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Finite-time Sliding Mode Control of Markovian Jump Systems Subject to Actuator Faults

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Abstract: This paper addresses the problem of finite-time boundedness (FTB) for a class of Markovian jump systems (MJSs) via sliding mode control (SMC) technique, in which there may happen actuator faults in all of control channels and mismatched external disturbance. By means of the available boundary information of actuator faults, a suitable sliding mode controller is designed such that state trajectories are driven to sliding surface before a specified finite (possibly *short*) time interval. Furthermore, a partitioning strategy is introduced to derive the sufficient conditions for ensuring the FTB of the closed-loop systems over the whole specified finite-time interval including the reaching phase and the sliding motion phase. Finally, a practical example are provided from an F-404 aircraft engine system to illustrate the proposed method.

Keywords: Actuator faults, finite-time boundedness(FTB), Markovian jump systems(MJSs), sliding mode control(SMC).

1. INTRODUCTION

The actuator faults often occur in actual physical systems, which can cause performance deterioration or even instability of the systems. Thus, it is necessary to develop an effective control technique to maintain a receivable stability for the closed-loop systems against actuator failures, see [1–7] and the references therein.

On the other hand, MJSs have received considerable attention in the past decades, since MJSs can effectively model many practice systems experiencing abrupt changes in their structures, e.g., the random failures of the components, sudden disturbances and variations of the environment [8, 9]. More recently, the SMC technique has witnessed increasing applications in MJSs [10–15]. Among them, Chen, et al in [11] proposed the on-line estimation mechanism on the loss effectiveness of actuators, by which an adaptive sliding mode controller was designed to attenuate the effect of the actuator degradation. The stabilization problem for MJSs subject to actuator and sensor faults was also addressed in [15], and a fault tolerant control strategy was designed by means of a sliding mode observer to estimate the disturbances, actuator and sensor faults simultaneously. It is worthy of noting that the aforementioned works were based on Lyapunov stability, that is, the behavior of the controlled system was considered within a sufficiently long (in principle infinite) time interval. However, in a lot of industrial process, FTB is more practical for analyzing the transient behavior within a finite (possibly short) interval.

A system is said to be finite-time stable (FTS) if, given a bound on the initial condition, its state (weighted) norm does not exceed a certain threshold during the specified time interval [16], which was further extended to the concept of FTB in [17] when the initial condition and external disturbances were concerned. Recently, some interesting results were obtained on FTS/FTB for MJSs, in [18, 19], etc. The problem of FTB based on SMC technique was investigated in [20] and a partitioning strategy was proposed to analyze the FTB over the reaching phase and the sliding motion phase. However, to the authors' best knowledge, few results have been obtained on the FTB of *Markovian jump systems* via SMC technique, especially, the case subject to actuator faults. Moreover, the existing works involving in FTB cannot be extended to the above case, due to the specific structures of MJSs and SMC. This motivates our present study. Apparently, when analyzing FTB of MJSs subject to actuator faults via SMC approaches, the following questions had to be answered, which just make the present research be challenging:

Q-1: For a given finite time T, how to guarantee that the state trajectories can arrive at the designed sliding surface within $[0, T^*]$ with $T^* \leq T$ and maintain in it within $[T^*, T]$ despite the effect of actuator faults?

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Q-2: Under the effect of the jumping among modes, how to guarantee FTB over the whole finite-time interval $[0,T]$?

The present work will answer the above questions. The main contributions of this work are highlighted as follows: 1) The problem of FTB for a class of MJSs with actuator faults and external disturbance via SMC technique is investigated. 2) A scalar selection criterion-dependent SMC law is developed to guarantee the reachability of sliding surface before the specified finite-time interval. 3) A partitioning strategy is introduced to obtain the sufficient conditions for the FTB of the closed-loop systems over the whole finite-time interval.

Notation: The notation $M>(<)0$ is used to denote a symmetric positive-definite(negative-definite) matrix. $\lambda_{max}(\cdot)$ and $\lambda_{min}(\cdot)$ denote the maximum and minimum eigenvalues of the corresponding matrix. $|\cdot|$ and $\|\cdot\|$ refer to the 1-norm and Euclidean vector norm, respectively. In symmetric block matrices, the symbol "$*$" is used as an ellipsis for terms induced for symmetry. The "*wrt*" is an abbreviation of "with respect to". $(\Omega,\mathcal{F},\mathcal{P})$ is a probability space with Ω the sample space, and $\mathcal{F}$ the σ-algebra of subsets of the sample space, and $\mathcal{P}$ is the probability measure. $\mathbf{E}\{\cdot\}$ denotes the expectation operator *wrt* probability measure $\mathcal{P}$. $\lfloor\cdot\rfloor$ refers to the floor of a decimal. The symbol He(X) is used to represent $X+X^T$. Throughout this paper, if not explicitly stated, matrices are assumed to have compatible to have compatible dimensions.

