un_x+\frac{1}{2}n_x(n_x+1)),$ $\mathfrak{L}=2+Nn_x+(LN+1)(5n_x+2n_w+n_y+n_u+q)+(N+1)(n_x+n_u)$	Problem (13)-(14): $\mathfrak{K}=3(N+1),$ $\mathfrak{L}=1+2n_w+n_x+LN(1+2n_w++n_x)$

$$\lambda_1(k)=\frac{1}{2}\frac{\varphi_1(y)-\varphi_1(-\bar{\psi})}{\varphi_1(\bar{\psi})-\varphi_1(-\bar{\psi})},D_l=[0.0006,0.0141]^{\mathrm{T}},$$
$$\lambda_2(k)=\frac{1}{2}\frac{\varphi_1(\bar{\psi})-\varphi_1(y)}{\varphi_1(\bar{\psi})-\varphi_1(-\bar{\psi})},C_l=[0\ \ 1],\ E_l=0.5,$$
$$\varphi_1(y)=7.2\times 10^{10}e^{-\frac{8750}{y+350}},\Delta_l=0.05,\ l\in\mathbb{Z}_{[1,4]},$$
$$\lambda_3(k)=\frac{1}{2}\frac{\varphi_2(y)-\varphi_2(-\bar{\psi})}{\varphi_2(\bar{\psi})-\varphi_2(-\bar{\psi})},\bar{u}=10,\Psi=I,$$
$$\lambda_4(k)=\frac{1}{2}\frac{\varphi_2(\bar{\psi})-\varphi_2(y)}{\varphi_2(\bar{\psi})-\varphi_2(-\bar{\psi})},\bar{\psi}=[0.5,10]^{\mathrm{T}},$$
$$\varphi_2(y)=3.6\times 10^{10}\left(e^{-\frac{8750}{y+350}}-e^{-\frac{8750}{350}}\right)/y.$$

In the above system, the system output $y(k)$, and the current scheduling parameter are known and have bounded rate of their changes. The control object is to regulate x_2 by manipulating the control input satisfying the input constraints. Select $\theta=0.02$ and solve problem (11)-(12) to obtain P_{e0} and the observer gain. In problem (33), select $\alpha_1=0.05$, $P_w=1$, $N=4$. Choose $\mathscr{Q}=25$ and $\mathscr{R}=1$, which means that we prefer the convergence of system output, which is also related to the system state x_2. The disturbance sequence w is randomly generated from the interval $[-1,1]$. Let $e(0)=[0.12,1.2]^{\mathrm{T}}$ and $\eta(0)=17.525$.

We compare Algorithm 1 with Algorithm 1 in [23] and Algorithm 3 in [24]. Fig. 1 shows the region of attraction for the compared algorithms. Here, the region of attraction denoted by $\hat{\mathcal{X}}$ is the region of $\hat{x}(0)$ such that whenever $\hat{x}(0)\in\hat{\mathcal{X}}$, the optimization problems of the compared algorithms are feasible at time $k=0$. It is shown that Algorithm 1 in the paper has the largest region of attraction. To compare the algorithms, two cases on the initial values of Algorithm 1 (Algorithm 1 for cases (a) and (b)) are considered. Considering that Algorithm 1 in [23], Algorithm 3 in [24] and Algorithm 1 are feasible, take $\hat{x}(0)=[0.18,3.6]^{\mathrm{T}}$ and $x(0)=[0.3,4.8]^{\mathrm{T}}$. In Algorithm 1 for case (b), take $\hat{x}(0)=[0.29,5.2]^{\mathrm{T}}$ and $x(0)=[0.41,6.4]^{\mathrm{T}}$, where the initial estimated state is near the boundary of the region of attraction for Algorithm 1. In this case, Algorithm 1 is feasible, while Algorithm 1 in [23] and Algorithm 3 in [24] are infeasible. The responses of $x(k)$, $\hat{x}(k)$, and the bounds of estimation error sets for the compared algorithms are shown in Figs. 2-4, where the dash (solid) lines with symbols are the responses of the estimated (true) states, and the ellipsoids are the estimation error sets. Fig. 5 shows the state trajectories of the augmented closed-loop system. Figs. 6 and 7 show the responses of the states $\hat{x}_1,x_1$ and $\hat{x}_2,x_2$, respectively. In Fig. 8, the control inputs in Algorithm 1 reach the bounds of control input constraints. From the simulation results, it can be concluded that the proposed multi-step OFRMPC enlarges the region of attraction and improves the control performance. The simulation time spent on Algorithm 1 (Algorithm 1 in [23] and Algorithm 3 in [24]) is 8.84 s (3.68 s, 28.63 s). Matlab 9.3 (Intel i5-7200U 2.5GHz, 8G Memory) is utilized for the simulations.

