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# LP Conditions for Stability and Stabilization of Positive 2D Discrete State-delayed Roesser Models

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**Abstract:** This paper deals with the stability and stabilization problems for positive 2D systems described by a linear discrete-time Roesser model with delays. A linear programming (LP) approach is used to establish the necessary and sufficient conditions for asymptotic stability of the positive 2D state delayed Roesser model. Furthermore, a design procedure for memory, non-negative memory and memoryless state feedback controllers is given by solving a certain LP problem. Two examples are included to illustrate the effectiveness of the proposed results.

**Keywords:** Discrete linear state-delayed 2D systems, linear programing, memory controller, memoryless controller, positive 2D Roesser model, stability, stabilization.

## 1. INTRODUCTION

In the literature, different 2D state-space models have been proposed, the most popular 2D linear discrete-time systems were introduced by Attasi [1], Roesser [2], Fornasini and Marchesini [3] and Kurek [4]. The extension of 2D Roesser model to positive 2D Roesser model has been introduced in [5]. Positive 2D Roesser systems are characterized by two non-negative independent variables propagating information in two independent directions, and have found applications in iteration learning control [6, 7], digital data filtering [8, 9], distributed and parallel computing [10], analysis of iterative algorithms [11], river pollution and self-purification process [12] and image processing [2].

A great number of results on the stability analysis for positive 2D systems have been obtained in the literature [13–19]. The choise of the Lyapunov functions for positive 2D Roesser model has been investigated in [13]. The internal stability of positive 2D systems have been investigated in [17]. LMI approach to checking stability of positive 2D systems have been proposed in [15, 16]. The stability of positive 2D systems described by the Roesser model and the synthesis of state-feedback controllers have been considered in [14, 20]. However, few results have reported in literature on positive 2D time-delay systems [18, 19].

In this paper, we are concerned with the control problem of positive 2D state delayed systems described by the Roesser model. Firstly, by transforming the original positive 2D state-delayed Roesser model into a system without delays, a necessary and sufficient condition on the augmented system matrix was derived for the asymptotic stability of positive 2D state-delayed Roesser model. On the other hand, we propose a simple numerical method for a complete treatment of the stabilization problem of positive 2D state-delayed Roesser systems. This method is based on the Linear Programming (LP) framework, which has been successfully applied for checking asymptotic stability, design of state feedback controllers and observers construction for positive 1D systems [21–23]. In addition, based on this approach, we also provide LP necessary and sufficient conditions for the stabilization problems with memory, non-negative memory and memoryless controllers. However, the stabilization problem is not fully investigated and still not completely solved.

Based on numerical experience, when dealing with matrices of high dimensions or with large time delays, the LP approach becomes computationally efficient and better than the LMI approach. Another advantage of the LP approach is the design of memoryless feedback controller, since, it is more difficult to impose such a controller structure in the augmented state space models [24]. In addition, our approach can be also applied to 2D state delayed Roesser models which are not positive in open-loop.

This paper is organized as follows: In Section 2 basic definitions and theorems concerning positive 2D Roesser systems without delays are given. Section 3, gives delay dependent and delay independent necessary and sufficient conditions in terms of LP problem for asymptotic stability of positive 2D Roesser systems with delays. Section 4, contains our main results, and provides synthesis of memory, non-negative memory and memoryless controllers for forced 2D Roesser system with delays. Section 5 gives

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numerical examples. Concluding remarks are given in Section 6.

The following notation will be used: $\mathbb{R}^{n\times m}$ denotes the set of $n\times m$ real matrices, the set of real $n\times m$ matrices with non-negative entries will be denoted by $\mathbb{R}_+^{n\times m}$ and the set of non-negative integers by $\mathbb{Z}_+$; the $n\times n$ identity matrix will be denoted by $I_n$, the $n\times m$ zero matrix will be denoted by $0_{n\times m}$, $1_n$ denotes a column vector of $n$-entries equal to one and **vec**(M) denotes the vector column of a matrix $M$. $\rho(M)$ denotes the spectral radius of a matrix $M\in\mathbb{R}^{n\times n}$ and is defined as: $\rho(M)=\max\{|\lambda_1|,\dots,|\lambda_n|\}$, where $\lambda_1,\dots,\lambda_n$ are the eigenvalues of $M$.

## 2. PRELIMINARIES RESULTS

Consider the autonomous 2D Roesser model described by the following state space equation

$$\begin{bmatrix} x^h_{i+1,j} \\ x^v_{i,j+1} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x^h_{i,j} \\ x^v_{i,j} \end{bmatrix}, \quad i,j\in\mathbb{Z}_+, \tag{1}$$

where $x^h_{i,j}\in\mathbb{R}^{n_1}$ and $x^v_{i,j}\in\mathbb{R}^{n_2}$ are the horizontal and vertical state vectors at the point $(i,j)$, and $A_{kl}\in\mathbb{R}^{n_k\times n_l}$, $k$, $l=1,2$, are known matrices.