2. PROBLEM FORMULATION

Consider the following MJSs:

$$\begin{aligned}\dot{x}(t)=&(A(r_t)+\Delta A(r_t))x(t)+B(r_t)(u(t)+f(x(t),r_t))\\&+D(r_t)w(t),\end{aligned}\tag{1}$$

where $x(t)\in\mathbb{R}^n$ is the state; $u(t)\in\mathbb{R}^m$ is the control input; $w(t)\in\mathbb{R}^r$ is the external disturbance; $f(x(t),r_t)\in\mathbb{R}^p$ is an unknown nonlinear function. $\{r_t,t\geq 0\}$ is a right-continuous Markovian chain on the probability space $(\Omega,\mathcal{F},\mathcal{P})$ taking values in a finite state space $\mathbb{S}=\{1,2,\dots,N\}$ with generator $\Pi=(\pi_{ij})_{N\times N}$ given by:

$$Pr\{r_{t+\Delta}=j|r_t=i\}=\begin{cases}\pi_{ij}\Delta+o(\Delta), & i\neq j,\\ 1+\pi_{ij}\Delta+o(\Delta), & i=j,\end{cases}$$

where $\Delta>0$ and $\lim_{\Delta\to 0}\frac{o(\Delta)}{\Delta}=0$, π_{ij} is the transition rate from i to j and satisfies $\pi_{ij}>0, i\neq j$, and $\pi_{ii}=-\sum_{i\neq j}\pi_{ij}<0$ for $\forall i,j\in\mathbb{S}$. $A(r_t)\in\mathbb{R}^{n\times n}$, $B(r_t)\in\mathbb{R}^{n\times m}$ and $D(r_t)\in\mathbb{R}^{n\times r}$ are system matrices, for each $i\in\mathbb{S}$, $A(r_t)=A_i$, $B(r_t)=B_i$, $D(r_t)=D_i$, $\Delta A(r_t)=A_i(t)$, $f(x(t),r_t)=f(x(t),i)$.

Thus, system (1) can be rewritten as

$$\dot{x}(t)=(A_i+\Delta A_i)x(t)+B_i(u(t)+f(x(t),i))+D_iw(t).\tag{2}$$

Here, the parameter uncertainty $\Delta A_i(t)$ is norm-bounded, i.e., $\Delta A_i(t)=M_iF_i(t)N_i$, where M_i and N_i are known real constant matrices, and $F_i(t)$ is an unknown matrix function satisfying $F_i(t)^TF_i(t)\leq I$.

Assumption 1: The nonlinear function $f(x(t),i)$ satisfies the following condition:

$$\|f(x(t),i)\|\leq\varepsilon_i\|x(t)\|,\tag{3}$$

where $\varepsilon_i>0$ is a known scalar.

Assumption 2: The disturbance $w(t)$ is assumed to be bounded peak signal over an assigned finite-time interval $[t_1,t_2]$, i.e.,

$$W_{[t_1,t_2],\delta}\triangleq\left\{w^T(t)w(t)\leq\delta^2,\ \forall t\in[t_1,t_2]\right\},\tag{4}$$

where $\delta>0$ is a known scalar.

In this work, it is assumed that the actuator faults may happen according to the following model:

$$u^F(t)=(I-\rho)u(t),\tag{5}$$

with $\rho=\text{diag}\{\rho_1,\dots,\rho_m\}$ satisfying:

$$0\leq\underline{\rho}_k\leq\rho_k\leq\bar{\rho}_k<1,\qquad k=1,2,\dots,m,\tag{6}$$

where the unknown parameter $\rho_k(k=1,\dots,m)$ denotes the loss of effectiveness of the kth actuator. Moreover, it is assumed $\underline{\rho}=\text{diag}\{\underline{\rho}_1,\dots,\underline{\rho}_m\}$, $\bar{\rho}=\text{diag}\{\bar{\rho}_1,\dots,\bar{\rho}_m\}$.

Thus, the system (2) subject to actuator faults (5) is described by:

$$\begin{aligned}\dot{x}(t)=&(A_i+\Delta A_i)x(t)+B_i(I-\rho)u(t)\\&+B_if(x(t),i)+D_iw(t).\end{aligned}\tag{7}$$

Remark 1: The actuator model in (5) is normal as $\underline{\rho}_k=\bar{\rho}_k=0$ and partly faulted as $0<\underline{\rho}_k\leq\bar{\rho}_k<1$. Hence, the case in this work is more general.

The following definitions are generalized from [21], [22], and [23].

Definition 1: Given a time interval $[t_1,t_2]$, two positive scalars c_1,c_2, with $c_1<c_2$, and a weighted matrix $R>0$. Systems (1) with $u(t)=0$ is said to be FTB with respect to $(c_1,c_2,[t_1,t_2],R,W_{[t_1,t_2],\delta})$, if, $\forall t\in[t_1,t_2]$,

$$\mathbf{E}\{x^T(t_1)Rx(t_2)\}\leq c_1\Rightarrow\mathbf{E}\{x^T(t)Rx(t)\}<c_2.$$

Remark 2: In Definition 1, the scalars c_1,c_2 and matrix R are known. In practice, the scalar c_2 and the matrix R are determined directly according to the transient constraints of physical systems, and the scalar c_1 is chosen according to the initial conditions (see [20–23] and the given simulation example later).

Definition 2 (Weak Infinitesimal Operator)**:** Let $C_2(\mathbb{R}^n\times\mathbb{S};\mathbb{R}_+)$ denote the family of all nonnegative functions $V(x(t),i)$ on $\mathbb{R}^n\times\mathbb{S}$ which are continuously twice differentiable in $x(t)$. For $V \in C_2(\mathbb{R}^n \times \mathbb{S}; \mathbb{R}_+)$ and $r_t = i$, define an infinitesimal operator $\mathscr{L}V(x(t), i)$ by

$$\mathscr{L}V(x(t),i) = \lim_{\Delta \to 0^+} \frac{1}{\Delta}[\mathbf{E}\{V(x(t+\Delta), r_{t+\Delta})\} - V(x(t),i)].$$

The objective of this work is to design a sliding mode controller such that the FTB of the MJSs (7) can be attained despite actuator faults and external disturbance.

3. SLIDING SURFACE AND SMC LAW

In this section, a set of mode-dependent sliding functions will be firstly constructed as follows:

$$s(x(t),i) = G_i x(t), \quad i \in \mathbb{S}, \tag{8}$$

where $G_i = B_i^T X_i$ and the matrix X_i is chosen such that $B_i^T X_i B_i$ is nonsingular for each $i \in \mathbb{S}$, which can be easily ensured by choosing $X_i > 0$ due to the full column of B_i.

