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Delay Dependent Local Stabilization Conditions for Time-delay Nonlinear Discrete-time Systems Using Takagi-Sugeno Models

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Abstract: We propose convex conditions for stabilization of nonlinear discrete-time systems with time-varying delay in states through a fuzzy Takagi-Sugeno (T-S) modeling. These conditions are developed from a fuzzy Lyapunov-Krasovskii function and they are formulated in terms of linear matrix inequalities (LMIs). The results can be applied to a class of nonlinear systems that can be exactly represented by T-S fuzzy models inside a specific region called the region of validity. As a consequence, we need to provide an estimate of the set of safe initial conditions called the region of attraction such that the closed-loop trajectories starting in this set are assured to remain in the region of validity and to converge asymptotically to the origin. The estimate of the region of attraction is done with the aid of two sets: one dealing with the current state, and the other concerning the delayed states. Then, we can obtain the feedback fuzzy control law depending on the current state, x_k, and the maximum delayed state vector, $x_{k-\bar{d}}$. It is shown that such a control law can locally stabilize the nonlinear discrete-time system at the origin. We also develop convex optimization procedures for the computation of the fuzzy control gains that maximize the estimates of the region of attraction. We present two examples to demonstrate the efficiency of the developed approach and to compare it with other approaches in the literature.

Keywords: Fuzzy Lyapunov-Krasovskii function, LMIs, nonlinear discrete-time systems, Takagi-Sugeno fuzzy models, time-varying delay in states.

1. INTRODUCTION

The problem of controlling delayed systems has received a significant amount of attention from both academics and industrial engineers in the last decades. Its relevance relies on the fact that most of the industrial process involve transfer of mass, energy, or information. Such a transfer is usually associated with the presence of delay in real systems yielding performance deterioration or even loss of stability [1]. In special, the control problem of delayed nonlinear systems has been an important topic in recent researches as can be seen, for example, in [2] for continuous-time systems and in [3–5], and [6], for discrete-time systems. For applications of discrete-time delayed systems see for instance [7] where an industrial furnace is modeled and controlled considering delayed states. In this work, we deal with the problem of local stabilization of nonlinear discrete-time systems with delayed states described by

$$x_{k+1} = f(x_k)x_k + f_d(x_k)x_{k-d_k} + g(x_k)u_k, \qquad (1)$$

where $x_k \in \mathbb{R}^n$ is the state vector, $u_k \in \mathbb{R}^m$ is the control input vector, and $d_k \in \mathbb{N}^*$ denotes the time-varying delay satisfying $1 \le \underline{d} \le d_k \le \bar{d}$, with $\underline{d}$ and $\bar{d}$ being the lower and upper bound of the delay, respectively. The functions $f(\cdot):\mathbb{R}^n \to \mathbb{R}^{n\times n}$, $f_d(\cdot):\mathbb{R}^n \to \mathbb{R}^{n\times n}$, and $g(\cdot):\mathbb{R}^n \to \mathbb{R}^{n\times m}$ are continuous and bounded in the region of operation. We assume that the initial conditions of the nonlinear system (1) are given by a sequence $\varphi_{\bar{d},0}$ defined later.

One way to handle the nonlinear system (1) is through the approach based on fuzzy Takagi-Sugeno (T-S) models which has been found a great success to model and control nonlinear systems (see for example [8–10]). An advantage of this approach is that the nonlinear system can be exactly represented by a T-S fuzzy model in a local region, inside the domain of operation, here called *region of validity* of the fuzzy T-S model. Such a local region may be a consequence of physical or security limits of the process. Note that the use of T-S modeling has been widely exploited in many contributions in the literature such as [8,11–13]. For other recent approaches in the context of fuzzy control see, for instance, [14] for adaptive fuzzy control of nonlinear

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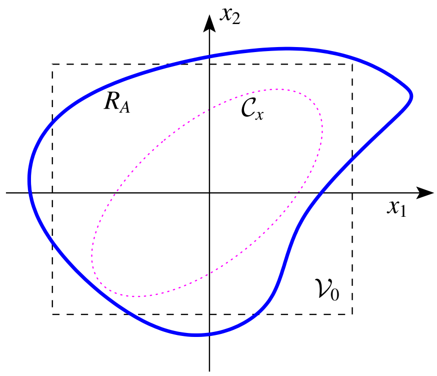

Fig. 1. Region of operation ($\mathcal{V}_0$, dashed line), region of attraction (R_A, continuous line), and an estimate of the region of attraction ($\mathcal{C}_x$, dotted continuous line).

network systems with input delay, and [15] for fault estimation and tolerant control of continuous time fuzzy T-S systems with time-varying delay. In this paper, we are concerned with the region of validity (region of operation) of the fuzzy T-S model as illustrated in Fig. 1: suppose a nonlinear system described by (1) with $n = 2$, the equilibrium point of interest being at the origin, at an initial instant $k = 0$, and with initial condition where only the current state, x_0, is not null. Also suppose that this system has been represented by a T-S fuzzy model such that its region of validity is indicated by the set $\mathcal{V}_0$ shown by dashed lines. There exists a set of possible values x_0 such that the trajectories emanating from these points converges to the origin. This is the region of attraction, R_A, indicated by continuous line in Fig. 1. This region may be nonconvex and its characterization is widely recognized as a challenging task. Therefore, we pursue the objective of finding an estimate of R_A which is indicated by $\mathcal{C}_x \subseteq R_A$ (dotted line) of safe initial conditions.

Due to the delayed states, such an estimate is a more complicated task than that in the delay free case. For instance, if the norm of the delayed states is increased, the weight of $f_d(x_k)x_{k-d_k}$ becomes more relevant on the dynamics of (1) and, thus, the size of the set $\mathcal{C}_x$ can be affected. Thus, it is necessary to characterize the region of safe initial conditions for all initial states in the sequence $\varphi_{\bar{d},0}$. We provide an estimate of the region of attraction, i.e. a safe region from where the closed-loop trajectories can start and are ensured to stay in the region of validity and be asymptotically stable. Considering such an estimate in the controller design is a fundamental issue that is often ignored in fuzzy T-S control approaches with some exceptions as in [16–19] for delay free systems and [20–23] for systems with delayed states. Moreover, when time delay is present the estimate of the region of attraction is more difficult than that for delay free systems because the initial conditions are defined by a sequence of states instead of only one initial state vector. It is noted that works dealing with the problems of stability analysis and controller synthesis for time-varying delay discrete-time systems represented by T-S fuzzy models do not consider the local stability issue (see, for instance [24–27]).

In this paper, we develop delay-dependent convex conditions based on fuzzy Lyapunov-Krasovskii (L-K) functions leading to our main contributions include: *i)* a convex condition to design a fuzzy memory state feedback control $u_k = K(\alpha_k)x_k + \bar{K}(\alpha_k)x_{k-\bar{d}}$, where $\bar{d}$ is the upper bound on the delay variation interval is developed without using on line knowledge of d_k; *ii)* the obtained T-S fuzzy controller ensures the local stabilization of the nonlinear system (1) with trajectories confined in the region of validity; and *iii)* an estimate of the region of attraction is also given. It should be noted that delay-dependent conditions are formulated in terms of linear matrix inequalities (LMIs). The present approach differs from those in [22] and [23] where the real time knowledge of the delay d_k and the associated delayed state (x_{k-d_k}) are required. Such a difference leads to an easier controller implementation. The estimation of the region of attraction is based on two sets: one describing a region of allowed values for the current state and the other one defining the region for the delayed states. Based on the developed conditions, convex optimization procedures are proposed to synthesize fuzzy controllers that maximize the region of attraction with respect to the region of validity of the T-S fuzzy model, where the dynamics of the nonlinear closed-loop system are allowed to evolve. We present two numerical examples to demonstrate the efficiency of the proposed approach and to compare it with other approaches found in the literature.

In Section 2 some definitions and the problem formulation are provided. In Section 3 we show some preliminary results. In Section 4 we present the main results: convex conditions for the synthesis of T-S state feedback control gains and the optimization procedures to compute the control law maximizing the region of attraction. In Section 5, we show two numerical examples to demonstrate effectiveness of the proposed approach. Some conclusions are presented in Section 6.

Notations: The ℓ-th row of the matrix L is denoted as $L_{(\ell)}$. The symbol $\star$ represents the transposed blocks with respect to the diagonal of real square and symmetric matrices. The matrices $\mathbf{I}$ and $\mathbf{0}$ denote the identity and the null matrices of appropriate dimensions, respectively. The sets of real numbers, non-negative real numbers, integer numbers, and integers numbers excluding the zero are denoted by $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{N}$, and $\mathbb{N}^*$, respectively. The set of integer numbers belonging to the interval from $a \in \mathbb{N}$ to $b \in \mathbb{N}$, $b \geq a$, is denoted by $\mathcal{I}[a,b]$. If ϑ is a sequence, its j-th element is denoted by $[\vartheta]_j$. We define two sequences: the first one is $\phi_{d,k} \in E_\phi$, with $E_\phi = E_1 \times E_2 \times \cdots \times E_d$, $E_j \subseteq \mathbb{R}^n$, and the j-th element of $\phi_{d,k}$ is $[\phi_{d,k}]_j = x_{k+j-(d+1)} \in E_j$, $j \in \mathcal{I}[1,d]$; thus, $\phi_{d,k} = \{x_{k-d}, x_{k-(d-1)}, \ldots, x_{k-1}\}$. The second sequence is $\varphi_{d,k} \in E_\varphi$, with $E_\varphi = E_\phi \times E_{d+1}$, $E_{d+1} \subseteq \mathbb{R}^n$, and the j-th element of $\varphi_{d,k}$ is $[\varphi_{d,k}]_j = x_{k+j-(d+1)} \in E_j$, $j \in \mathcal{I}[1,d+1]$; thus, $\varphi_{d,k} = \{\phi_{d,k}, x_k\}$. The sequence of difference of delayed initial conditions is defined as $\Delta\phi_{d,k} = \{\phi_{d,k-d+1} - \phi_{d,k-d}, \dots, \phi_{d,k} - \phi_{d,k-1}\}$. The norm of the sequence of vectors, ϑ with d elements, is defined as $\|\vartheta\|_d = \sup_{j \in \mathcal{I}[1,d]} \|[\vartheta]_j\|$, where $\|\cdot\|$ is the euclidean norm. $\lambda_{\max}(M)$ is the maximum eigenvalue of the (symmetric) matrix M with real entries. For a real number r, $\mathtt{round}(r)$ returns the nearest integer.