The Boundary conditions for (1) have the form

$$\begin{cases} x^h_{0,j}\in\mathbb{R}^{n_1}, \forall j\in\mathbb{Z}_+ \\ x^v_{i,0}\in\mathbb{R}^{n_2}, \forall i\in\mathbb{Z}_+. \end{cases} \tag{2}$$

**Definition 1:** System (1) is called a positive 2D Roesser model if all the trajectories generated by (1), with non-negative boundary conditions (2) remain non-negative.

**Definition 2:** A real matrix $M$ is called a non-negative matrix ($M\in\mathbb{R}_+^{n\times q}$) if all its elements are non-negative $m_{ij}\geq 0$, $i=1,\dots,n$, $j=1,\dots,q$.

**Proposition 1** [25]**:** The 2D Roesser model (1) is positive if and only if the matrix $\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ is non-negative, or equivalently, the matrices $A_{11}$, $A_{12}$, $A_{21}$ and $A_{22}$ are non-negative.

**Definition 3** [26]**:** A positive 2D Roesser system described by (1) is called asymptotically stable if the state evolution corresponding to any set of non-negative boundary conditions (2) asymptotically tends to zero, i.e.,

$$\lim_{i,j\to\infty} x_{i,j}=0.$$

**Theorem 1** [14, 27]**:** Assume that system (1) is positive. Then the following statements are equivalent

**(i)** System (1) is asymptotically stable.

**(ii)** $\rho\left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\right)<1$.

**(iii)** There exist vectors $\lambda_1\in\mathbb{R}^{n_1}$ and $\lambda_2\in\mathbb{R}^{n_2}$ such that

$$\begin{cases} (A_{11}-I_{n_1})\lambda_1+A_{12}\lambda_2<0, \\ (A_{22}-I_{n_2})\lambda_2+A_{21}\lambda_1<0, \\ \lambda_1>0, \lambda_2>0. \end{cases} \tag{3}$$

## 3. POSITIVE 2D ROESSER MODEL WITH DELAYS

In this section, we address the problem of positivity and asymptotic stability for positive 2D Roesser systems with time-delays.

Next, consider the following autonomous 2D Roesser model with $q$ delays

$$\begin{bmatrix} x^h_{i+1,j} \\ x^v_{i,j+1} \end{bmatrix} = \sum_{t=0}^{q} \begin{bmatrix} A^t_{11} & A^t_{12} \\ A^t_{21} & A^t_{22} \end{bmatrix} \begin{bmatrix} x^h_{i-t,j} \\ x^v_{i,j-t} \end{bmatrix}, \quad i,j\in\mathbb{Z}_+, \tag{4}$$

where $x^h_{i,j}\in\mathbb{R}^{n_1}$ and $x^v_{i,j}\in\mathbb{R}^{n_2}$ are the horizontal and vertical state vectors at the point $(i,j)$. The matrices $A^t_{11}$, $A^t_{12}$, $A^t_{21}$, $A^t_{22}$, $t=0,\dots,q$ are known with appropriate dimension.

The most natural method to analyze a positive 2D system with delays is the transformation of such 2D system into an equivalent non-delayed 2D system, and then inspect the augmented matrix. Therefore, the 2D system represented by (4) can be transformed in the following

$$\begin{bmatrix} \tilde{x}^h_{i+1,j} \\ \tilde{x}^v_{i,j+1} \end{bmatrix} = \tilde{A} \begin{bmatrix} \tilde{x}^h_{i,j} \\ \tilde{x}^v_{i,j} \end{bmatrix}, \quad i,j\in\mathbb{Z}_+, \tag{5}$$

by using the augmented vectors

$$\tilde{x}^h_{i,j} = \begin{bmatrix} x^h_{i,j} \\ x^h_{i-1,j} \\ \vdots \\ x^h_{i-q,j} \end{bmatrix} \quad \text{and} \quad \tilde{x}^v_{i,j} = \begin{bmatrix} x^v_{i,j} \\ x^v_{i,j-1} \\ \vdots \\ x^v_{i,j-q} \end{bmatrix},$$

where, the matrix $\tilde{A}$ is given by

$$\tilde{A} = \left[\begin{array}{ccc|ccc} A^0_{11} & \dots & A^q_{11} & A^0_{12} & \dots & A^q_{12} \\ I_{q*n_1} & & 0 & 0 & & 0 \\ \hline A^0_{21} & \dots & A^q_{21} & A^0_{22} & \dots & A^q_{22} \\ 0 & & 0 & I_{q*n_2} & & 0 \end{array}\right] \in\mathbb{R}^{N\times N}. \tag{6}$$

Finally, the 2D Roesser model with $q$ delays (4) has been reduced to an equivalent 2D Roesser system without delays, but with higher dimension $N=(q+1)(n_1+n_2)$.