For the given finite-time interval $[0, T]$, a suitable sliding mode controller should be designed to drive the states trajectories onto the specified sliding surface $s(x(t),i) = 0$ during $[0, T^*]$ with $T^* \leq T$ and maintain there for the rest time interval $[T^*, T]$. To this end, we design the following SMC law:

$$u(t) = -K_i x(t) - \gamma_i(t)\mathrm{sgn}(s(x(t),i)), \tag{9}$$

with

$$\gamma_i(t) = \rho_i + \mu_i \delta, \tag{10}$$

$$\mu_i = \frac{\|(B_i^T X_i B_i)^{-1} B_i^T X_i D_i\|}{1 - \bar{\rho}_{max}}, \tag{11}$$

and $\rho_i > 0$ a adjustable parameter to be further described in Theorem 1, and $\bar{\rho}_{max} = \max\{\bar{\rho}_1, ..., \bar{\rho}_m\}$.

4. THE REACHABILITY WITH $T^* \leq T$

In this section, it will be proven that the SMC law (9)-(11) can ensure the reachability of the specified sliding surfaces $s(x(t),i) = 0$.

Theorem 1: For a given finite-time (possibly *short*) interval T, the SMC law is designed as (9)-(11) and the adjustable parameters $\rho_i > 0$ satisfies

$$\rho_i \geq \frac{\kappa_i + \chi_i \sqrt{c_2/\lambda_{min}(R)}}{1 - \bar{\rho}_{max}}, \tag{12}$$

where

$$\kappa_i \geq \frac{\lambda_{max}[(B_i^T X_i B_i)^{-1}]}{T} \|B_i^T X_i x(0)\|, \tag{13}$$

$$\begin{aligned}\chi_i =& \|(B_i^T X_i B_i)^{-1} B_i^T X_i A_i\| + \|I - \underline{\rho}\| \|K_i\| \\ &+ \|(B_i^T X_i B_i)^{-1} B_i^T X_i M_i\| \|N_i\| + \varepsilon_i \\ &+ \frac{1}{2} \|\sum_{j=1}^{N} [\lambda_{ij}(B_j^T X_j B_j)^{-1}] B_i^T X_i\|,\end{aligned} \tag{14}$$

then the state trajectories can be driven onto the sliding surface $s(x(t),i) = 0$ during the interval $[0, T^*]$ with $T^* \leq T$ and remain there in the subsequent time.

Proof: Choose the Lyapunov function:

$$V_1(x(t),i) = s^T(x(t),i)(B_i^T X_i B_i)^{-1} s(x(t),i). \tag{15}$$

Then, by Definition 2, we obtain the infinitesimal operator as

$$\begin{aligned}&\mathscr{L}V_1(x(t),i) \\ &= 2s^T(x(t),i)(B_i^T X_i B_i)^{-1}(B_i^T X_i M_i F_i(t) N_i + B_i^T X_i A_i)x(t) \\ &\quad + 2s^T(x(t),i) f(x(t),t,i) \\ &\quad + 2s^T(x(t),i)(B_i^T X_i B_i)^{-1} B_i^T X_i D_i w(t) \\ &\quad + s^T(x(t),i) \sum_{j=1}^{N} [\lambda_{ij}(B_j^T X_j B_j)^{-1}] s(x(t),i) \\ &\quad + 2s^T(x(t),i)(I - \rho)u(t).\end{aligned} \tag{16}$$

Substituting the control law (9)-(14) into (16) and noting $\|s(x(t),i)\| \leq |s(x(t),i)|$, we can obtain

$$\begin{aligned}\mathscr{L}V_1(x(t),i) \leq& 2\|s(x(t),i)\| [\chi_i \|x(t)\| \\ &- ((1 - \bar{\rho}_{max})\rho_i - \kappa_i) - \kappa_i].\end{aligned} \tag{17}$$

If the adjustable parameter ρ_i satisfies the expression (12), one has $(1 - \bar{\rho}_{max})\rho_i - \kappa_i \geq 0$ and

$$\frac{(1 - \bar{\rho}_{max})\rho_i - \kappa_i}{\chi_i} \geq \sqrt{\frac{c_2}{\lambda_{min}(R)}}. \tag{18}$$

Now, define the following domains:

$$\Phi_1 = \{x(t) : \chi_i \|x(t)\| \leq (1 - \bar{\rho}_{max})\rho_i - \kappa_i\}, \tag{19}$$

$$\Phi_2 = \{x(t) : \sqrt{\lambda_{min}(R)} \|x(t)\| \leq \sqrt{c_2}\}. \tag{20}$$

Apparently, one has $\Phi_2 \subseteq \Phi_1$. It will be proven in Theorem 2 later that the FTB of the system (1) over the finite-time interval $[0, T]$ can be ensured, that is, one has $x^T(t)Rx(t) < c_2$ in the mean square. That implies that one has $\|x(t)\| \leq \sqrt{c_2/\lambda_{min}(R)}$ during the interval $[0, T]$ in the mean square. Thus, the state trajectories will remain within the set Φ_1 in the interval $[0, T]$ in the mean square, which yields from (17):

$$\mathscr{L}V_1(x(t),i) \leq -2\kappa_i \|s(x(t),i)\|. \tag{21}$$

The following will further give the reaching instant T^*. To this end, by Rayleigh's inequality, we obtain from (15) $V_1(x(t),i) \leq \lambda_{max}[(B_i^T X_i B_i)^{-1}] \|s(x(t),i)\|^2$ Then, we have from (21):

$$\mathscr{L}V_1(x(t),i) \leq -\frac{2\kappa_i}{\sqrt{\lambda_{\max}[(B_i^T X_i B_i)^{-1}]}} \sqrt{V_1(x(t),i)}. \tag{22}$$

Integrating (22) from 0 to t, one can ensure that there exists an instant T^* satisfying

$$T^* \leq \frac{\sqrt{\lambda_{\max}[(B_i^T X_i B_i)^{-1}]}}{\kappa_i}\sqrt{V_1(x(0),i)}, \tag{23}$$

such that $\mathbf{E}\{V_1(x(t),i)\}=0$ for $t \geq T^*$, and consequently $s(x(t),i)=0$ (in the mean square).