2. PROBLEM STATEMENT

Consider a class of discrete-time nonlinear systems with time-varying delay in states given by (1), which can be exactly represented by a Takagi-Sugeno (T-S) fuzzy model with $N = 2^p$ rules as follows (see [13] for details):

$$\begin{aligned} &\text{Rule } i: \\ &\quad \text{IF } z_1(k) \text{ is } M_{i1} \text{ and } \cdots \text{ and } z_p(k) \text{ is } M_{ip}, \\ &\quad \text{THEN } x_{k+1} = A_i x_k + A_{di} x_{k-d_k} + B_i u_k, \end{aligned} \tag{2}$$

where $z_j(k)$, $j \in \mathcal{I}[1,p]$, are the scalar premise variables which depend only on the state x_k, M_{ij}, $i \in \mathcal{I}[1,N]$, are the fuzzy sets, and p is the number of premise variables. The matrices $A_i \in \mathbb{R}^{n\times n}$, $A_{di} \in \mathbb{R}^{n\times n}$ and $B_i \in \mathbb{R}^{n\times m}$ are supposed to be known. The delay satisfies $1 \le \underline{d} \le d_k \le \bar{d}$, where $\underline{d}$ and $\bar{d}$ are known natural numbers related to the minimal and maximal values assumed by the delay, respectively. Note that if at least one of the nonlinear functions in (1) depends on x_{k-d_k}, the real-time knowledge of the d_k would be necessary to ensure the synchronism of the premise variables. The initial conditions of (1) are given by a sequence $\varphi_{\bar{d},0} \in E_\varphi$.

The defuzzification process of the model (2) can be represented as [12]

$$x_{k+1} = A(\alpha_k)x_k + A_d(\alpha_k)x_{k-d_k} + B(\alpha_k)u_k, \tag{3}$$

where $\alpha_{k(i)} = w_i(z(k))/\sum_{j=1}^N w_j(z(k))$ with $w_i = \prod_{j=1}^p M_{ij}(z_j(k))$, and $z(k) = \begin{bmatrix} z_1(k) & z_2(k) & \dots & z_p(k)\end{bmatrix}^T$. The membership function vector $\alpha_k \in \Xi$ is a state-dependent time-varying parameter that is supposed to be available in real time, and Ξ is the unitary simplex:

$$\Xi = \left\{\alpha_k \in \mathbb{R}^N;\ \sum_{i=1}^N \alpha_{k(i)} = 1,\ \alpha_{k(i)} \ge 0,\ i \in \mathcal{I}[1,N]\right\}. \tag{4}$$

The matrices in (3) can be rewritten as:

$$\begin{aligned} &\begin{bmatrix} A(\alpha_k) & A_d(\alpha_k) & B(\alpha_k)\end{bmatrix} \\ &= \sum_{i=1}^N \alpha_{k(i)} \begin{bmatrix} A_i & A_{di} & B_i\end{bmatrix},\ \alpha_k \in \Xi. \end{aligned} \tag{5}$$

It is worth highlighting that the exact representation of the nonlinear system (1) by the T-S fuzzy model (2)–(5) inside a bounded region in the state space can be established by using the same modeling T-S fuzzy technique as in [12]. In this technique, the domain of each premise variable z_i is a bounded region in the state space whose limits are considered in the fuzzy T-S modeling process (see [17] for details). Besides, it is often required in real systems to consider a domain of operation that may include safety operational conditions, physical constraints, levels of energy consumption, etc. Motivated by these requirements, we consider a nonempty and possibly open subregion $\mathcal{V}_0 \subset \mathbb{R}^n$, called the region of validity, characterized by the intersection of a finite number of half spaces:

$$\mathcal{V}_0 = \{x_k \in \mathbb{R}^n;\ |L_{(\ell)}x_k| \le \eta_{(\ell)}\}, \tag{6}$$

where $\eta_{(\ell)} > 0$, $L_{(\ell)} \in \mathbb{R}^{1\times n}$, $\ell \in \mathcal{I}[1,\kappa]$ and κ represents the number of constraints that characterize the allowed state-space region for the closed-loop system. For instance, consider a system with a state vector given by $x_k = \begin{bmatrix} x_{1,k} & x_{2,k}\end{bmatrix}^T$, where $x_{1,k}$ is constrained to $-0.05 \le x_{1,k} \le 0.05$, and $x_{2,k}$ is free. Then, the region of validity $\mathcal{V}_0$ can be characterized by (6) with $\kappa = \ell = 1$,

$$L = \begin{bmatrix} 1 & 0 \end{bmatrix},\ \text{and } \eta_{(1)} = 0.05. \tag{7}$$

Thus, any state vector verifying $|L_{(1)}x_k| \le 0.05$ belongs to $\mathcal{V}_0$. Thus, through the fuzzy modeling considered in this paper, we can exactly represent a real plant modeled by the nonlinear system (1) taking into consideration physical and operational constraints described in (6). Note that whenever the state trajectories evolve inside the region of validity, $x_k \in \mathcal{V}_0$, it is verified that $\alpha_k \in \Xi$.

Using the parallel distributed compensation, we propose a feedback fuzzy control law as:

$$u_k = K(\alpha_k)x_k + \bar{K}(\alpha_k)x_{k-\bar{d}}, \tag{8}$$

where the matrices of the controller depend on the membership function as follows:

$$\begin{bmatrix} K(\alpha_k) & \bar{K}(\alpha_k)\end{bmatrix} = \sum_{i=1}^N \alpha_{k(i)} \begin{bmatrix} K_i & \bar{K}_i\end{bmatrix},\ \alpha_k \in \Xi, \tag{9}$$

with $K_i \in \mathbb{R}^{m\times n}$ and $\bar{K}_i \in \mathbb{R}^{m\times n}$ being the control gains associated with the i-th rule in (2).

Remark 1: The control law (8)–(9) does not need the real time knowledge of d_k, because only the maximum delay, $\bar{d}$, is used. Thus, (8)–(9) is simpler to be implemented than the control law found in [21–23] where x_{k-d_k} is used, requiring the real time knowledge of d_k. Moreover, the use of the delayed term into (8) adds more information on the control law, which is known to lead to less conservative control action [25].

By replacing (8) into (3), we get the closed-loop system as:

$$\begin{aligned} x_{k+1} =& (A(\alpha_k)+B(\alpha_k)K(\alpha_k))x_k + A_d(\alpha_k)x_{k-d_k} \\ &+ B(\alpha_k)\bar{K}(\alpha_k)x_{k-\bar{d}}. \end{aligned} \tag{10}$$

Our interest is to characterize a region $\Upsilon_\varphi \subseteq \mathcal{V}_0$, such that the trajectories of the nonlinear system (1) with control law (8)–(9) and initial conditions given by the sequence $\varphi_{\bar{d},0}$ with elements $[\varphi_{\bar{d},0}]_j$, $j \in \mathcal{I}[1,\bar{d}+1]$, belonging to Υ_φ remain confined in $\mathcal{V}_0$. Also note that the mathematical description of such a region is a challenging task. Therefore, we can formulate the main problem in this work as follows.

Problem 1: Determine the T-S fuzzy gains K_i and $\bar{K}_i$, for $i \in \mathcal{I}[1,N]$, of the controller (8)–(9) and characterize an estimate of the region of attraction, Υ_φ, such that trajectories of the closed-loop system consisting of (1), (8)–(9) remain in $\mathcal{V}_0$ and converge asymptotically to the origin, for any initial condition given by the sequence $\varphi_{\bar{d},0}$ with elements $[\varphi_{\bar{d},0}]_j \in \Upsilon_\varphi$, $j \in \mathcal{I}[1,\bar{d}+1]$.