8. CONCLUSIONS

For the LPV systems with bounded changes of scheduling parameters and disturbance, the multi-step OFRMPC approach is investigated, where predictions of future bounds of estimation error and future parametric uncer-

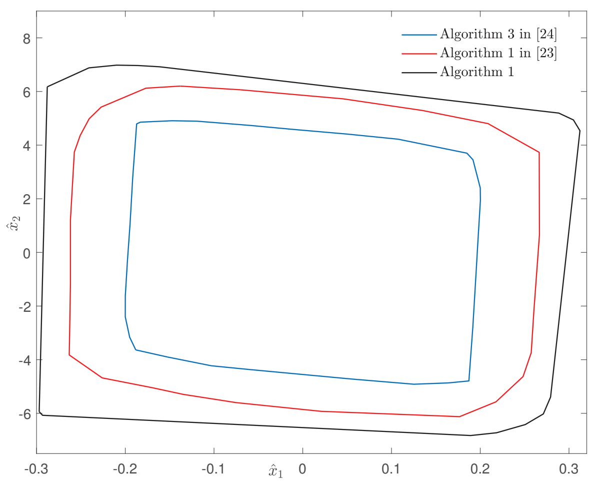

Fig. 1. The region of attraction for the compared algorithms.

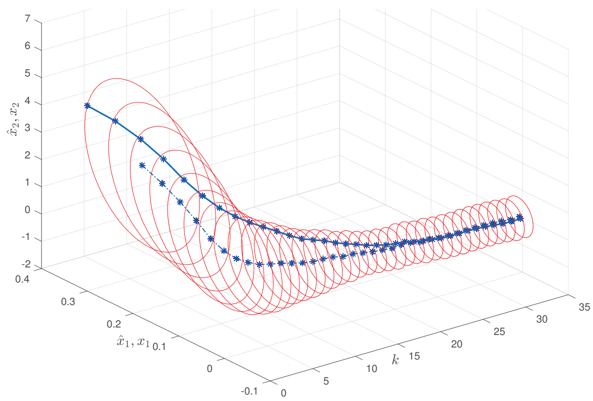

Fig. 2. The responses of $\hat{x}(k)$, $x(k)$, and the evolution of the bounds of $e(k)$, Algorithm 1 for case (a).

tain sets are considered. The proposed algorithm guarantees robust UUB of the estimation error and robust stability of the observer system. The multi-step OFRMPC method reduces the conservatism in the output feedback controller design and introduces more degree of freedom for the optimization problem. However, more optimization variables and LMI conditions are introduced in the optimization problem, which increases the on-line computational burden. Our future work on this topic would be reducing the on-line computational burden.

APPENDIX A

A.1. Proof of Lemma 2

Since $||w(k+i)||^2_{P_w} \le 1$, $e^{\mathrm{T}}(i|k)P_e e(i|k) \ge 1$ is equivalent to $e^{\mathrm{T}}(i|k)P_e e(i|k) \ge ||w(k+i)||^2_{P_w}$. Equation (10) is equivalent to

$$e^{\mathrm{T}}(i|k)P_e e(i|k) \ge ||w(k+i)||^2_{P_w}$$

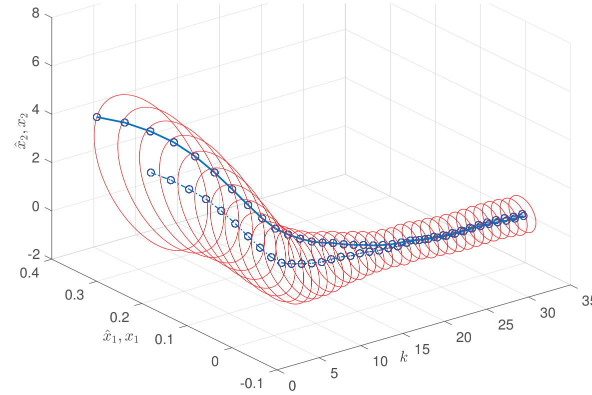

Fig. 3. The responses of $\hat{x}(k)$, $x(k)$ and the evolution of the bounds of $e(k)$, Algorithm 1 in [23].