Note that one has by Rayleigh's inequality and the fact $s(x(0),i)=B_i^T X_i x(0)$:

$$T^* \leq \frac{\lambda_{\max}[(B_i^T X_i B_i)^{-1}]}{\kappa_i}\|B_i^T X_i x(0)\|. \tag{24}$$

Thus, from (12) and (24), one obtains $T^* \leq T$, which means that the state trajectories of system (1) will be driven onto the specified sliding surface $s(x(t),i)=0$ in finite time T^* with $T^* \leq T$. □

Remark 3: It should be pointed out that the designed SMC law (9)-(11) depends on the given time T and the bounds of actuator faults $\bar{\rho}$ via the selection on the scalar ρ_i. Hence, the proposed SMC law can effectively attenuate the effect of actuator faults during the specified finite-time interval. This just answers the first question (**Q-1**).

5. FTB OVER THE WHOLE FINITE-TIME INTERVAL $[0,T]$

As is well known, there exist two phases for the state trajectories of SMC systems: the reaching phase within $[0,T^*]$ and the sliding motion phase within $[T^*,T]$. In this section, by analyzing FTB problem of the MJSs (7) during the reaching phase and the sliding motion phase respectively, a sufficient condition will be given to guarantee FTB over the whole finite-time interval $[0,T]$. By substituting SMC law (9) into (7), we obtain the following closed-loop system:

$$\begin{aligned}\dot{x}(t) =& [A_i+\Delta A_i(t)-B_i(I-\rho)K_i]x(t)+B_i f(x(t),t,i)\\ &+D_i w(t)-B_i(I-\rho)\gamma_{si}(t),\end{aligned} \tag{25}$$

where $\gamma_{si}(t)=\gamma_i(t)\mathrm{sgn}(s(x(t),i))$.

In the following, a partitioning strategy is generalized from the nonlinear system in [20] to MJSs case:

Lemma 1: (Partitioning Strategy) For the system (1) with the specified parameters $(c_1,c_2,[0,T],R,W_{[0,T],\delta})$, the closed-loop system (25) is FTB *wrt* $(c_1,c_2,[0,T],R,W_{[0,T],\delta})$, if only if there exists an auxiliary scalar c_i^* satisfying $c_1<c_i^*<c_2$ such that this system is FTB *wrt* $(c_1,c^*,[0,T^*],R,W_{[0,T],\delta})$ during reaching phase and FTB with respect to $(c^*,c_2,[T^*,T],R,W_{[0,T],\delta})$ during sliding motion phase, where $c^*=\max_{i\in\mathbb{S}}\{c_i^*\}$.

Remark 4: It is shown from Lemma 1 that the reaching phase within $[0,T^*]$ and the sliding motion phase within $[T^*,T]$ can be connected by parameters c^* and T^*. Therefore, when FTB during the reaching phase and the sliding motion phase are guaranteed simultaneously, the FTB over the whole finite-time interval $[0,T]$ will be attained. This answers the second question(**Q-2**).

In the following theorem, the sufficient condition is derived via the above partitioning strategy to guarantee FTB over the whole finite-time interval $[0,T]$.

Theorem 2: For the given the parameters $(c_1,c_2,[0,T],R,W_{[0,T],\delta})$ and a feasible scalar η_i, if there exist positive constants $c^*,\vartheta,\alpha_i,\beta_i,\zeta_i,\xi_i$, and matrices $\mathcal{P}_i>0$ and $\mathcal{L}_i$ for any $i\in\mathbb{S}$, satisfying the following linear matrix inequalities (LMIs):

$$\begin{bmatrix}\Lambda_{11i} & \Lambda_{12i}\\ * & \Lambda_{22i}\end{bmatrix}<0, \tag{26}$$

$$\begin{bmatrix}\Omega_{11i} & \Omega_{12i}\\ * & \Omega_{22i}\end{bmatrix}<0, \tag{27}$$

$$c_1<c^*<c_2, \tag{28}$$

$$\vartheta R^{-1}<\mathcal{P}_i<R^{-1}, \tag{29}$$

$$\begin{bmatrix}-e^{-\eta_i T}c^*+(2\rho_i^2 T+(1+2\mu_i^2)\delta^2 T)\eta_i & \sqrt{c_1}\\ * & -\vartheta I\end{bmatrix}<0, \tag{30}$$

$$-e^{-\eta_i T}c_2\vartheta+c^*+\eta_i\delta^2 T\vartheta<0, \tag{31}$$

where

$$\Lambda_{11i}=\begin{bmatrix}\Xi_{1i} & D_i\\ * & -\eta_i I\end{bmatrix},$$

$$\Lambda_{12i}=\begin{bmatrix}-B_i(I-\bar{\rho}) & \alpha_i M_i & \mathcal{P}_i N_i^T & \beta_i B_i & \varepsilon_i\mathcal{P}_i I & B_i & \mathcal{L}_i^T\bar{\rho}^T & M_i^k\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix},$$