3. PRELIMINARIES AND TECHNICAL LEMMAS

Inspired by [26], we rewrite the T-S fuzzy model (3) with the aid of the variable $\omega_{d_k} = x_{k-d_k} - x_{k-\bar{d}}$, yielding:

$$\begin{aligned} x_{k+1} =& A(\alpha_k)x_k + A_d(\alpha_k)x_{k-\bar{d}} + A_d(\alpha_k)\omega_{d_k} \\ &+ B(\alpha_k)u_k. \end{aligned} \tag{11}$$

By using the fuzzy control law (8) into (11), the closed-loop system given by (10) can be equivalently written as:

$$x_{k+1} = \hat{A}(\alpha_k)x_k + \hat{A}_d(\alpha_k)x_{k-\bar{d}} + A_d(\alpha_k)\omega_{d_k}, \tag{12}$$

where the matrices $\hat{A}(\alpha_k)$ and $\hat{A}_d(\alpha_k)$ are

$$\begin{aligned} \hat{A}(\alpha_k) &= A(\alpha_k) + B(\alpha_k)K(\alpha_k) \\ &= \sum_{i=1}^{N}\sum_{j=i}^{N} \mu_{ij}\alpha_{k(i)}\alpha_{k(j)} \frac{A_i + B_iK_j + A_j + B_jK_i}{2}, \end{aligned} \tag{13}$$

and

$$\begin{aligned} \hat{A}_d(\alpha_k) &= A_d(\alpha_k) + B(\alpha_k)\bar{K}(\alpha_k) \\ &= \sum_{i=1}^{N}\sum_{j=i}^{N} \mu_{ij}\alpha_{k(i)}\alpha_{k(j)} \frac{A_{di} + B_i\bar{K}_j + A_{dj} + B_j\bar{K}_i}{2}, \end{aligned} \tag{14}$$

with

$$\mu_{ij} = 2 \text{ if } i \neq j, \text{ or } \mu_{ij} = 1 \text{ if } i = j. \tag{15}$$

Consider now the fuzzy L-K function candidate $V(\varphi_{\bar{d},k},\alpha_k) : E_\varphi \times \Xi \mapsto \mathbb{R}^+$:

$$V(\varphi_{\bar{d},k},\alpha_k) = \sum_{v=1}^{4} V_v(\varphi_{\bar{d},k},\alpha_k) > 0, \tag{16}$$

where $V_1(\varphi_{\bar{d},k},\alpha_k) = x_k^T Q^{-1}(\alpha_k)x_k$, $V_2(\varphi_{\bar{d},k},\alpha_k) = \sum_{j=k-d_k}^{k-1} x_j^T S(\alpha_j)x_j + \sum_{\ell=2-\bar{d}}^{1-\underline{d}} \sum_{j=k+\ell-1}^{k-1} x_j^T S(\alpha_j)x_j$, $V_3(\varphi_{\bar{d},k}) = \sum_{j=k-\bar{d}}^{k-1} x_j^T R x_j$, $V_4(\varphi_{\bar{d},k}) = x_{k-1}^T Z_1 x_{k-1} + (\bar{d}-1)\sum_{\ell=-\bar{d}}^{-2} \sum_{m=k+\ell}^{k-2} (x_{m+1}-x_m)^T Z_2 (x_{m+1}-x_m)$, $Q(\alpha_k) = \sum_{i=1}^{N} \alpha_{k(i)} Q_i$, and $S(\alpha_k) = \sum_{i=1}^{N} \alpha_{k(i)} S_i$, where $Q_i = Q_i^T \in \mathbb{R}^{n\times n}$, $S_i = S_i^T \in \mathbb{R}^{n\times n}$, $R = R^T \in \mathbb{R}^{n\times n}$, $Z_1 = Z_1^T \in \mathbb{R}^{n\times n}$, and $Z_2 = Z_2^T \in \mathbb{R}^{n\times n}$. This fuzzy L-K function candidate is used to search for solutions to the Problem 1. Then, we define the estimate of the region of attraction and present three lemmas used to obtain the main results of this work.

Definition 1: The estimate of the region of attraction $\Upsilon_\varphi \subseteq E_\varphi$ is defined as

$$\Upsilon_\varphi = \Bigg\{ \varphi_{\bar{d},0} \in E_\varphi;\ \forall j \in \mathcal{I}[1,\bar{d}+1], \\ [\varphi_{\bar{d},0}]_j \in \left\{ \begin{array}{ll} \mathcal{C}_x, & j = \bar{d}+1, \\ \mathcal{B}_\phi, & \text{otherwise.} \end{array} \right\} \Bigg\}, \tag{17}$$

where

$$\mathcal{C}_x = \left\{ x_0 \in E_{\bar{d}+1};\ V_1(x_0,\alpha_0) \leq c(\phi_{\bar{d},0}) \right\} \subseteq \mathcal{V}_0, \tag{18}$$

$$\begin{aligned} \mathcal{B}_\phi = \big\{ &[\phi_{\bar{d},0}]_j \in E_j,\ j \in \mathcal{I}[1,\bar{d}];\ \|\phi_{\bar{d},0}\|_{\bar{d}} \leq r_1 \\ &\|\Delta\phi_{\bar{d},0}\|_{\bar{d}} \leq r_2, \text{ and } [\phi_{\bar{d},0}]_j \in \mathcal{V}_0,\ j \in \mathcal{I}[1,\bar{d}] \big\}, \end{aligned} \tag{19}$$

$c(\phi_{\bar{d},0})$ is a function $E_\phi \mapsto \mathbb{R}^+$ and $r_i \in \mathbb{R}^+$, for $i = 1,2$.

In the characterization of Υ_φ, set $\mathcal{C}_x$ describes a region of allowable values for the current state while the set $\mathcal{B}_\phi$ defines the region for the elements $[\phi_{\bar{d},0}]_j \in E_j$, for $j \in \mathcal{I}[1,\bar{d}]$. The case discussed in the Section "I. Introduction," with the aid of Fig. 1 has an Υ_φ given by $\mathcal{C}_x$ and $\mathcal{B}_\phi = \{0\}$. The next lemma presents a tool necessary to compute the set $\mathcal{C}_x$.

Lemma 1: Assume that function (16) is a fuzzy L-K function. An associated level set is given by the intersection of ellipsoidal sets related to matrices $Q_i^T = Q_i > \mathbf{0}$, $i \in \mathcal{I}[1,N]$:

$$\begin{aligned} \mathcal{L}_{V_1}(c) &= \left\{ \mathcal{E}(Q_i^{-1},c),\ \forall \alpha_k \in \Xi \right\} \\ &= \bigcap_{\alpha_k \in \Xi} \mathcal{E}(Q^{-1}(\alpha_k),c) \\ &= \bigcap_{i \in \mathcal{I}[1,N]} \mathcal{E}(Q_i^{-1},c) \subseteq \mathcal{V}_0, \end{aligned} \tag{20}$$

where c is a positive scalar, and $\mathcal{E}(Q_i^{-1},c)$, for $i \in \mathcal{I}[1,N]$, denotes the ellipsoidal sets defined as follows:

$$\mathcal{E}(Q_i^{-1},c) = \left\{ x_k \in \mathbb{R}^n;\ x_k^T Q_i^{-1} x_k \leq c \right\}. \tag{21}$$

The proof of Lemma 1 can be found in [28, Lemma 4].

Equations (20)–(21) are used to characterize the sets where corresponding trajectories of the system remain confined when they start in Υ_φ. In case of $c=1$, we use the simplified notation $\mathcal{L}_{V_1}=\mathcal{L}_{V_1}(1)$ and $\mathcal{E}(Q_i^{-1})=\mathcal{E}(Q_i^{-1},1)$. The following lemma allows the computation and the definition of parameters to characterize the sets $\mathcal{C}_x$ and $\mathcal{B}_\phi$, and, thus, Υ_φ.

Lemma 2: Assume that function (16) is a fuzzy L-K function. The set $\mathcal{C}_x$ is characterized as

$$\begin{aligned}\mathcal{C}_x=\big\{&x_0\in\mathbb{R}^n;\ V_1(x_0,\alpha_0)\le 1-\rho_1\|\phi_{\bar d,0}\|_{\bar d}^2\\&-\rho_2\|\Delta\phi_{\bar d,0}\|_{\bar d}^2\big\}\subseteq\mathcal{V}_0,\end{aligned}\tag{22}$$

with

$$\begin{aligned}\rho_1=&\max_{i\in\mathcal{I}[1,N]}(\lambda_{\max}(S_i))\left(\delta+\frac{(\delta-1)^2-(\delta-1)}{2}\right)\\&+\lambda_{\max}(R)\bar d+\lambda_{\max}(Z_1),\end{aligned}\tag{23}$$

$$\rho_2=\lambda_{\max}(Z_2)\frac{\bar d(\bar d-1)^2}{2},\tag{24}$$

and $\delta=\bar d-\underline{d}+1$. Then, we can choose r_1 and r_2 in (19) satisfying

$$0\le r_1\le\rho_1^{-\frac12},\ 0\le r_2\le\rho_2^{-\frac12},\ \text{and}\ \rho_1r_1^2+\rho_2r_2^2\le 1.\tag{25}$$

Proof: Consider Lemma 1 with $c=1-\rho_1\|\phi_{\bar d,0}\|_{\bar d}^2-\rho_2\|\Delta\phi_{\bar d,0}\|_{\bar d}^2$. Then, we have

$$\begin{aligned}\mathcal{C}_x&=\mathcal{L}_{V_1}(1-\rho_1\|\phi_{\bar d,0}\|_{\bar d}^2-\rho_2\|\Delta\phi_{\bar d,0}\|_{\bar d}^2)\\&=\{x_0\in\mathbb{R}^n;\\&\quad x_0^TQ^{-1}(\alpha_0)x_0\le 1-\rho_1\|\phi_{\bar d,0}\|_{\bar d}^2-\rho_2\|\Delta\phi_{\bar d,0}\|_{\bar d}^2\}\\&\subseteq\mathcal{V}_0.\end{aligned}\tag{26}$$