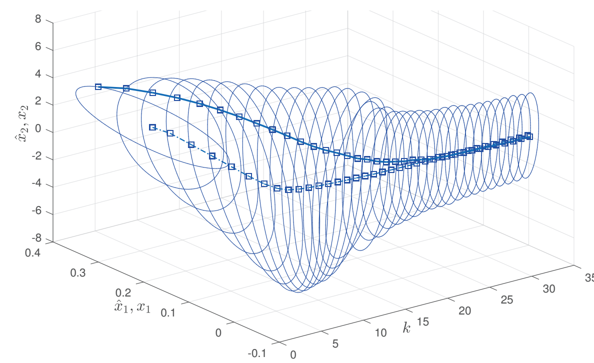

Fig. 4. The responses of $\hat{x}(k)$, $x(k)$ and the evolution of the bounds of $e(k)$, Algorithm 3 in [24].

$$\Rightarrow e^{\mathrm{T}}(i|k)P_e e(i|k) - e^{\mathrm{T}}(i+1|k)P_e e(i+1|k) \ge 0. \tag{A.1}$$

By applying the S-procedure, equation (A.1) is satisfied if there exists a scalar $\theta \ge 0$ such that

$$\begin{aligned} &e^{\mathrm{T}}(i|k)P_e e(i|k) - e^{\mathrm{T}}(i+1|k)P_e e(i+1|k) \\ &\ge \theta(e^{\mathrm{T}}(i|k)P_e e(i|k) \ge ||w(k+i)||^2_{P_w}). \end{aligned} \tag{A.2}$$

For $l \in \mathbb{Z}_{[1,L]}$, equation (A.2) is rearranged as

$$\begin{bmatrix} e(i|k) \\ w(k+i) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \Pi_{1l} & \star \\ \Pi_{2l} & \Pi_{3l} \end{bmatrix} \begin{bmatrix} e(i|k) \\ w(k+i) \end{bmatrix} \ge 0, \tag{A.3}$$

where

$$\begin{aligned} \Pi_{1l} &= (1-\theta)P_e - (A_l - L_p C_l)^{\mathrm{T}} P_e (A_l - L_p C_l), \\ \Pi_{2l} &= -(D_l - L_p E_l)^{\mathrm{T}} P_e (A_l - L_p C_l), \\ \Pi_{3l} &= \theta P_w - (D_l - L_p E_l)^{\mathrm{T}} P_e (D_l - L_p E_l). \end{aligned}$$

Equation (A.3) is satisfied for any possible $e(i|k)$ and $w(k+i)$ if

$$\begin{bmatrix} \Pi_{1l} & \star \\ \Pi_{2l} & \Pi_{3l} \end{bmatrix} \ge 0. \tag{A.4}$$

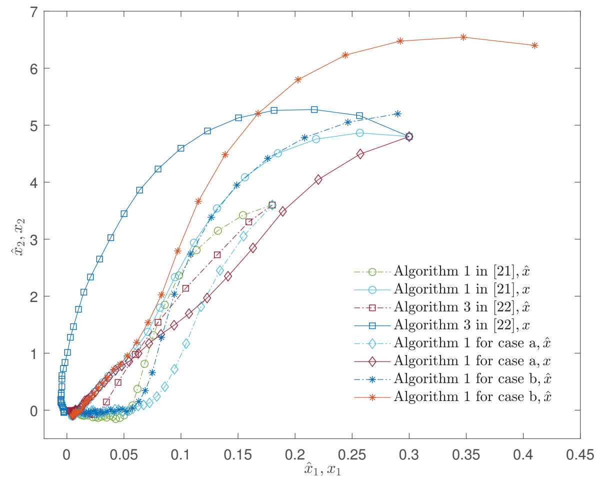

Fig. 5. The state trajectories of the system for the compared algorithms.