$$\Lambda_{22i}=-\mathrm{diag}\{\eta_i I,\alpha_i I,\alpha_i I,\beta_i I,\beta_i I,\zeta_i I,\zeta_i I,N_i^k\},$$

$$\Xi_{1i}=\mathcal{P}_i A_i^T+A_i\mathcal{P}_i-\mathcal{L}_i^T B_i^T-B_i\mathcal{L}_i+(\lambda_{ii}-\eta_i)\mathcal{P}_i,$$

$$\Omega_{11i}=\begin{bmatrix}\Xi_{2i} & \Gamma_i D_i\\ * & -\eta_i I\end{bmatrix},\quad \Omega_{12i}=\begin{bmatrix}\xi_i\Gamma_i M_i & \mathcal{P}_i N_i^T & M_i^k\\ 0 & 0 & 0\end{bmatrix},$$

$$\Omega_{22i}=-\mathrm{diag}\{\xi_i I,\xi_i I,N_i^k\},\quad \Gamma_i=I-B_i(B_i^T X_i B_i)^{-1}B_i^T X_i,$$

$$\Xi_{2i}=\mathcal{P}_i A_i^T\Gamma_i^T+\Gamma_i A_i\mathcal{P}_i+(\lambda_{ii}-\eta_i)\mathcal{P}_i,$$

$$M_i^k=[\sqrt{\lambda_{i1}}\mathcal{P}_i,\sqrt{\lambda_{i2}}\mathcal{P}_i,\ldots,\sqrt{\lambda_{iN}}\mathcal{P}_i],$$

$$N_i^k=\mathrm{diag}\{\mathcal{P}_1,\mathcal{P}_2,\ldots,\mathcal{P}_N\}, i\neq j,$$

the closed-loop system (25) is FTB *wrt* $(c_1,c_2,[0,T],R,W_{[0,T],\delta})$ and the control gain in (9) is given by $K_i=\mathcal{L}_i\mathcal{P}_i^{-1}$.

Proof: We shall analyze the FTB during the two phases, respectively, over the finite-time interval $[0,T]$.

Case 1: FTB during reaching phase within $[0,T^*]$.

For the following Lyapunov functional candidate:

$$V_2(x(t),i)=x^T(t)P_i x(t), \tag{32}$$

we obtain its infinitesimal operator as

$$\mathscr{L}V_2(x(t),i)$$

$$
\begin{aligned}
&= x^T(t)\left(\bar{A}_i^T P_i + P_i\bar{A}_i + \sum_{j=1}^{N}(\lambda_{ij}P_j)\right)x(t) \\
&\quad + w^T(t)D_i^T P_i x(t) + x^T(t)P_iD_iw(t) \\
&\quad + 2x^T(t)P_i\Delta A_i(t)x(t) + 2x^T(t)P_iB_i\rho K_ix(t) \\
&\quad - \gamma_{si}^T(t)(I-\rho)^T B_i^T P_i x(t) - x^T(t)P_iB_i(I-\rho)\gamma_{si}(t) \\
&\quad + 2x^T(t)P_iB_if(x(t),t,i), \qquad (33)
\end{aligned}
$$

where $\bar{A}_i = A_i - B_iK_i$.

For $\alpha_i > 0$, $\beta_i > 0$ and $\zeta_i > 0$, one has

$$
\begin{aligned}
2x^T(t)P_i\Delta A_i(t)x(t) &\le \alpha_i x^T(t)P_iM_iM_i^T P_i x(t) \\
&\quad + \alpha_i^{-1}x^T(t)N_i^T N_i x(t), \qquad (34)\\
2x^T(t)P_iB_if(x(t),t,i) &\le \beta_i x^T(t)P_iB_iB_i^T P_i x(t) \\
&\quad + \beta_i^{-1}\varepsilon_i^2 x^T(t)x(t), \qquad (35)\\
2x^T(t)P_iB_i\rho K_ix(t) &\le \zeta_i^{-1}x^T(t)K_i^T\bar{\rho}^T\bar{\rho}K_ix(t) \\
&\quad + \zeta_i x^T(t)P_iB_iB_i^T P_i x(t). \qquad (36)
\end{aligned}
$$

Define an auxiliary function as follows:

$$
\begin{aligned}
J_1(x(t),i) =& \mathscr{L}V_2(x(t),i) - \eta_iV_2(x(t),i) \\
&- \eta_i w^T(t)w(t) - \eta_i\gamma_{si}^T(t)\gamma_{si}(t). \qquad (37)
\end{aligned}
$$

We have $J_1(x(t),i) \le \varpi^T(t)\Psi_{1i}\varpi(t)$ from (33)-(36), where $\varpi(t) = [x(t)\ w(t)\ \gamma_{si}(t)]^T$.

Let $\mathcal{P}_i \triangleq P_i^{-1}$ and $\mathcal{L}_i \triangleq K_i\mathcal{P}_i$. it is easily shown that $\Psi_{1i} < 0$ can be ensured by (26).