Applying Proposition 1 to the 2D system (5), we obtain the following result.

**Proposition 2:** The 2D Roesser model with $q$ delays (4) is positive if and only if the matrix $\tilde{A}\in\mathbb{R}^{N\times N}$ is non-negative, or equivalently, the matrices $A^t_{11}$, $A^t_{12}$, $A^t_{21}$ and $A^t_{22}$ are non-negative $\forall t=0,\dots,q$.

Also, by considering the new 2D system (5), we are in place to announce the following necessary and sufficient conditions for asymptotic stability of the positive 2D state delayed Roesser model (4). From Theorem 1 applied to the 2D system (5), we have the following result.

**Theorem 2:** The positive 2D Roesser model (4) with $q$ delays is asymptotically stable if and only if one of the following equivalent conditions holds.

**(i)** The positive 2D Roesser model (5) is asymptotically stable.

**(ii)** $\rho(\tilde{A}) < 1$.

**(iii)** There exist a vector $\lambda \in \mathbb{R}^{(q+1)*(n_1+n_2)}$ such that

$$\begin{cases} (\tilde{A} - I_N)\lambda < 0, \\ \lambda > 0. \end{cases} \tag{7}$$

In what follows, we present delay dependent necessary and sufficient conditions with regard to the asymptotic stability of the positive 2D system described by the Roesser model (4).

**Theorem 3:** The positive 2D Roesser system (4) with $q$ delays is asymptotically stable if and only if the following LP problem in the variables $\lambda_1^0 \in \mathbb{R}^{n_1}$, ..., $\lambda_1^q \in \mathbb{R}^{n_1}$, $\lambda_2^0 \in \mathbb{R}^{n_2}$, ..., $\lambda_2^q \in \mathbb{R}^{n_2}$ is feasible.

$$\begin{cases} (A_{11}^0 - I_{n_1})\lambda_1^0 + \sum_{t=1}^{q} A_{11}^t \lambda_1^t + \sum_{t=0}^{q} A_{12}^t \lambda_2^t < 0, \\ (A_{22}^0 - I_{n_2})\lambda_2^0 + \sum_{t=1}^{q} A_{22}^t \lambda_2^t + \sum_{t=0}^{q} A_{21}^t \lambda_1^t < 0, \\ \lambda_1^t < \lambda_1^{t+1}, \ t = 0, \ldots, q-1, \\ \lambda_2^t < \lambda_2^{t+1}, \ t = 0, \ldots, q-1, \\ \lambda_1^t > 0, \ t = 0, \ldots, q, \\ \lambda_2^t > 0, \ t = 0, \ldots, q. \end{cases} \tag{8}$$

**Proof:** To show this, we take into account that the 2D system (4) is positive and asymptotically stable. Note that, the positive 2D state-delayed Roesser model (4) is asymptotically stable if and only if the 2D system (5) is asymptotically stable. Then, applying to the 2D system (5) Theorem 2, we have that the 2D system (4) is asymptotically stable if and only if there exists $\lambda \in \mathbb{R}^N$ such that LP conditions (7) holds. Now, by using the expression of $\tilde{A}$ given in (6) and defining $\lambda = [\lambda_1^0 \ldots \lambda_1^q \, \lambda_2^0 \ldots \lambda_2^q]^T$, with this change of variable, the inequalities (7) are effectively the same inequalities in the LP constraints (8). Finally, the reverse implication can be trivially obtained by the simple matrix manipulation shown above. Thus, the proof is complete. □

Now, we give necessary and sufficient delay independent conditions for the asymptotic stability of positive 2D system described by the Roesser model (4).

**Theorem 4:** The positive Roesser system (4) with $q$ delays is asymptotically stable if and only if the following LP problem in the variables $\lambda_1 \in \mathbb{R}^{n_1}$ and $\lambda_2 \in \mathbb{R}^{n_2}$ is feasible.

$$\begin{cases} \left(\sum_{t=0}^{q} A_{11}^t - I_{n_1}\right)\lambda_1 + \sum_{t=0}^{q} A_{12}^t \lambda_2 < 0, \\ \left(\sum_{t=0}^{q} A_{22}^t - I_{n_2}\right)\lambda_2 + \sum_{t=0}^{q} A_{21}^t \lambda_1 < 0, \\ \lambda_1 > 0, \ \lambda_2 > 0. \end{cases} \tag{9}$$