From V_i, $i=2,4$, using $\hat s\equiv\max_{i\in\mathcal{I}[1,N]}\lambda_{\max}S_i$, $\hat r\equiv\lambda_{\max}R$, $\hat z_1\equiv\lambda_{\max}Z_1$, $\hat z_2\equiv\lambda_{\max}Z_2$, and $\sum_{i=1}^N\alpha_{i,j}\mathbf{I}=\mathbf{I}$, we have that

$$\begin{aligned}&V_2(\varphi_{\bar d,k},\alpha_k)+V_3(\varphi_{\bar d,k})+V_4(\Delta\phi_{\bar d,k})\\&\le\sum_{j=-d_k}^{-1}[\phi_{\bar d,0}]_{\bar d+1+j}^TS(\alpha_j)[\phi_{\bar d,0}]_{\bar d+1+j}\\&\quad+\sum_{\ell=2-\bar d}^{1-\underline{d}}\sum_{j=\ell-1}^{-1}[\phi_{\bar d,0}]_{\bar d+1+j}^TS(\alpha_j)[\phi_{\bar d,0}]_{\bar d+1+j}\\&\quad+\sum_{j=-\bar d}^{-1}[\phi_{\bar d,0}]_{\bar d+1+j}^TR[\phi_{\bar d,0}]_{\bar d+1+j}+[\phi_{\bar d,0}]_{\bar d}^TZ_1[\phi_{\bar d,0}]_{\bar d}\\&\quad+(\bar d-1)\sum_{\ell=-\bar d}^{-2}\sum_{m=\ell}^{-2}[\Delta\phi_{\bar d,0}]_{\bar d+1+j}^TZ_2[\Delta\phi_{\bar d,0}]_{\bar d+1+j}\\&\le\sum_{j=-d_k}^{-1}[\phi_{\bar d,0}]_{\bar d+1+j}^T(\hat s\sum_{i=1}^N\alpha_{i,j}\mathbf{I})[\phi_{\bar d,0}]_{\bar d+1+j}\\&\quad+\sum_{\ell=2-\bar d}^{1-\underline{d}}\sum_{j=\ell-1}^{-1}[\phi_{\bar d,0}]_{\bar d+1+j}^T(\hat s\sum_{i=1}^N\alpha_{i,j}\mathbf{I})[\phi_{\bar d,0}]_{\bar d+1+j}\\&\quad+\sum_{j=-\bar d}^{-1}[\phi_{\bar d,0}]_{\bar d+1+j}^T(\hat r\mathbf{I})[\phi_{\bar d,0}]_{\bar d+1+j}+[\phi_{\bar d,0}]_{\bar d}^T(\hat z_1\mathbf{I})[\phi_{\bar d,0}]_{\bar d}\\&\quad+(\bar d-1)\sum_{\ell=-\bar d}^{-2}\sum_{m=\ell}^{-2}[\Delta\phi_{\bar d,0}]_{\bar d+1+j}^T(\hat z_2\mathbf{I})[\Delta\phi_{\bar d,0}]_{\bar d+1+j}\\&\le\delta\hat s\|\phi_{\bar d,0}\|_{\bar d}^2+(\delta^2-3\delta+2)\hat s/2\|\phi_{\bar d,0}\|_{\bar d}^2+\bar d\hat r\|\phi_{\bar d,0}\|_{\bar d}^2\\&\quad+\hat z_1\|\phi_{\bar d,0}\|_{\bar d}^2+(\bar d(\bar d-1)^2)\hat z_2/2\|\Delta\phi_{\bar d,0}\|_{\bar d}^2\\&=\rho_1\|\phi_{\bar d,0}\|_{\bar d}^2+\rho_2\|\Delta\phi_{\bar d,0}\|_{\bar d}^2,\end{aligned}\tag{27}$$

where ρ_1 and ρ_2 are given in (23) and (24), respectively, and $\delta=\bar d-\underline{d}+1$. Furthermore, it is necessary that $0\le c\le 1$ and this is possible only if in (19) $0\le r_1\le\rho_1^{-\frac12}$, $0\le r_2\le\rho_2^{-\frac12}$, and $\rho_1r_1^2+\rho_2r_2^2\le 1$. □

The following lemma establishes the relationship between the sets $\mathcal{C}_x$ and $\mathcal{B}_\phi$ in terms of the confinement of trajectories in $\mathcal{L}_{V_1}$ and local asymptotic stability.

Lemma 3: If $V(\varphi_{\bar d,k},\alpha_k)$ given by (16) is a L-K function, then for all $x_0\in\mathcal{C}_x=\mathcal{L}_{V_1}(1-\rho_1\|\phi_{\bar d,0}\|_{\bar d}^2-\rho_2\|\Delta\phi_{\bar d,0}\|_{\bar d}^2)\subseteq\mathcal{V}_0$ and $[\phi_{\bar d,0}]_j\in\mathcal{B}_\phi$, $j\in\mathcal{I}[1,\bar d]$, it is ensured that $x_k\in\mathcal{L}_{V_1}$, for all $k\ge 0$ and $\lim_{k\to\infty}[\varphi_{\bar d,k}]_j=\mathbf{0}$, $j\in\mathcal{I}[1,\bar d+1]$.

Proof: If (16) is a L-K function then it verifies $\Delta V(\varphi_{\bar d,k},\alpha_k)<0$, $\alpha_k\in\Xi$, $k>0$. By using Lemma 2, we have

$$\begin{aligned}V(\varphi_{\bar d,k},\alpha_k)<&V(\varphi_{\bar d,0},\alpha_0)\\\le&x_0^TQ^{-1}(\alpha_0)x_0\\&+\rho_1\|\phi_{\bar d,0}\|_{\bar d}^2+\rho_2\|\Delta\phi_{\bar d,0}\|_{\bar d}^2.\end{aligned}\tag{28}$$

Therefore, we can check that if $x_0^TQ^{-1}(\alpha_0)x_0\le 1-\rho_1\|\phi_{\bar d,0}\|_{\bar d}^2-\rho_2\|\Delta\phi_{\bar d,0}\|_{\bar d}^2$, then $x_k^TQ^{-1}(\alpha_k)x_k\le 1$. Because (16) is a L-K function, then $\lim_{k\to\infty}[\varphi_{\bar d,k}]_j=\mathbf{0}$, $j\in\mathcal{I}[1,\bar d+1]$. and thus the local asymptotic stability of the closed-loop system is ensured. □

4. MAIN RESULTS

In this section, we present our main contributions which are summarized in the following theorem.

Theorem 1: Suppose there exist symmetric positive definite matrices $Q_i\in\mathbb{R}^{n\times n}$, $\tilde S_i\in\mathbb{R}^{n\times n}$, $i\in\mathcal{I}[1,N]$, $\tilde R\in\mathbb{R}^{n\times n}$, $\tilde Z_1\in\mathbb{R}^{n\times n}$, $\tilde Z_2\in\mathbb{R}^{n\times n}$, and matrices $H\in\mathbb{R}^{n\times n}$, $U\in\mathbb{R}^{n\times n}$, $\tilde X_j\in\mathbb{R}^{n\times n}$, for $j\in\mathcal{I}[1,4]$, $Y_i\in\mathbb{R}^{m\times n}$, and $\bar Y_i\in\mathbb{R}^{m\times n}$ satisfying $\forall i,p,q\in\mathcal{I}[1,N]$, $j\in\mathcal{I}[i,N]$, and $\forall\ell\in\mathcal{I}[1,\kappa]$ the LMIs:

$$\Pi_{ijpq}<\mathbf{0},\tag{29}$$

and

$$\begin{bmatrix} -Q_i & Q_i L_{(\ell)}^T \\ \star & -\eta_{(\ell)}^2 \end{bmatrix} \leq \mathbf{0}, \tag{30}$$

with

$$\Pi_{ijpq} = \begin{bmatrix} -H-H^T & 0.5(A_iU+B_iY_j+A_jU+B_jY_i) & \mathbf{0} & \mathbf{0} & \begin{matrix} 0.5\,(A_{di}U+B_i\bar{Y}_j \\ +A_{dj}U+B_j\bar{Y}_i) \end{matrix} & 0.5(A_{di}+A_{dj})U & H^T \\ \star & \begin{matrix} 0.5(Q_i+Q_j)-U^T-U \\ +0.5\delta(\tilde{S}_i+\tilde{S}_j)+\tilde{R} \\ +\tilde{Z}_1+(\bar{d}-1)^2\tilde{Z}_2 \end{matrix} & -(\bar{d}-1)^2\tilde{Z}_2 & \tilde{X}_1 & -\tilde{X}_1 & -\tilde{X}_1 & \mathbf{0} \\ \star & \star & \bar{d}(\bar{d}-2)\tilde{Z}_2-\tilde{Z}_1 & \tilde{X}_2 & \tilde{Z}_2-\tilde{X}_2 & -\tilde{X}_2 & \mathbf{0} \\ \star & \star & \star & \tilde{R}-\tilde{S}_p+\tilde{X}_3+\tilde{X}_3^T & -\tilde{R}-\tilde{X}_3+\tilde{X}_4^T & -\tilde{X}_3 & \mathbf{0} \\ \star & \star & \star & \star & -\tilde{Z}_2-\tilde{X}_4-\tilde{X}_4^T & -\tilde{X}_4 & \mathbf{0} \\ \star & \star & \star & \star & \star & -\tilde{R} & \mathbf{0} \\ \star & \star & \star & \star & \star & \star & -Q_q \end{bmatrix}.$$

Then, the controller matrices (9) obtained with

$$K_i = Y_iU^{-1} \quad \text{and} \quad \bar{K}_i = \bar{Y}_iU^{-1} \tag{31}$$

ensure that the origin of the closed-loop control system consisting of (1), (8), and (9) subject to constraints (6) is asymptotically stable for every initial conditions $\varphi_{\bar{d},0}$, $[\varphi_{\bar{d},0}]_j \in \Upsilon_\varphi$, $j \in \mathcal{I}[1,\bar{d}+1]$, with the sets $\mathcal{C}_x$ and $\mathcal{B}_\phi$ (see (18) and (19)) with parameters ρ and r computed according to Lemma 2, and also that the respective trajectories remain in $\mathcal{L}_{V_1} \subseteq \mathcal{V}_0$. Besides, (16) is a fuzzy L-K function with the matrices Q_i, $S_i = U^{-T}\tilde{S}_iU^{-1}$, $i \in \mathcal{I}[1,N]$, and $R = U^{-T}\tilde{R}U^{-1}$, that are obtained in (29)–(30) and appear in (16).