Moreover, it follows from $J_1(x(t),i) < 0$ that

$$
\begin{aligned}
\mathscr{L}V_2(x(t),i) <& \eta_iV_2(x(t),i) + \eta_i w^T(t)w(t) \\
&+ \eta_i\gamma_{si}^T(t)\gamma_{si}(t). \qquad (38)
\end{aligned}
$$

Besides, $\gamma_{si}(t)$ satisfies the following inequality:

$$
\gamma_{si}^T(t)\gamma_{si}(t) = \gamma_i^T(t)\gamma_i(t) \le 2\rho_i^2 + 2\mu_i^2 w^T(t)w(t). \quad (39)
$$

Multiplying both sides of expression (38) by $e^{-\eta_i t}$ and integrating the obtained inequality from 0 to t with $t \in [0,T^*]$, we have

$$
\begin{aligned}
&e^{-\eta_i t}\mathbf{E}\{V_2(x(t),i)\} \\
&< V_2(0) + 2\rho_i^2\eta_i\int_0^t e^{-\eta_i\tau}\mathrm{d}\tau \\
&\quad + (1+2\mu_i^2)\eta_i\int_0^t e^{-\eta_i\tau}w^T(\tau)w(\tau)\mathrm{d}\tau \\
&\le \bar{\sigma}_P c_1 + 2\rho_i^2 T\eta_i + (1+2\mu_i^2)\eta_i\delta^2 T. \qquad (40)
\end{aligned}
$$

On the other hand, it follows from (32) that

$$
e^{-\eta_i t}\mathbf{E}\{V_2(x(t),i)\} \ge e^{-\eta_i T}\underline{\sigma}_P\mathbf{E}\{x^T(t)Rx(t)\}. \quad (41)
$$

It yields from (40) and (41) that

$$
\mathbf{E}\{x^T(t)Rx(t)\} \le \frac{\bar{\sigma}_P c_1 + 2\rho_i^2 T\eta_i + (1+2\mu_i^2)\eta_i\delta^2 T}{e^{-\eta_i T}\underline{\sigma}_P}, \quad (42)
$$

where

$$
\begin{aligned}
\bar{\sigma}_P &= \max_{i\in\mathbb{S}}(\lambda_{\max}(R^{-1/2}P_iR^{-1/2})), \\
\underline{\sigma}_P &= \min_{i\in\mathbb{S}}(\lambda_{\min}(R^{-1/2}P_iR^{-1/2})).
\end{aligned}
$$

Further, if there exists scalar $c^* \in (c_1,c_2)$ satisfying

$$
\frac{\bar{\sigma}_P c_1 + 2\rho_i^2 T\eta_i + (1+2\mu_i^2)\eta_i\delta^2 T}{\underline{\sigma}_P} < e^{-\eta_i T}c^*, \quad (43)
$$

we obtain from (42) that $\mathbf{E}\{x^T(t)Rx(t)\} < c^*$ for all $t \in [0,T^*]$. Thus, according to Definition 2, the closed-loop system (25) is FTB *wrt* $(c_1, c^*, [0,T^*], R, W_{[0,T^*],\delta})$. By the condition (29), one has $1 < \underline{\sigma}_P$, $\bar{\sigma}_P < \frac{1}{\vartheta}$. Thus, it is easily shown that the inequality (43) can be ensured by the inequality (30).

Case 2: FTB over sliding motion phase within $[T^*,T]$.

According to SMC theory, when the state trajectories maintain in the sliding surface $s(t) = 0$, we have the following equivalent controller:

$$
\begin{aligned}
u_{eq}(t) =& -(I-\rho)^{-1}(B_i^T X_iB_i)^{-1}B_i^T X_i \\
&\times\left[(A_i + \Delta A_i(t))x(t) + D_iw(t)\right] \\
&-(I-\rho)^{-1}f(x(t),t,i). \qquad (44)
\end{aligned}
$$

By substituting (44) into (7), we get the following system:

$$
\dot{x}(t) = (\tilde{A}_i + \Delta\tilde{A}_i(t))x(t) + \tilde{D}_iw(t), \quad (45)
$$

where $\tilde{A}_i = \Gamma_iA_i$, $\Delta\tilde{A}_i(t) = \Gamma_i\Delta A_i(t)$ and $\tilde{D}_i = \Gamma_iD_i$.

Choose the following Lyapunov functional candidate:

$$
V_2(x(t),i) = x^T(t)P_ix(t). \quad (46)
$$

We obtain the infinitesimal operator as

$$
\begin{aligned}
&\mathscr{L}V_2(x(t),i) \\
&= x^T(t)\left(\mathrm{He}(P_i(\tilde{A}_i + \Delta\tilde{A}_i)) + \sum_{j=1}^{N}(\lambda_{ij}P_j)\right)x(t) \\
&\quad + w^T(t)\tilde{D}_i^T P_ix(t) + x^T(t)P_i\tilde{D}_iw(t). \qquad (47)
\end{aligned}
$$

For $\xi_i > 0$, one has

$$
\begin{aligned}
2x^T(t)P_iT_i\Delta A_i(t)x(t) \le& \xi_i x^T(t)P_iT_iM_iM_i^T T_i^T P_ix(t) \\
&+ \xi_i^{-1}x^T(t)N_i^T N_ix(t). \qquad (48)
\end{aligned}
$$

Define an auxiliary function as follows:

$$
\begin{aligned}
J_2(x(t),i) =& \mathscr{L}V_2(x(t),i) - \eta_iV_2(x(t),i) \\
&- \eta_i w^T(t)w(t). \qquad (49)
\end{aligned}
$$

We have $J_1(x(t),i) \le \varsigma^T(t)\Psi_{2i}\varsigma(t)$ from (47)-(48), where $\varsigma(t) = [x(t)\ w(t)]^T$.

It is easily shown that Ψ_{2i} can be ensured by (27).