**Proof:** Form Theorem 3 we have that, the positive 2D system (4) is asymptotically stable if and only if there exists vectors $\lambda_1^t > 0$ and $\lambda_2^t > 0$, $\forall t = 0, \ldots, q$ such that LP conditions (8) holds. Now, by taking $\lambda_1 = \lambda_1^0$ and $\lambda_2 = \lambda_2^0$, and by using the fact that $\lambda_1 < \lambda_1^t$ and $\lambda_2 < \lambda_2^t$, $\forall t = 1, \ldots, q$ combined with the non-negativity of the matrices $A_{11}^t$, $A_{12}^t$, $A_{21}^t$, $A_{22}^t$, $t = 0, \ldots, q$, we can show that

$$\begin{cases} \left(\sum_{t=0}^{q} A_{11}^t - I_{n_1}\right)\lambda_1 + \sum_{t=0}^{q} A_{12}^t \lambda_2 \\ < (A_{11}^0 - I_{n_1})\lambda_1^0 + \sum_{t=1}^{q} A_{11}^t \lambda_1^t + \sum_{t=0}^{q} A_{12}^t \lambda_2^t \\ < 0, \\ \left(\sum_{t=0}^{q} A_{22}^t - I_{n_2}\right)\lambda_2 + \sum_{t=0}^{q} A_{21}^t \lambda_1 \\ < (A_{22}^0 - I_{n_2})\lambda_2^0 + \sum_{t=1}^{q} A_{22}^t \lambda_2^t + \sum_{t=0}^{q} A_{21}^t \lambda_1^t \\ < 0, \\ \lambda_1 > 0, \ \lambda_2 > 0. \end{cases} \tag{10}$$

Finally, the reverse implication can be trivially obtained. Thus, the proof is complete. □

**Remark 1:** When $A_{11}^i = 0$, $A_{12}^i = 0$, $A_{21}^i = 0$ and $A_{22}^i = 0$, $\forall i = 1, \ldots, q$, the derived conditions in Theorem 4 ensures the asymptotic stability of positive 2D Roesser model without delays.

**Remark 2:** Theorem 4 shows that, the magnitude of delays has no any effect on the asymptotic stability of the positive 2D Roesser model (4).

**Remark 3:** Theorem 4 reveals the important difference between positive 2D Roesser model with delays and general 2D Roesser model with delays in terms of asymptotic stability. Since, the asymptotic stability for general 2D Roesser model with delays is closely related to the magnitude of delays.

**Remark 4:** It is now well-established that an LP program can be solved in polynomial time. Then, the computational complexity of inequalities (9) and (8) is polynomial time due to the fact that the corresponding equations are all presented by LP problem.

**Remark 5:** The number of decision variables in LP conditions (9) of Theorem 4 is $n_1+n_2$, and, it is fewer than those in LP conditions (8) (or (7)) of Theorem 3 (or Theorem 2) respectively, which is equal to $(q+1)*(n_1+n_2)$ in LP conditions (8) (or (7)).

## 4. STABILIZATION OF 2D ROESSER SYSTEMS WITH DELAYS

In this section, we suppose that all the states are available, and we develop the main results of stabilization for 2D state delayed Roesser model. We restrict our attention to memory state feedback controllers, and we develop necessary and sufficient conditions for positivity and asymptotic stability of closed-loop system.

Next, let consider the following forced 2D state delayed Roesser model

$$\begin{bmatrix} x^h_{i+1,j} \\ x^v_{i,j+1} \end{bmatrix} = \sum_{t=0}^{q} \begin{bmatrix} A^t_{11} & A^t_{12} \\ A^t_{21} & A^t_{22} \end{bmatrix} \begin{bmatrix} x^h_{i-t,j} \\ x^v_{i,j-t} \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u_{i,j}, \tag{11}$$

where $i,\, j \in \mathbb{Z}_+$, $x^h_{i,j} \in \mathbb{R}^{n_1}$ and $x^v_{i,j} \in \mathbb{R}^{n_2}$ are the horizontal and the vertical state vectors at the point $(i,j)$ and $u(i,j) \in \mathbb{R}^m$ is the control input at the point $(i,j)$. The matrices $A^t_{11}, A^t_{12}, A^t_{21}, A^t_{22}$, $t=0,\dots,q$, $B_1$ and $B_2$ are known with appropriate dimension.

### 4.1. Memory controller

The problem addressed in the following, is that of designing a memory state feedback controller of the form

$$u_{i,j} = \sum_{t=0}^{q} \begin{bmatrix} K^t_1 & K^t_2 \end{bmatrix} \begin{bmatrix} x^h_{i-t,j} \\ x^v_{i,j-t} \end{bmatrix} \tag{12}$$

for which the closed-loop system is positive and asymptotically stable.