Proof: Firstly, we show the asymptotic stability of the closed-loop system consisting of (1), (8), (9) with (31) whenever $\alpha_k \in \Xi$. Secondly, we demonstrate the inclusion of the contractive set $\mathcal{L}_{V_1}$ in $\mathcal{V}_0$. This is used to ensure the closed-loop system trajectories remain in $\mathcal{V}_0$ whenever they evolve from Υ_φ.

Considering the positivity of $\tilde{Z}_1$ and $\tilde{Z}_2$, if (29) is satisfied, then matrices $Q_i \in \mathbb{R}^{n\times n}$, $\tilde{S}_i \in \mathbb{R}^{n\times n}$, $i \in \mathcal{I}[1,N]$, and $\tilde{R} \in \mathbb{R}^{n\times n}$ are symmetric definite positive and, therefore, $V(\varphi_{\bar{d},k},\alpha_k)$ is positive verifying (16). Replace Y_i and $\bar{Y}_i$ by K_iU and $\bar{K}_iU$ in (29), respectively, multiply the resulting inequality successively by $\alpha_{k(i)}$, $\alpha_{k(j)}$, $\alpha_{k-d_k(p)}$, and $\alpha_{k+1(q)}$, and sum up on $i \in \mathcal{I}[1,N]$, $j \in \mathcal{I}[i,N]$, $q \in \mathcal{I}[1,N]$, $p \in \mathcal{I}[1,N]$. The resulting inequality can be over bounded by replacing into block $(2,2)$ the term $Q(\alpha_k)-U^T-U$ by $-U^TQ(\alpha_k)^{-1}U$ [29] which allows to apply the Schur's complement to get $\tilde{\Theta}_k < \mathbf{0}$, with

$$\tilde{\Theta}_k = \begin{bmatrix} \begin{matrix} H^TQ^{-1}(\alpha_k^+)H \\ -H-H^T \end{matrix} & \hat{A}(\alpha_k)U & \mathbf{0} & \mathbf{0} & \hat{A}_d(\alpha_k)U & A_d(\alpha_k)U \\ \star & \begin{matrix} -U^TQ^{-1}(\alpha_k)U+\delta\tilde{S}(\alpha_k) \\ +\tilde{R}+\tilde{Z}_1+(\bar{d}-1)^2\tilde{Z}_2 \end{matrix} & -(\bar{d}-1)^2\tilde{Z}_2 & \tilde{X}_1 & -\tilde{X}_1 & -\tilde{X}_1 \\ \star & \star & \bar{d}(\bar{d}-2)\tilde{Z}_2-\tilde{Z}_1 & \tilde{X}_2 & \tilde{Z}_2-\tilde{X}_2 & -\tilde{X}_2 \\ \star & \star & \star & \tilde{R}-\tilde{S}(\alpha_k^-)+\tilde{X}_3+\tilde{X}_3^T & -\tilde{R}-\tilde{X}_3+\tilde{X}_4^T & -\tilde{X}_3 \\ \star & \star & \star & \star & -\tilde{Z}_2-\tilde{X}_4-\tilde{X}_4^T & -\tilde{X}_4 \\ \star & \star & \star & \star & \star & -\tilde{R} \end{bmatrix},$$

where the shorthands $\alpha_k^- \equiv \alpha_{k-d_k}$ and $\alpha_k^+ \equiv \alpha_{k+1}$ were used, and the matrices $\hat{A}(\alpha_k)$ and $\hat{A}_d(\alpha_k)$ are given in (13) and (14), respectively.

Then, by taking into account the regularity of H and U, and $X_i = U^{-T}\tilde{X}_iU^{-1}$, for $i \in \mathcal{I}[1,4]$, and $F = H^{-T}$, let us consider the congruence transformation $\Theta_k = \mathcal{T}^T\tilde{\Theta}_k\mathcal{T}$ with $\mathcal{T} = \mathrm{diag}\{H^{-1},U^{-1},U^{-1},U^{-1},U^{-1},U^{-1}\}$. To verify $\Delta V(\varphi_{\bar{d},k},\alpha_k) < 0$, note that we can assume $\xi_k = \begin{bmatrix} x_{k+1}^T & x_k^T & x_{k-1}^T & x_{k-d_k}^T & x_{k-\bar{d}}^T & \omega_{dk}^T \end{bmatrix}^T$ and take $\Omega_k = \xi_k^T\Theta_k\xi_k$ which leads to

$$\begin{aligned} \Omega_k = & x_{k+1}^TQ^{-1}(\alpha_{k+1})x_{k+1} + x_k^T\left[-Q^{-1}(\alpha_k)\right. \\ & \left.+\delta S(\alpha_k)+R+Z_1+(\bar{d}-1)^2Z_2\right]x_k \\ & +2x_k^T[-(\bar{d}-1)^2Z_2]x_{k-1} \\ & +x_{k-1}^T[\bar{d}(\bar{d}-2)Z_2-Z_1]x_{k-1}+2x_{k-1}^TZ_2x_{k-\bar{d}} \\ & +x_{k-d_k}^T[-S(\alpha_{k-d_k})+R]x_{k-d_k}+2x_{k-d_k}^T[-R]x_{k-\bar{d}} \\ & +x_{k-\bar{d}}[-Z_2]x_{k-\bar{d}}+\omega_{dk}^T[-R]\omega_{dk}. \end{aligned} \tag{32}$$

By considering $\omega_{d_k} = x_{k-d_k} - x_{k-\bar{d}}$, the feasibility of LMIs (29) ensures that $\Delta V(\varphi_{\bar{d},k},\alpha_k) \leq \Omega_k < 0$ is verified whenever $\alpha_k \in \Xi$, with $\Delta V(\varphi_{\bar{d},k},\alpha_k)$ computed as done in

[30]. Then, we can also conclude about the negativity of $V(\varphi_{\bar{d},k},\alpha_k)$, $\forall\ \alpha_k \in \Xi$.

Then, we can conclude that the feasibility of (29) provides the negativity of $\Delta V(\varphi_{\bar{d},k},\alpha_k)$. However, via the Lyapunov-Krasovskii's theorem [1], only the local the asymptotic stability of the closed-loop system (10) can be ensured with the control gains given by (9) and (31) by guaranteeing that the state trajectories evolve only inside the set $\mathcal{V}_0$, where $\alpha_k \in \Xi$. Thus, consider that besides inequalities (29), the LMIs in (30) are also verified. Then, we multiply (30) by $\alpha_{k(i)}$ and sum up on $i \in \mathcal{I}[1,N]$, getting:

$$\Lambda = \begin{bmatrix} -Q(\alpha_k) & Q(\alpha_k)L_{(\ell)}^T \\ \star & -\eta_{(\ell)}^2 \end{bmatrix} \leq \mathbf{0}. \tag{33}$$

By using the congruence transformation $\mathcal{F}^T\Lambda\mathcal{F} = \tilde{\Lambda}$ with $\mathcal{F} = \mathrm{diag}\{Q^{-1}(\alpha_k), 1\}$, we get

$$\tilde{\Lambda} = \begin{bmatrix} -Q^{-1}(\alpha_k) & L_{(\ell)}^T \\ \star & -\eta_{(\ell)}^2 \end{bmatrix} \leq \mathbf{0}, \tag{34}$$

which is equivalent, by Schur's complement, to $L_{(\ell)}^T\eta_{(\ell)}^{-2}L_{(\ell)} - Q^{-1}(\alpha_k) \leq \mathbf{0}$. In this inequality, we can pre- and post-multiply by x_k^T and x_k, getting

$$x_k^T L_{(\ell)}^T \eta_{(\ell)}^{-2} L_{(\ell)} x_k \leq 1, \forall x_k \in \mathcal{L}_{V_1} = x_k^T Q^{-1}(\alpha_k) x_k \leq 1. \tag{35}$$

Thus, we prove that $\mathcal{L}_{V_1} \subseteq \mathcal{V}_0$. From lemmas 1 and 3, one can conclude that any trajectory starting in Υ_φ remains in $\mathcal{L}_{V_1}$ and, therefore, the local stability of the closed-loop control system consisting of (1), (8), and (9) with the gains given by (31) is ensured. □

By means of Theorem 1, we provided a solution to Problem 1, allowing the implementation of the control law (8) which is simpler than that used, for instance, in [3–6, 21–23]. Furthermore, by means of the feasibility of the LMIs in Theorem 1 we can compute the set Υ_φ that characterizes initial conditions from which the closed loop trajectories are asymptotically stable and do not leave the region of validity, $\mathcal{V}_0$.