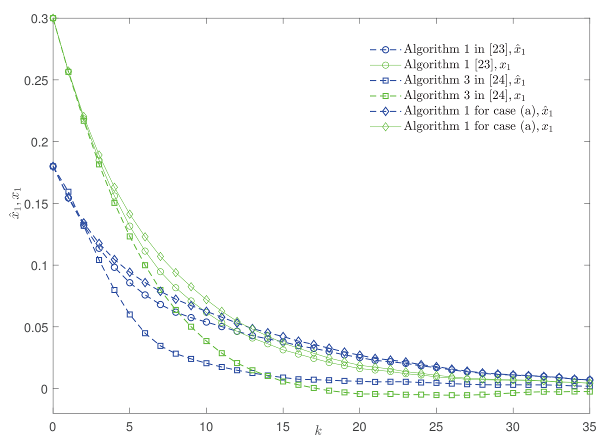

Fig. 6. The responses of state $\hat{x}_1$ and x_1.

By applying the Schur complement and considering the convexity of the polytopic description of system (1), equation (A.4) is equivalent to (12). By maximizing the trace(P_e) subject to (12), the matrix P_{e0} with the largest trace can guarantee (12) is obtained, i.e., the matrix P_{e0} is the shape matrix of the minimal RPI set for the estimation error.

A.2. Proof of Lemma 3

Suppose that at time $k+i$, $i \in \mathbb{Z}_+$, the estimation error and bounded disturbance satisfy

$$e^{\mathrm{T}}(i|k)P_{e0}e(i|k) \leq \eta(k+i), \tag{A.5}$$

$$w^{\mathrm{T}}(k+i)P_w w(k+i) \leq 1. \tag{A.6}$$

Based on (9), the estimation error constraint at time $k+i+1$ satisfies

$$e^{\mathrm{T}}(i+1|k)P_{e0}e(i+1|k) \leq \eta(k+i+1). \tag{A.7}$$

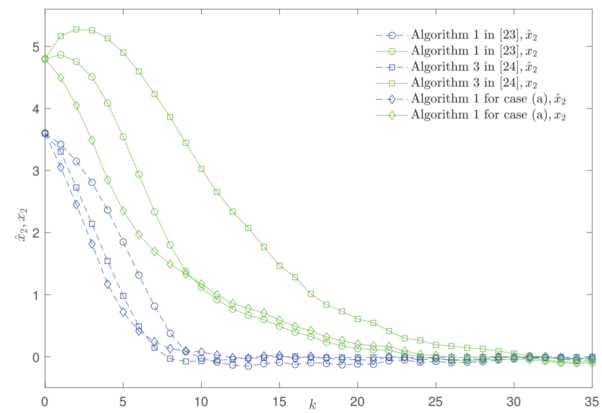

Fig. 7. The responses of state $\hat{x}_2$ and x_2.

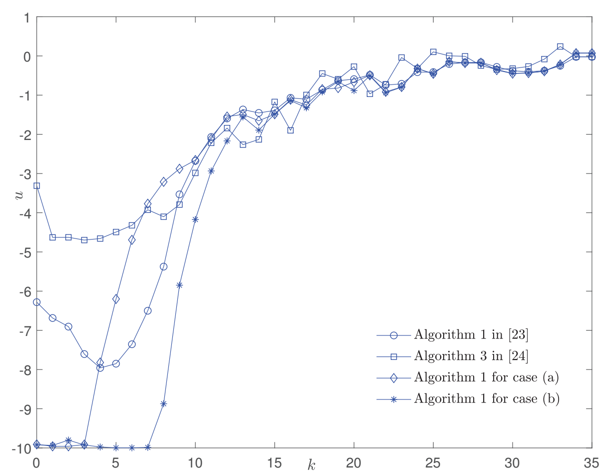

Fig. 8. The comparison of control inputs.