Applying the control (12) to the 2D system (11) yields the closed-loop system

$$\begin{bmatrix} x^h_{i+1,j} \\ x^v_{i,j+1} \end{bmatrix} = \sum_{t=0}^{q} \begin{bmatrix} A^t_{11}+B_1K^t_1 & A^t_{12}+B_1K^t_2 \\ A^t_{21}+B_2K^t_1 & A^t_{22}+B_2K^t_2 \end{bmatrix} \begin{bmatrix} x^h_{i-t,j} \\ x^v_{i,j-t} \end{bmatrix}. \tag{13}$$

Our objective for designing the controller (12) is to simultaneously stabilize and guarantee the positivity of (13). Hence, with regards to the results in Proposition 2 and Theorem 4, we need to find necessary and sufficient conditions on matrices $A^t_{11}, A^t_{12}, A^t_{21}, A^t_{22}$, $t=0,\dots,q$, $B_1$ and $B_2$, such that there exists matrices $K^t_1$ and $K^t_2$, $t=0,\dots,q$ satisfying positivity and asymptotic stability of the closed-loop system (13).

In the following, necessary and sufficient conditions are developed for positivity and asymptotic stability of the closed-loop system (13).

**Theorem 5:** The closed-loop system (13) is positive and asymptotically stable if and only if the following LP problem in the variables $\lambda_1 \in \mathbb{R}^{n_1}$, $\lambda_2 \in \mathbb{R}^{n_2}$, $Z^0_1 \in \mathbb{R}^{m\times n_1}$, $\dots$, $Z^q_1 \in \mathbb{R}^{m\times n_1}$, $Z^0_2 \in \mathbb{R}^{m\times n_2}$, $\dots$, $Z^q_2 \in \mathbb{R}^{m\times n_2}$ is feasible.

$$\begin{cases} M_0\lambda_1 + \sum_{t=0}^{q} A^t_{12}\lambda_2 + B_1\left(\sum_{t=0}^{q} Z^t_1 1_{n_1} + \sum_{t=0}^{q} Z^t_2 1_{n_2}\right) \\ <0, \\ M_1\lambda_2 + \sum_{t=0}^{q} A^t_{21}\lambda_1 + B_2\left(\sum_{t=0}^{q} Z^t_1 1_{n_1} + \sum_{t=0}^{q} Z^t_2 1_{n_2}\right) \\ <0, \\ A^t_{11}\mathbf{diag}(\lambda_1) + B_1Z^t_1 \geq 0,\ t=0,\dots,q, \\ A^t_{12}\mathbf{diag}(\lambda_2) + B_1Z^t_2 \geq 0,\ t=0,\dots,q, \\ A^t_{21}\mathbf{diag}(\lambda_1) + B_2Z^t_1 \geq 0,\ t=0,\dots,q, \\ A^t_{22}\mathbf{diag}(\lambda_2) + B_2Z^t_2 \geq 0,\ t=0,\dots,q, \\ \lambda_1 > 0,\ \ \lambda_2 > 0, \end{cases} \tag{14}$$

where $M_0 = \sum_{t=0}^{q} A^t_{11} - I_{n_1}$ and $M_1 = \sum_{t=0}^{q} A^t_{22} - I_{n_2}$.

Moreover, the gain matrices $K^t_1$ and $K^t_2$, $t=0,\dots,q$ are computed as

$$K^t_1 = Z^t_1\mathbf{diag}(\lambda_1)^{-1},\ \ K^t_2 = Z^t_2\mathbf{diag}(\lambda_2)^{-1},$$
$$t=0,\dots,q.$$

**Proof:** We take into account that the closed-loop system (13) is positive and stable. By using Theorem 4, we have that the closed-loop system (13) is stable if and only if there exists $\lambda_1 \in \mathbb{R}^{n_1}$, $\lambda_2 \in \mathbb{R}^{n_2}$ such that

$$\begin{cases} \left(\sum_{t=0}^{q}(A^t_{11}+B_1K^t_1) - I_{n_1}\right)\lambda_1 + \sum_{t=0}^{q}(A^t_{12}+B_1K^t_2)\lambda_2 \\ <0, \\ \left(\sum_{t=0}^{q}(A^t_{22}+B_2K^t_2) - I_{n_2}\right)\lambda_2 + \sum_{t=0}^{q}(A^t_{21}+B_2K^t_1)\lambda_1 \\ <0, \\ \lambda_1 > 0,\ \ \lambda_2 > 0. \end{cases}$$