The estimated region of attraction Υ_φ of the closed-loop nonlinear system consisting of (1), (8), and (9) – see Definition 2 and Theorem 1 – can be maximized by a convex optimization algorithm. For this purpose, we consider the maximization of an ellipsoidal set included in the level set $\mathcal{L}_{V_1}$ as follows,

$$\mathcal{E}(W) = \left\{x \in \mathbb{R}^n; x^T W x \leq 1\right\} \subseteq \mathcal{L}_{V_1}. \tag{36}$$

This condition is equivalent to:

$$\begin{bmatrix} W & \mathbf{I} \\ \mathbf{I} & Q_i \end{bmatrix} \geq \mathbf{0},\ i \in \mathcal{I}[1,N]. \tag{37}$$

Thus, the following convex optimization procedure is proposed aiming to maximize the set $\mathcal{C}_x$ and, in consequence, Υ_φ (see [23]):

$$\mathcal{P}_W \begin{cases} \quad \min & \mathrm{trace}(W) \\ \text{subject to} & (29), (30),\ \text{and}\ (37). \end{cases} \tag{38}$$

Remark 2: A quadratic stabilization condition may be obtained from Theorem 1 by imposing $Q_i = Q$ and $\tilde{S}_i = \tilde{S}$, for $i \in \mathcal{I}[1,N]$. In this case, functions (16) are not dependent on the membership function α_k, and thus they are not a fuzzy function anymore. This leads to the well known quadratic stability approach [31] which can yield conservative results.

Remark 3: Whenever functions $f(\cdot)$ and $f_d(\cdot)$ in (1) depend on the delayed state, x_{k-d_k}, the membership functions also depend on d_k which is supposed to be unknown. In this case, by imposing $Y_i = Y$ and $\bar{Y}_i = \bar{Y}$, $i \in \mathcal{I}[1,N]$, in Theorem 1, constant gain matrices K and $\bar{K}$ can be obtained. This is equivalent to removing the weights α_k from the control law (8) and usually leads to more conservative results.

Remark 4: The conditions in Theorem 1 can recover the case where the delay is time-invariant and uncertain. To this end, one can impose $\bar{d} = \underline{d}$ which results in $\delta = 1$.

The proposed procedure can be summarized as follows:
Algorithm 1: (Fuzzy control design)

1) Determine the region of operation of the nonlinear system (1) and obtain a T-S fuzzy model as in (2)–(5).
2) Write constraints by using (6) to describe the region of validity, $\mathcal{V}_0$.
3) Solve the convex optimization procedure (38) to compute the fuzzy control gains for the control law (8)–(9) and the estimate of the region of attraction Υ_φ (17)–(19).
4) If step 3) fails, then return to the step 1) and, if possible, determine a smaller region of operation and the respective T-S fuzzy model.

5. NUMERICAL EXAMPLES

We present two examples to illustrate our approach and also to compare it with other approaches in the literature. The first example is motivated by the model of a magnetic suspension and the second one has been randomly generated.

5.1. Example 1

Consider the model of a magnetic suspension system investigated in [33]:

$$\dot{x}_1(t) = x_2(t),$$
$$\dot{x}_2(t) = \frac{g\mu(\mu x_1(t) + 2\mu y_0 + 2)x_1(t)}{(1+\mu(x_1(t)+y_0))^2}x_1(t)$$

$$-\frac{K_m}{m}x_2(t)+\frac{\lambda\mu}{2m(1+\mu(x_1(t)+y_0))^2}u(t), \tag{39}$$

where $x_1(t)$ and $x_2(t)$ are the ball deviation around its desired position and vertical velocity, respectively, and $y_0 = 0.05m$ is the desired position. From [33], the physical parameters are: $m = 0.068kg$ the mass of the suspended ball, $g = 9.8ms^{-2}$ the gravity acceleration, $K_m = 0.001Nsm^{-1}$ the viscous friction coefficient, $\lambda = 0.46H$ the inductance, and $\mu = 2m^{-1}$ the coefficient of inductance variation. The following discretized version of model (39) can be obtained through Euler discretization with sampling period $T = 0.01s$,

$$\begin{aligned}
x_{1,k+1} &= x_{1,k}+cTx_{2,k}+(1-c)Tx_{2,k-d_k},\\
x_{2,k+1} &= \frac{Tg\mu(\mu x_{1,k}+2\mu y_0+2)x_{1,k}}{(1+\mu(x_{1,k}+y_0))^2}x_{1,k}\\
&\quad +c\left(1-\frac{TK_m}{m}\right)x_{2,k}\\
&\quad +(1-c)\left(1-\frac{TK_m}{m}\right)x_{2,k-d_k}\\
&\quad +\frac{T\lambda\mu}{2m(1+\mu(x_{1,k}+y_0))^2}u_k,
\end{aligned} \tag{40}$$

where a delay in state x_2 is included to match, for example, practical sensor dynamics in the measured velocity, the time varying delay is assumed to be bounded by $\underline{d} = 1$ and $\bar{d} = 4$, i.e., $d_k \in \mathcal{I}[1,4]$. The parameter c was arbitrarily chosen as $c = 0.7$. The physical structure of the assembling imposes that $0 \le \bar{x}_{1,k} \le 0.1$, where $\bar{x}_{1,k} = x_{1,k}+y_0$ and, thus, we have $-0.05 \le x_{1,k} \le 0.05$ in (40). The region of validity $\mathcal{V}_0$ for this system can be modeled by (6), with $L = \begin{bmatrix}1 & 0\end{bmatrix}$ and $\eta = 0.05$ as in (7). Considering such a region, an exact T-S model for (40) can be obtained with $i \in \mathcal{I}[1,4]$, $p = 1,2$, $B_1 = B_2 = \begin{bmatrix}0 & 0.6765\end{bmatrix}^T$, $B_3 = B_4 = \begin{bmatrix}0 & 0.0676\end{bmatrix}^T$, $A_1 = A_3 = \begin{bmatrix}1 & 0.007\\ 0.0157 & 0.6999\end{bmatrix}$, $A_2 = A_4 = \begin{bmatrix}1 & 0.007\\ -0.02068 & 0.6999\end{bmatrix}$, $A_{d1} = A_{d2} = A_{d3} = A_{d4} = \begin{bmatrix}0 & 0.003\\ 0 & 0.3\end{bmatrix}$. The fuzzy sets M_{ip} are computed as $M_{11}(z_1(x_{1,k})) = M_{31}(z_1(x_{1,k})) = \frac{z_1(x_{1,k})-a_2}{a_1-a_2}$, $M_{21}(z_1(x_{1,k})) = M_{41}(z_1(x_{1,k})) = \frac{a_1-z_1(x_{1,k})}{a_1-a_2}$, $M_{12}(z_2(x_{1,k})) = M_{22}(z_2(x_{1,k})) = \frac{z_2(x_{1,k})-b_2}{b_1-b_2}$, $M_{32}(z_2(x_{1,k})) = M_{42}(z_2(x_{1,k})) = \frac{b_1-z_2(x_{1,k})}{b_1-b_2}$, where $z_1(x_{1,k}) = \frac{Tg\mu(\mu x_{1,k}+2\mu y_0+2)x_{1,k}}{(1+\mu(x_{1,k}+y_0))^2}$, $a_1 = \max(z_1(x_{1,k}))$, $a_2 = \min(z_1(x_{1,k}))$, $z_2(x_{1,k}) = \frac{T\lambda\mu}{2m(1+\mu(x_{1,k}+y_0))^2}$, $b_1 = \max(z_2(x_{1,k}))$, and $b_2 = \min(z_2(x_{1,k}))$. From these fuzzy sets, it is possible to compute the membership function $\alpha_k : \alpha_{ki} = M_{i1}(z_1(x_{1,k}))M_{i2}(z_2(x_{1,k}))$, $i \in \mathcal{I}[1,4]$.

By using the optimization procedure (38), we obtain

$$W = \begin{bmatrix}402.7012 & 2.7203\\ 2.7203 & 2.7416\end{bmatrix},$$

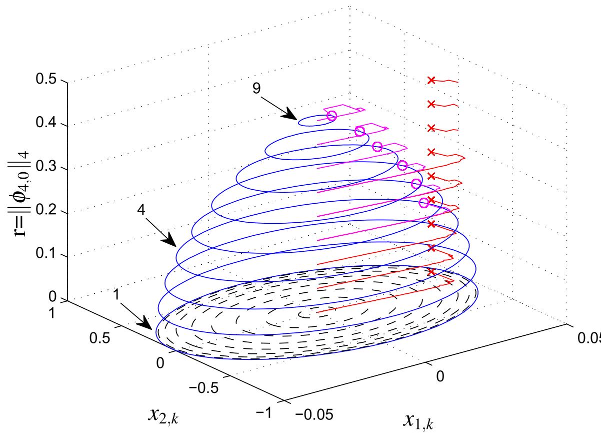

Fig. 2. Estimate sets $\mathcal{C}_x$ as a function of $\|\phi_{4,0}\|_4$.

and the following gains for the control law (8)–(9):

$$\begin{aligned}
K_1 &= -\begin{bmatrix}23.5892 & 10.1660\end{bmatrix}, & \bar{K}_1 &= \begin{bmatrix}0.0095 & -0.0949\end{bmatrix},\\
K_2 &= -\begin{bmatrix}22.9383 & 10.1551\end{bmatrix}, & \bar{K}_2 &= \begin{bmatrix}0.0100 & -0.0917\end{bmatrix},\\
K_3 &= -\begin{bmatrix}33.7967 & 14.6231\end{bmatrix}, & \bar{K}_3 &= \begin{bmatrix}0.0136 & -0.1308\end{bmatrix},\\
K_4 &= -\begin{bmatrix}33.1612 & 14.6206\end{bmatrix}, & \bar{K}_4 &= \begin{bmatrix}0.0137 & -0.1336\end{bmatrix}.
\end{aligned} \tag{41}$$