By means of $J_2(x(t),i) < 0$, we obtain

$$\mathscr{L}V_2(x(t),i) < \eta_i V_2(x(t),i) + \eta_i w^T(t)w(t). \tag{50}$$

It is shown in **Case 1** that the closed-loop system is FTB wrt $(c_1, c^*, [0,T^*], R, W_{[0,T^*],\delta})$ over the reaching phase, which implies that the initial condition of sliding motion phase at instant T^* satisfies $\mathbf{E}\{x^T(T^*)Rx(T^*)\} < c^*$. Thus, by multiplying both sides of expression (50) by $e^{-\eta_i t}$ and integrating the obtained inequality from T^* to t with $t \in [T^*, T]$, we have

$$\begin{aligned} & e^{-\eta_i t}\mathbf{E}\{V_2(x(t),i)\} \\ & < e^{-\eta_i T^*}\mathbf{E}\{V_2(x(T^*),i)\} + \eta_i \int_{T^*}^{t} e^{-\eta_i \tau} w^T(\tau)w(\tau)\mathrm{d}\tau \\ & < x^T(T^*)P_i x(T^*) + \eta_i \delta^2 T \\ & \leq \bar{\sigma}_P c^* + \eta_i \delta^2 T. \end{aligned} \tag{51}$$

In addition, it follows from (46) that

$$e^{-\eta_i t}\mathbf{E}\{V_2(x(t),i)\} \geq e^{-\eta_i T}\underline{\sigma}_P \mathbf{E}\{x^T(t)Rx(t)\}. \tag{52}$$

Putting together (51) and (52), we have

$$\mathbf{E}\{x^T(t)Rx(t)\} \leq \frac{\bar{\sigma}_P c^* + \eta_i \delta^2 T}{e^{-\eta_i T}\underline{\sigma}_P}. \tag{53}$$

Further, if $c^* \in (c_1, c_2)$ satisfies

$$\frac{\bar{\sigma}_P c^* + \eta_i \delta^2 T}{\underline{\sigma}_P} < e^{-\eta_i T} c_2, \tag{54}$$

we obtain from (53) that $\mathbf{E}\{x^T(t)Rx(t)\} < c_2$ for all $t \in [T^*, T]$. Moreover, it is easily shown that the inequalities (54) can be ensured by (31). □

Remark 5: When solving LMIs (28)-(31) in Theorem 2, the parameters ρ_i and η_i should be given in advance. However, it is seen from (12) that ρ_i is dependent on the norm of the control gain K_i. Since K_i will be obtain only after solving LMIs in Theorem 2, it is difficult to choose the parameter ρ_i according to expression (12). On the other hand, it can be seen from (28), (29), and (31) that the scalar η_i should be selected in the range of $(0, \frac{1}{T}\ln\frac{c_2}{c_1})$. Apparently, a desirable η_i within $(0, \frac{1}{T}\ln\frac{c_2}{c_1})$ should make the obtained control gain matrix K_i to have minimum norm. To this end, this work will give a Search Algorithm for suitable parameters ρ_i and η_i in this following.

Search Algorithm: find parameters ρ_i and η_i simultaneously.

1: Choose an initial parameter candidate $\rho_i^c = \frac{\kappa_i + \chi_{1i}\sqrt{c_2/\lambda_{min}(R)}}{1-\bar{\rho}_{max}} + 0.1$, where $\chi_{1i} = \chi_i - \|I - \underline{\rho}\|\|K_i\|$ with χ_i as in (14), that is, χ_{1i} is equal to χ_i without the term $\|I - \underline{\rho}\|\|K_i\|$.
2: Let $L = \frac{1}{T}\ln\frac{c_2}{c_1}$ and $\rho_i = \rho_i^c$, and substitute ρ_i into LMIs (26)-(31).
3: **for** $j = 1$ **to** $\lfloor 100L \rfloor$
4: Let $\eta_i^j = 0.01j$ and $\eta_i = \eta_i^j$, and substitute η_i into LMIs (26)-(31).
5: Solve LMIs (26)-(31).
6: **if** LMIs (26)-(31) is feasible.
7: Calculate $\|K_i\|$ by $K_i = \mathcal{L}_i(\mathcal{P}_i)^{-1}$ where $\mathcal{L}_i$ and $\mathcal{P}_i$ is the solution of LMIs (26)-(31).
8: Let $K_i^j = K_i$.
9: **else**
10: $\|K_i^j\| = \inf$
11: **end if**
12: **end for**
13: $K_i = \min\{\|K_i^1\|, \|K_i^2\|, \ldots, \|K_i^{\lfloor 100L \rfloor}\|\}$. Let the superscript of the minimum be *min*.
14: Let $\eta_i = \eta_i^{min}$
15: **if** $\rho_i^c \geq \frac{\kappa_i + \chi_i\sqrt{c_2/\lambda_{min}(R)}}{1-\bar{\rho}_{max}}$ where κ_i and χ_i are given as in (13)-(14).
16: $\rho_i = \rho_i^c$
17: **else**
18: Set $\rho_i^c = \rho_i^c + 0.1$. Go to step 2.
19: **end if**

6. SIMULATION

In this section, consider the linearized model of the GE F-404 aircraft engine system as a modified version of the example in [12], whose system matrix is taken as follows:

$$A(t) = \begin{bmatrix} -1.46 & 0 \\ 0.1643 + 0.5g(t) & -0.4 + g(t) \end{bmatrix}, \tag{55}$$

where $g(t)$ is an uncertain model parameter, whose value is taken as -0.5 or -2 according to a Markovian process $r_t \in \{1, 2\}$ with the transition rate matrix as:

$$\Pi = \begin{bmatrix} -0.5 & 0.5 \\ 0.7 & -0.7 \end{bmatrix},$$

Thus, we have the following two subsystems:

Subsystem 1:

$$A_1 = \begin{bmatrix} -1.46 & 0 \\ -0.0857 & -0.9 \end{bmatrix}, B_1 = \begin{bmatrix} 0.05 \\ 0.1 \end{bmatrix}, D_1 = \begin{bmatrix} 0.02 \\ 0.03 \end{bmatrix},$$

$$M_1 = \begin{bmatrix} 0.01 \\ 0.02 \end{bmatrix}, N_1 = \begin{bmatrix} 0.01 & 0.01 \end{bmatrix}, F_1(t) = 0.5\cos(t),$$