Now, define $K^t_1 = Z^t_1\mathbf{diag}(\lambda_1)^{-1}$ and $K^t_2 = Z^t_2\mathbf{diag}(\lambda_2)^{-1}$, $\forall t=0,\dots,q$, with these change of variables, the above inequalities are effectively the first two inequalities in the LP constraints (14). The other inequalities in the LP constraints (14) are obtained as follows. Note that the matrices $A^t_{11}+B_1K^t_1$, $A^t_{12}+B_1K^t_2$, $A^t_{21}+B_2K^t_1$, $A^t_{22}+B_2K^t_2$, $t=0,\dots,q$ are non-negative if $(A^t_{11}+B_1K^t_1)\mathbf{diag}(\lambda_1)$, $(A^t_{12}+B_1K^t_2)\mathbf{diag}(\lambda_2)$, $(A^t_{21}+B_2K^t_1)\mathbf{diag}(\lambda_1)$ and $(A^t_{22}+B_2K^t_2)\mathbf{diag}(\lambda_2)$, $t=0,\dots,q$ are non-negative matrices (because $\mathbf{diag}(\lambda_i) \geq 0,\ i=1,2$). Thus, by recalling that $K^t_1 = Z^t_1\mathbf{diag}(\lambda_1)^{-1}$ and $K^t_2 = Z^t_2\mathbf{diag}(\lambda_2)^{-1}$, $t=0,\dots,q$, the above inequalities are equivalent to the rest of inequalities in the LP constraints (14).

Finally, the reverse implication follows the same line of arguments and then is omitted. Thus, the proof is complete. □

### 4.2. Non-negative memory controller

In the following, a non-negative memory state feedback controller can be handled by using a similar LP approach.

**Theorem 6:** The closed-loop system (13) is positive and asymptotically stable with a non-negative memory state feedback controller if and only if the following LP problem in the variables $\lambda_1 \in \mathbb{R}^{n_1}$, $\lambda_2 \in \mathbb{R}^{n_2}$, $Z_1^0 \in \mathbb{R}^{m\times n_1}$, $\dots$, $Z_1^q \in \mathbb{R}^{m\times n_1}$, $Z_2^0 \in \mathbb{R}^{m\times n_2}$, $\dots$, $Z_2^q \in \mathbb{R}^{m\times n_2}$ is feasible.

$$\begin{cases} M_0\lambda_1 + \sum_{t=0}^{q} A_{12}^t\lambda_2 + B_1\left(\sum_{t=0}^{q} Z_1^t 1_{n_1} + \sum_{t=0}^{q} Z_2^t 1_{n_2}\right) \\ < 0, \\ M_1\lambda_2 + \sum_{t=0}^{q} A_{21}^t\lambda_1 + B_2\left(\sum_{t=0}^{q} Z_1^t 1_{n_1} + \sum_{t=0}^{q} Z_2^t 1_{n_2}\right) \\ < 0, \\ A_{11}^t\mathbf{diag}(\lambda_1) + B_1 Z_1^t \geq 0,\ t = 0,\dots,q, \\ A_{12}^t\mathbf{diag}(\lambda_2) + B_1 Z_2^t \geq 0,\ t = 0,\dots,q, \\ A_{21}^t\mathbf{diag}(\lambda_1) + B_2 Z_1^t \geq 0,\ t = 0,\dots,q, \\ A_{22}^t\mathbf{diag}(\lambda_2) + B_2 Z_2^t \geq 0,\ t = 0,\dots,q, \\ Z_1^t \geq 0, t = 0,\dots,q, \\ Z_2^t \geq 0, t = 0,\dots,q, \\ \lambda_1 > 0,\ \ \lambda_2 > 0, \end{cases} \tag{15}$$

where $M_0 = \sum_{t=0}^{q} A_{11}^t - I_{n_1}$ and $M_1 = \sum_{t=0}^{q} A_{22}^t - I_{n_2}$.

Moreover, the gain matrices $K_1^t$ and $K_2^t$, $t = 0,\dots,q$ are computed as

$$K_1^t = Z_1^t\mathbf{diag}(\lambda_1)^{-1},\ \ K_2^t = Z_2^t\mathbf{diag}(\lambda_2)^{-1},\\ t = 0,\dots,q.$$

**Proof:** Note that the control law (12) is nonnegative if and only if the matrices $K_1^t$ and $K_2^t$, are nonnegative $\forall t = 1,\dots,q$. Then, by using the change of variables $K_1^t = Z_1^t\mathbf{diag}(\lambda_1)^{-1}$, $K_2^t = Z_2^t\mathbf{diag}(\lambda_2)^{-1}$, $t = 0,\dots,q$, we have necessarily that matrices $Z_1^t$ and $Z_2^t$ are nonnegative $\forall t = 0,\dots,q$. To complete this proof, we can follow the same line of arguments as in the Proof of Theorem 5. □

**Remark 6:** A negative state feedback control can be considered, by just imposing $Z_1^t \leq 0$ and $Z_2^t \leq 0$, $\forall t = 0,\dots,q$ instead $Z_1^t \geq 0$ and $Z_2^t \geq 0$, $\forall t = 0,\dots,q$ in the LP problem (15).