Furthermore, we obtain $\rho_1 = 5.4959$ and $\rho_2 = 0.8133$ by using (23) and (24), respectively. From these values and considering the set $\mathcal{V}_0$ given by (7), we can determine the value of r_1 and r_2 through Lemma 2 which yields $r_1 \le \rho_1^{-1/2} = 0.4266$ and $r_2 \le \rho_2^{-1/2} = 1.1089$. Therefore, we can choose the value of r_1 and r_2 verifying $0 \le \|\phi_{\bar{d},0}\|_4 \le r_1 \le 0.4266$, $0 \le \|\Delta\phi_{\bar{d},0}\|_4 \le r_2 \le 1.1089$, and $\phi_1 r_1^2+\rho_2 r_2^2 \le 1$, and, in consequence, we have the sets $\mathcal{B}_\phi$ given as in (19). It is important to say that the range obtained for $\|\phi_{\bar{d},0}\|_4$ and $\|\Delta\phi_{\bar{d},0}\|_4$ allows intersections between regions $\mathcal{V}_0$ and $\mathcal{B}_\phi$ such that the admissible velocity can be $|x_{2,k}| \ge 0.05m/s$. Note that the obtained result reduces the conservatism in the estimation of the region of stability, because a larger region is obtained. In the sequel we perform two kinds of analysis: one concerning the area of the region $\mathcal{C}_x$, the set of the current initial conditions, and the other regarding the state space trajectories of the fuzzy closed-loop system.

5.1.1 Analysis of $\mathcal{C}_x$

From the sets $\mathcal{B}_\phi$ computed in the previous paragraph, we can calculate the set $\mathcal{C}_x$, given in (18) and (22). This allows us to investigate the set $\mathcal{C}_x$ as a function of $\|\phi_{4,0}\|_4$ and $\|\Delta\phi_{4,0}\|_4$. The resulting level sets and their projections in the plane $x_{1k} \times x_{2k}$ are shown in Fig. 2, where it is considered $r_2 = 0$ yielding $\|\Delta\phi_{4,0}\|_4 = 0$. Note that, as expected, the area of $\mathcal{C}_x$ decreases whenever $\|\phi_{4,0}\|_4$ increases. If all the initial past states of the nonlinear system (40) are null, then we have $r_1 = 0$ yielding $\|\phi_{4,0}\|_4 = 0$

and $c(\phi_{4,0}) = 1$. In this case, the set $\mathcal{C}_x$ is the same as $\mathcal{L}_{V_1}$. We also present in Fig. 2 two sets of closed-loop trajectories: one with trajectories starting from $\times$ marks, where the current state is always given by $x_{0k} = \begin{bmatrix}0.045 & 0.12\end{bmatrix}^T$, and the other with trajectories starting from $\circ$ marks where the current state x_{0k} is always taken in the border of the respective $\mathcal{C}_x$ set. In both cases the the same sequence of delayed states given by $\phi_{4,0}$ where $[\phi_{4,0}]_j = r_1 \begin{bmatrix}0 & 1\end{bmatrix}^T$, $j \in \mathcal{I}[1,4]$ is used; therefore, due the structure of the elements of $\phi_{4,0}$, we have $\|\phi_{4,0}\| = r_1$. The trajectories are simulated for different values of $0 \le r_1 \le 0.42$. Considering the trajectories marked with $\times$, from bottom to top in Fig. 2, note that: in the first three x_{0k} belonging to $\mathcal{C}_x$, the trajectories are asymptotically stable as ensured by our approach. The fourth to sixth trajectories have x_{0k} outside of $\mathcal{C}_x$ but these trajectories still converge to the origin. The last three trajectories, from the seventh to the nine-th, have x_{0k} outside of set $\mathcal{C}_x$ and they go out of the region of validity. These simulations illustrate the relevance of the set of initial conditions and the influence of the delayed states on the convergence to the origin in the local stability. As a complement, we present the other set of trajectories starting at $\circ$ marks in Fig. 2: in all cases the x_{0k} is taken at the border of the respective set $\mathcal{C}_x$ with the same set of delayed states that has been used for the respective trajectories marked with $\times$. In these cases, the sequence of initial conditions, given by $\varphi_{4,0}$ matches the set Υ_φ and the asymptotic stability of the trajectories without leaving the region of validity is ensured by Theorem 1.

We can estimate the size of set $\mathcal{L}_{V_1}$, given as in (18) and (22), through the size of $\mathcal{E}(W)$. Such estimated can be computed by noting that the area of this ellipsoidal set, named $s_\mathcal{E}$, is proportional to $(\det(W))^{-1/2}$ [32]. Thus, we can compare the estimates of $\mathcal{L}_{V_1}$ obtained from this work with the ones obtained from the approaches in [22, 23]. The computed values for each estimate is normalized by the value achieved in the present approach, i.e., $\bar{s}_\mathcal{E} = 0.0302$ that corresponds to the maximum area of $\mathcal{C}_x$ reached with $\|\phi_{4,0}\|_4 = 0$. The normalized estimates are shown in Table 1.

The conditions presented in this paper are less restrictive as they do not need d_k as required in [22, 23]. As shown in Table 1, our approach yields the estimate of the size for $\mathcal{C}_x$ without requiring the knowledge of d_k, being 23.75% bigger than the region computed by [22, Theorem 1] and 2% smaller than the size of the region achieved by [23, Theorem 5] which requires the on line knowledge of d_k. This demonstrates that the choice of control law (8) does not introduce conservatism meanwhile it requires less prior knowledge. The main reason for our approach achieves a bigger estimation of the region of attraction is the candidate L-K function used and the slack variables in the present formulation, that relax the numerical optimization problems.

Table 1. Normalized areas of the regions of stability.

Opt. Procedures	Control law	Area
(38)	$u_k = K(\alpha_k)x_k + \bar{K}(\alpha_k)x_{k-\bar{d}}$	1
[23, Theorem 5]	$u_k = K(\alpha_k)x_k + K_d(\alpha_k)x_{k-d_k}$	1.02
[22, Theorem 1]	$u_k = K(\alpha_k)x_k + K_d(\alpha_k)x_{k-d_k}$	0.7476

5.1.2 Fuzzy closed-loop trajectories

We have used the results in [24], [27], and [26] to compare the behavior of the fuzzy closed-loop trajectories for different initial conditions. It should be noted that differently from our approach, all these approaches do not consider the region of validity, their control laws only rely on the current state, i. e., $u_k = K(\alpha_k)x_k$, and their design depend on a scalar design parameter ε which we adopt here $\varepsilon = 10$. For the approaches in [24] and [27], we found feasible solutions while for the approach in [26], no solution has been found even with other values of ε taken on a large interval of positive real numbers. We have tested two cases: one with null initial past states and the other with past states different from zero.

In the first case, we use the control law (8)–(9) with gains shown in (41) and assume $\|\phi_{4,0}\|_4 = 0$, i.e., all the delayed states are equal to zero. Thus, we have the sets $\mathcal{C}_x = \mathcal{L}_{V_1}$ and $\mathcal{B}_\phi = \{0\}$. In the Fig. 3, it is shown the set $\mathcal{C}_x = \mathcal{L}_{V_1}$ (dashed line) and the stable trajectories for seven initial conditions $\varphi_{4,0} = \{\phi_{4,0}, x_0\} \in E_\varphi$, $\phi_{4,0} \in E_\phi$, $x_0 \in \mathbb{R}^n$, $[\phi_{4,0}]_j = \begin{bmatrix}0 & 0\end{bmatrix}^T$, $j \in \mathcal{I}[1,4]$, indicated by $\times$.

In these simulations, we consider that $d_k = \mathtt{round}(2.5 - 1.5\cos(0.2k))$, $|x_{1,k}| \le 0.05$, and if $|x_{1,k}| = 0.05$, then $x_{2,k} = 0$ due to physical constraints in the system. Note that all these stable trajectories evolve inside the region of stability, $\mathcal{C}_x$. Furthermore, there is a trajectory, that does not go to the origin, marked by $\circ$. This trajectory starts outside the region of stability and it

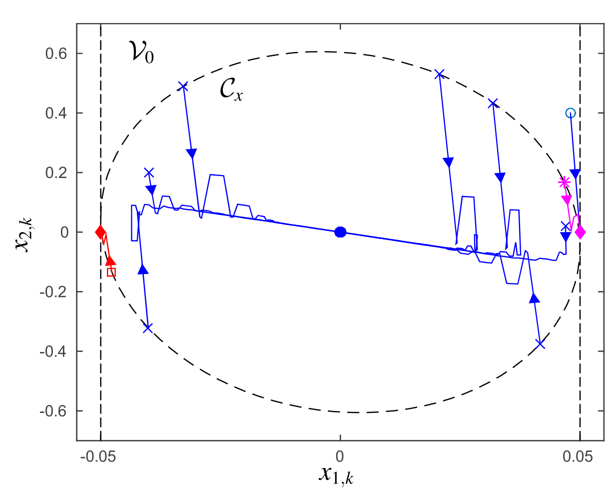

Fig. 3. $\mathcal{V}_0$, $\mathcal{C}_x$, and trajectories for case 1.