According to the S-procedure, the sufficient condition for "(A.5) and (A.6) $\Rightarrow$ (A.7)" to hold is that there exist nonnegative scalars $\phi_1(i)$ and $\phi_2(i)$, $i \in \mathbb{Z}_+$, such that

$$\begin{aligned} &\eta(k+i^+) - e^{\mathrm{T}}(i^+|k)P_{e0}e(i^+|k) \\ &\quad - \phi_1(i)(1 - w^{\mathrm{T}}(k+i)P_w w(k+i)) \\ &\quad - \phi_2(i)(\eta(k+i) - e^{\mathrm{T}}(i|k)P_{e0}e(i|k)) \geq 0. \end{aligned} \tag{A.8}$$

Define $\zeta(i|k) = [1, e(i|k), w(k+i)]^{\mathrm{T}} \in \mathbb{R}^{n_\zeta}$, where $n_\zeta = 1 + n_x + n_w$. Then for all possible $\zeta(i|k)$, by applying the Schur complement and considering the predicted system parametric uncertain sets in (5), equation (A.8) is equivalent to (14). Further by minimizing the scalar $\eta(k+i+1)$ subject to (14), the bounds of the estimation error set at time $k+i+1$ can be obtained.

A.3. Proof of Lemma 4

Since (19), (20), (22), (24) and (25) are satisfied, it can be inferred that $\hat{x}(i|k) \in \varepsilon_{Q_c^{-1}(k,i)}$, $i \in \mathbb{Z}_{[0,\infty)}$, with $Q_c^{-1}(k,i) = Q_c^{-1}(k,N)$ when $i \in \mathbb{Z}_{[N,\infty)}$. Consider the following conditions:

$$\begin{aligned}&\max_{i\geq 0} \mid \xi_s u(i|k) \mid^2 = \max_{i\geq 0} |\xi_s F(k+i)\hat{x}(i|k)|^2\\&\leq \max_{i\geq 0} \left\|\xi_s F(k+i)Q_c^{1/2}(k,i)\right\|^2 \left\|Q_c^{-1/2}(k,i)\hat{x}(i|k)\right\|^2\\&\leq \max_{i\geq 0} \left\|\xi_s F(k+i)Q_c^{1/2}(k,i)\right\|^2. \qquad (A.9)\end{aligned}$$

If the following constraints are satisfied:

$$\xi_s F(k+i)Q_c(k,i)[\xi_s F(k+i)]^{\mathrm{T}} \leq \bar{u}_s^2, s\in \mathbb{Z}_{[0,n_u]}, \qquad (A.10)$$

then $\mid u(i|k) \mid\leq \bar{u},\ i\in\mathbb{Z}_{[0,\infty)}$. By applying the Schur complement, the convergence transformation via diag$\{Q_c(k,i), I\}$ and $Y(k+i) = F(k+i)Q_c(k,i)$, equation (A.10) is equivalent to (31).

According to the prediction of the future true state in (18), and considering $\hat{x}(i|k)\in\varepsilon_{Q_c^{-1}(k,i)}$, $e(i|k)\in\varepsilon_{[P_{e0}/\eta(k+i)]}$, $w(k+i)\in\varepsilon_{P_w}$, then

$$\begin{aligned}&\max_{i\geq 0} \mid \Psi_t x(i+1|k) \mid^2\\&= \max_{i\geq 0}\left|\Psi_t\Pi_{4i}\begin{bmatrix}\hat{x}(i|k)\\ e(i|k)\\ w(k+i)\end{bmatrix}\right|^2\\&\leq 3\max_{i\geq 0}\left\|\Psi_t\Pi_{4i}\begin{bmatrix}Q_c^{\frac{1}{2}}(k,i) & 0 & 0\\ 0 & [\frac{P_{e0}}{\eta(k+i)}]^{-\frac{1}{2}} & 0\\ 0 & 0 & P_w^{-\frac{1}{2}}\end{bmatrix}\right\|^2,\\&\Pi_{4i} = [\Phi(k,i)\ \ A(k+i)\ \ D(k+i)].\end{aligned}$$

If the following constraints are satisfied:

$$\bar{\psi}_t^2 - 3\Psi_t\Pi_{4i}^{\mathrm{T}}\begin{bmatrix}Q_c^{-1}(k,i) & 0 & 0\\ 0 & [\frac{P_{e0}}{\eta(k+i)}]^{-1} & 0\\ 0 & 0 & P_w^{-1}\end{bmatrix}\Pi_{4i}\geq 0, \qquad (A.11)$$

then $|\Psi x(i+1|k)|\leq\bar{\psi}$. By applying the Schur complement, convergence transformation via diag$\{Q_c(k,i), I\}$ and $Y(k+i) = F(k+i)Q_c(k,i)$, and considering future prediction of system parametric uncertain sets in (5), it is shown that (A.11) is equivalent to (32).

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