### 4.3. Memoryless controller

In the case when we do not have access to the delayed states, or, when the delays are unknown, a memoryless state feedback controller of the form

$$u_{i,j} = \begin{bmatrix} K_1 & K_2 \end{bmatrix} \begin{bmatrix} x_{i-t,j}^h \\ x_{i,j-t}^v \end{bmatrix} \tag{16}$$

can be designed for the 2D Roesser system (11). In this case, the matrices $A_{11}^t$, $A_{12}^t$, $A_{21}^t$ and $A_{22}^t$ must be nonnegative $\forall t = 1,\dots,q$.

The following result can be derived from Theorem 5 by taking $K_1^t = 0$ and $K_2^t = 0$ for all $t = 1,\dots,q$.

**Theorem 7:** Assume that the matrices $A_{11}^t$, $A_{12}^t$, $A_{21}^t$ and $A_{22}^t$ are non-negative $\forall t = 1,\dots,q$. Then, there exist a memoryless state feedback controller of the form (16) such that the closed-loop system (13) is positive and asymptotically stable if and only if the following LP problem in the variables $\lambda_1$, $\lambda_2$, $Z_1$, $Z_2$, is feasible.

$$\begin{cases} \left(\sum_{t=0}^{q} A_{11}^t - I_{n_1}\right)\lambda_1 + \sum_{t=0}^{q} A_{12}^t\lambda_2 + B_1\left(Z_1 1_{n_1} + Z_2 1_{n_2}\right) \\ < 0, \\ \left(\sum_{t=0}^{q} A_{22}^t - I_{n_2}\right)\lambda_2 + \sum_{t=0}^{q} A_{21}^t\lambda_1 + B_2\left(Z_1 1_{n_1} + Z_2 1_{n_2}\right) \\ < 0, \\ A_{11}^0\mathbf{diag}(\lambda_1) + B_1 Z_1 \geq 0, \\ A_{12}^0\mathbf{diag}(\lambda_2) + B_1 Z_2 \geq 0, \\ A_{21}^0\mathbf{diag}(\lambda_1) + B_2 Z_1 \geq 0, \\ A_{22}^0\mathbf{diag}(\lambda_2) + B_2 Z_2 \geq 0, \\ \lambda_1 > 0,\ \ \lambda_2 > 0. \end{cases} \tag{17}$$

Moreover, the gain matrices $K_1$ and $K_2$ are computed as

$$K_1 = Z_1\mathbf{diag}(\lambda_1)^{-1},\ K_2 = Z_2\mathbf{diag}(\lambda_2)^{-1}.$$

**Remark 7:** In comparison to other methods, for example the one based on linear matrix inequalities (LMIs), the LP approach is easier, it leads to less conservative conditions than the LMI approach, and usually it possesses a numerical advantage on the computational complexity since the number of decision variables in LP conditions are usually much fewer than those in LMI conditions [28].

**Remark 8:** Many interior points methods have been devoted to solve LP problems (see for instance Sedumi solver). Also, there exist other solvers that can be used to solve large size LP problems such as Cplex.

**Remark 9:** Recently, the problems of 2D dissipative control and filtering have been investigated in [29] for a linear discrete-time Roesser model without delays, and the robust stochastic stability analysis has been analyzed in [30] for 2D discrete state-multiplicative noisy systems (SMNSs) in the Roesser form. The references mentioned above do not tackle the positive characteristics of the 2D

Roesser model. Also, these existing methods might lead to negative horizontal and vertical states, so they are not adequate for stabilizing positive 2D state delayed Roesser model that are inherently non-negative.

### 4.4. Standard LP form

Previously, we have seen that the provided stability tests (10) and stabilization results (14), (15) and (17) are formulated as linear matrix constraints. We would like to show that these LP's can be re-expressed in the well-known standard form, which involves vector constraints with a single unknown vector variable. This can be done by using the Kronecker product $\otimes$ and **vec** operation.