Subsystem 2:

$$A_2 = \begin{bmatrix} -1.46 & 0 \\ -0.8357 & -2.4 \end{bmatrix}, B_2 = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}, D_2 = \begin{bmatrix} 0.01 \\ 0.02 \end{bmatrix},$$

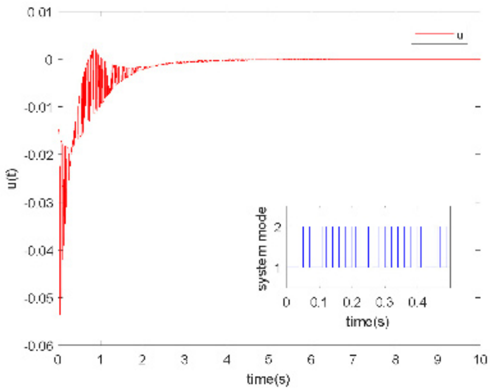

Fig. 1. Control signal $u(t)$ without fault ($\rho = 0$).

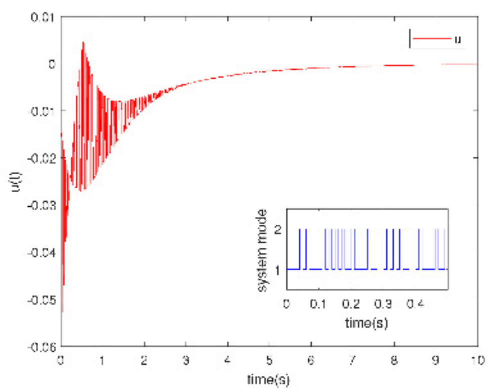

Fig. 2. Control signal $u(t)$ with faults.

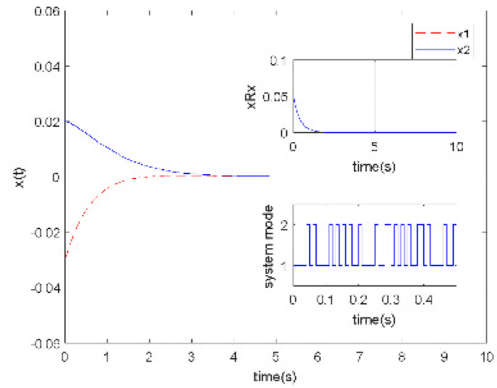

Fig. 3. Trajectories of state $x(t)$ with faults.

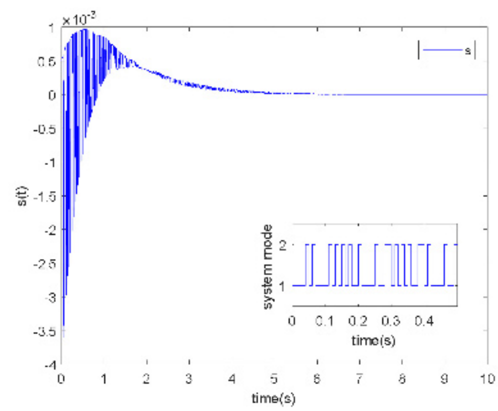

Fig. 4. Sliding variable $s(t)$ with faults.

$$M_2 = \begin{bmatrix} 0.01 \\ 0.02 \end{bmatrix}, \; N_2 = \begin{bmatrix} 0.02 & 0.02 \end{bmatrix}, \; F_1(t) = 0.5\sin(t),$$

and $f(x(t),t,i) = 0.02\sin(\sqrt{x_1^2 + x_2^2})(i = 1,2)$, $w(t) = 0.15e^{-t}$, with $\delta = 0.15$ and $\varepsilon_1 = \varepsilon_2 = 0.02$. The actuator faults occur with bounds $\underline{\rho} = 0.5$, $\bar{\rho} = 0.9$ in this example.

Now, the objective of this work is to design an SMC law to achieve the FTB with $c_1 = 2$, $c_2 = 12$, $T = 10$ and $R = \mathrm{diag}\{40, 36\}$ for the MJSs (1) subject to actuator faults. To this end, by using the **Search Algorithm**, the parameters ρ_i and η_i can be obtained as $\rho_1 = 0.2555$, $\rho_2 = 0.6488$, $\eta_1 = 0.03$, $\eta_2 = 0.01$, and the parameter μ_i in (11) is given as $\mu_1 = 3.2100$, $\mu_2 = 1.3433$. By solving the LMIs in Theroem 2, we obtain $c^* = 10.5979$ and

$$K_1 = [-0.0458 \;\; 0.0351], \; K_2 = [-0.2263 \;\; -0.3372].$$

Then, for the chosen matrices $X_1 = \mathrm{diag}\{4, 1\}$ and $X_2 = \mathrm{diag}\{1.5, 2\}$, the simulation results with initial state $x_0 = [-0.03 \;\; 0.02]^T$ are shown in Figs. 1-4, when the system is subject to actuator faults or not, respectively. It is seen from Fig. 2 the control signal has greater chattering when the actuator degradation occur. It is shown in Fig. 3 that the closed-loop system can not only attain the FTB over the interval $[0, 10]$, but also has a quick convergence despite the effect of mode switching and actuator faults.

7. CONCLUSION

In this paper, we have investigated the FTB problem via SMC methods for a class of MJSs subject to actuator faults. By using a partitioning strategy, the sufficient conditions achieving FTB during both reaching phase and sliding motion phase have been obtained, respectively. In practical application, the actuator faults may happen with stuck and outage as in [25], which is worth further investigating in future work.

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