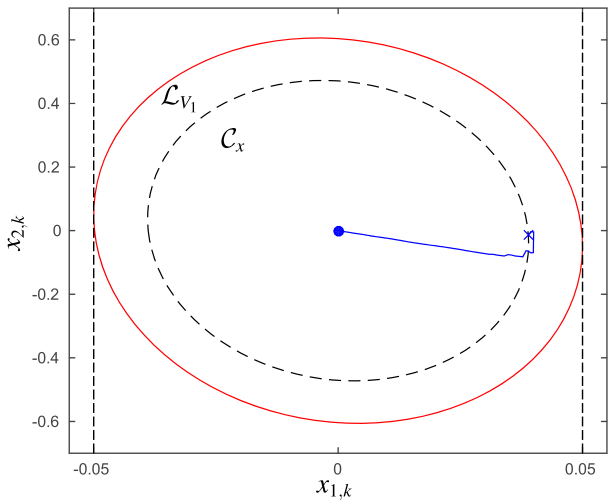

Fig. 4. $\mathcal{L}_{V_1}$, $\mathcal{C}_x$, and a trajectory for case 2.

goes to an equilibrium point outside the region of validity. Now, we consider the control law obtained by using [24, Theorem 2] and [27, Theorem 4]. We simulated the nonlinear closed-loop system with two initial conditions $\varphi_{4,0} = \{\phi_{4,0}, x_0\} \in E_\varphi$, $\phi_{4,0} \in E_\phi$, and $x_0 \in \mathbb{R}^n$, with $[\phi_{4,0}]_j = \begin{bmatrix}0 & 0\end{bmatrix}^T$, $j \in \mathcal{I}[1,4]$, and $x_0^i = \begin{bmatrix}x_{1,0} & x_{2,0}\end{bmatrix}^T$, for $i \in \mathcal{I}[1,2]$, where $x_0^1 = \begin{bmatrix}0.04679 & 0.1662\end{bmatrix}^T$ marked by $*$ for [24] and $x_0^1 = \begin{bmatrix}-0.04769 & -0.1331\end{bmatrix}^T$ marked by $\square$ for [27]. The two trajectories go to equilibrium points outside the region of validity as one can see in Fig. 3. Note that these initial conditions are inside the region of stability estimated by our work, but the trajectories are not asymptotically stable to the origin for those approaches in [24] and [27]. Note that, because the recent conditions proposed in [21] handle only constant and known delay, they are are not applicable to the system considered in this example.

For the second case, we consider $\|\phi_{4,0}\|_4 = 0.2$, therefore $\mathcal{C}_x \subset \mathcal{L}_{V_1}$, $\mathcal{B}_\phi$ is defined as (19) with $r_1 = 0.2$ and $r_2 = 0$. In this situation, we choose $\varphi_{4,0} = \{\phi_{4,0}, x_0\} \in E_\varphi$ with $[\phi_{4,0}]_j = \begin{bmatrix}0 & 0.2\end{bmatrix}^T$, $j \in \mathcal{I}[1,4]$ and $x_0 = \begin{bmatrix}0.03893 & -0.01499\end{bmatrix}^T$. In Fig. 4, it is shown that the resulting trajectory evolves outside $\mathcal{C}_x$, however always inside $\mathcal{L}_{V_1}$.

5.2. Example 2

Consider the T-S fuzzy model (3)–(5) randomly generated with $A_1 = \begin{bmatrix}-1.536 & 0.2844\\ -1.28 & 0.7396\end{bmatrix}$, $A_2 = \begin{bmatrix}1.536 & 0.2844\\ -1.28 & 0.7396\end{bmatrix}$, $A_3 = \begin{bmatrix}-1.536 & 0.16\\ -1.28 & 0.416\end{bmatrix}$, $A_4 = \begin{bmatrix}1.536 & 0.16\\ -1.28 & 0.416\end{bmatrix}$, $A_{d1} = A_{d3} = \begin{bmatrix}0.0672 & -0.1344\\ 0 & -0.0672\end{bmatrix}$, $A_{d2} = A_{d4} = \begin{bmatrix}-0.0672 & 0.1344\\ 0 & 0.0672\end{bmatrix}$, $B_i = \begin{bmatrix}1\\ 0.8\end{bmatrix}$, $d_k \in [1, \bar{d}]$, and the operational region for this system has been fixed by (6) with $L = \mathbf{I}$ and $\eta = \begin{bmatrix}0.8 & 4\end{bmatrix}^T$. The fuzzy sets are given by $M_{11}(x_{1,k}) = M_{31}(x_{1,k}) = \frac{1}{2}(1 + 1.25x_{1,k})$, $M_{21}(x_{1,k}) = M_{41}(x_{1,k}) = \frac{1}{2}(1 - 1.25x_{1,k})$. Besides, if $|x_{2,k}| < 3$ then $M_{12}(x_{2,k}) = M_{22}(x_{2,k}) = 0$ and $M_{32}(x_{2,k}) = M_{42}(x_{2,k}) = 1$, else $M_{12}(x_{2,k}) = M_{22}(x_{2,k}) = (x_{2,k}^2 - 9)/7$, $M_{32}(x_{2,k}) = M_{42}(x_{2,k}) = (16 - x_{2,k}^2)/7$. The membership function is $\alpha_k : \alpha_{ki} = M_{i1}(x_{1,k})M_{i2}(x_{2,k})$, $i \in \mathcal{I}[1,4]$. We have used the conditions proposed in Theorem 1 and in [27], [26], [24], and [25] to search for the maximum admissible delay $\bar{d}$ such that the closed loop system is asymptotically stable. The condition in [27] failed to design a controller even for $\bar{d} = 1$. From [26, Theorem 4] and [24, Theorem 2], we found feasible solutions for $\bar{d} \in \mathcal{I}[1,2]$ and $\bar{d} \in \mathcal{I}[1,3]$, respectively. On the other hand, by using Theorem 1, we found feasible solution for a wider interval of delay, $\bar{d} \in \mathcal{I}[1,4]$, which indicates that our approach is less conservative than those in [24,26,27]. Another condition presented in [25, Theorem 4] can be used to compute a non PDC control law that feedbacks the states with minimal delay and with maximal delay. By using such a condition one can find feasible solution for the same interval found by our proposed method. However the PDC control law obtained through the Theorem 1 is simpler to implement than that in [25]. It is worth to say that among all these works, only ours takes into account the region of validity of the T-S fuzzy model.

5.2.1 Time-invariant delay

As in Example 1, the conditions proposed in [21] are not applicable in this case because of the time-varying nature of the delay. Thus, to consider the approach presented in [21], that requires time-invariant and known delay, assume the specific case of known time-invariant delay equal $d = d_k = 4$, $\forall k \geq 0$. It is possible to verify that the synthesis conditions in [21, Theorem 2] lead to a smaller estimate of the region of attraction than that computed by our approach with $\delta = 1$. This shows that our approach is more general because in our case the delay is unknown and uncertain (please see Remark 4). This is illustrated in Fig. 5 where the estimates of the region of attraction computed by our approach (set $\mathcal{C}_x$, dashed line) and that by [21, Theorem 2] (set $\Omega(1)$, dotted line) are shown, assuming that only x_0 can differs from zero, i.e. all the delayed states are null. It is clear that our approach yields a bigger region (about 9.5 times bigger than that achieved by [21, Theorem 2]) illustrating the relevance and the efficacy of our approach.

6. CONCLUSIONS

In this paper, we developed convex local delay-dependent conditions for synthesis of fuzzy stabilizing

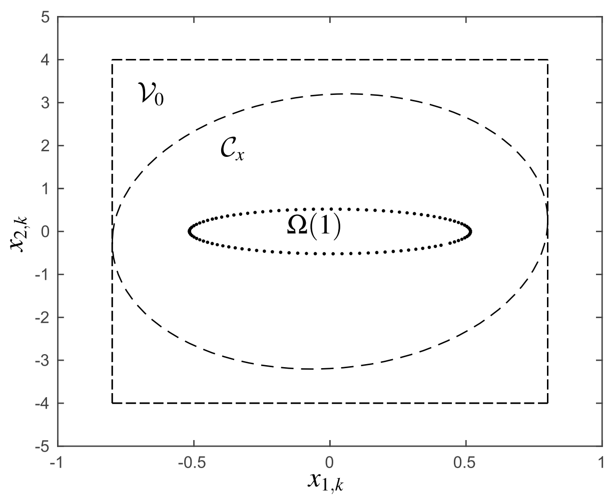

Fig. 5. Estimates of the region of attraction for null delayed states ($\mathcal{B}_\phi = \{0\}$) computed by our approach with $\delta = 1$ (set $\mathcal{C}_x$, dashed line) and, with known time-invariant delay by [21, Theorem 2] (set $\Omega(1)$, dotted line).

feedback controllers. The conditions were based on a fuzzy Lyapunov-Krasovskii function candidate and they were formulated in terms of convex optimization procedures, which can be efficiently solved in a polynomial time. The upper bound $\bar{d}$ of the time varying delay is used in the state feedback control law instead of more restrictive on line delay value. We provided an estimate of the region of attraction, such that the trajectories emanating from this region are guaranteed to asymptotically converge to the origin. The estimate of the region of attraction was characterized through two sets, where the first one describes a region of allowed values for the current states and the other one defines the region for the elements of the delayed initial conditions. From the proposed convex optimization procedures, we can compute controller gains maximizing the estimate of the region of attraction. Finally, we showed two examples to demonstrate the efficiency of the developed approach and to compare it with other approaches in the literature.

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