In what follows, we propose a standard LP form for the problem (17)

$$\begin{bmatrix} \sum_{t=0}^{q} A_{11}^t - I_{n_1} & \sum_{t=0}^{q} A_{12}^t & 1_{n_1}^T \otimes B_1 & 1_{n_2}^T \otimes B_1 \\ \sum_{t=0}^{q} A_{21}^t & \sum_{t=0}^{q} A_{22}^t - I_{n_2} & 1_{n_1}^T \otimes B_2 & 1_{n_2}^T \otimes B_2 \\ -I_{n_1} & 0_{n_1\times n_2} & 0_{n_1\times r_3} & 0_{n_1\times r_4} \\ 0_{n_2\times n_1} & -I_{n_2} & 0_{n_2\times r_3} & 0_{n_2\times r_4} \end{bmatrix} w < 0,$$

$$\begin{bmatrix} F_0 & 0_{r_1\times n_2} & -I_{n_1}\otimes B_1 & 0_{r_1\times r_4} \\ 0_{r_5\times n_1} & F_1 & 0_{r_5\times r_3} & -I_{n_2}\otimes B_1 \\ F_2 & 0_{r_5\times n_2} & -I_{n_1}\otimes B_2 & 0_{r_5\times r_4} \\ 0_{r_2\times n_1} & F_3 & 0_{r_2\times r_3} & -I_{n_2}\otimes B_2 \end{bmatrix} w \le 0, \quad (18)$$

where $F_0 = -\sum_{i=1}^{n_1} e_i e_i^T \otimes A_{11}^0 e_i$, $F_1 = -\sum_{i=1}^{n_2} e_i e_i^T \otimes A_{12}^0 e_i$, $F_2 = -\sum_{i=1}^{n_1} e_i e_i^T \otimes A_{21}^0 e_i$, $F_3 = -\sum_{i=1}^{n_2} e_i e_i^T \otimes A_{22}^0 e_i$, $r_1 = n_1 * n_1$, $r_2 = n_2 * n_2$, $r_3 = n_1 * m$, $r_4 = n_2 * m$, $r_5 = n_1 * n_2$, $e_i$ is the canonical vector of $\mathbb{R}^n$ and the new vector variable $w$ is defined as

$$w = [\lambda_1 \quad \lambda_2 \quad \mathbf{vec}(Z_1) \quad \mathbf{vec}(Z_2)]^T.$$

## 5. NUMERICAL EXAMPLES

All numerical examples provided in this section have been solved by using `linprog` function in `Matlab` environment.

Next, we give two examples to illustrate the effectiveness of the proposed methods.

### 5.1. Example 1

We are looking to check the asymptotic stability of the positive 2D state delayed Roesser model (1) with $q = 1$ and the following system matrices

$$\begin{bmatrix} A_{11}^0 & A_{21}^0 \\ A_{21}^0 & A_{22}^0 \end{bmatrix} = \left[\begin{array}{cc|c} 0.1 & 0.1 & 0.1 \\ 0 & 0.2 & 0.2 \\ \hline 0 & 0.2 & 0.1 \end{array}\right],$$

$$\begin{bmatrix} A_{11}^1 & A_{21}^1 \\ A_{21}^1 & A_{22}^1 \end{bmatrix} = \left[\begin{array}{cc|c} 0.2 & 0.1 & 0 \\ 0 & 0.1 & 0.1 \\ \hline 0 & 0.5 & 0.11 \end{array}\right].$$

Firstly, by applying Theorem 3, the following conditions must be satisfied:

$$\begin{bmatrix} -0.9 & 0.1 & 0.2 & 0.1 & 0.1 & 0 \\ 0 & -0.8 & 0 & 0.1 & 0.2 & 0.1 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0.2 & 0 & 0.5 & -0.9 & 0.11 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{bmatrix} < 0.$$

One feasible solution is given by solving the above LP problem or equivalently the LP problem (8) and the vectors $\lambda_1^0, \lambda_1^1, \lambda_2^0$ and $\lambda_2^1$ are $\lambda_1^0 = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 103.6654 \\ 45.8934 \end{bmatrix}$, $\lambda_1^1 = \begin{bmatrix} d_3 \\ d_4 \end{bmatrix} = \begin{bmatrix} 159.1628 \\ 65.7955 \end{bmatrix}$, $\lambda_2^0 = d_5 = 82.7639$ and $\lambda_2^1 = d_6 = 130.0384$.

Secondly, we can also check the asymptotic stability of the above positive 2D system by using Theorem 4. For this purpose, it suffices to use the result of Theorem 4, thus looking for a solution that fulfills the following LP conditions:

$$\begin{bmatrix} -0.7 & 0.2 & 0.1 \\ 0 & -0.7 & 0.3 \\ 0 & 0.7 & -0.79 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} < 0.$$

In this case, the above LP problem or equivalently the LP problem (9) is feasible and the vectors $\lambda_1$ and $\lambda_2$ are $\lambda_1 = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 111.0473 \\ 7.1884 \end{bmatrix}$ and $\lambda_2 = y_3 = 106.0928$.

Finally, if we want to check the asymptotic stability of the above positive 2D system by using Theorem 2, we have to calculate the spectral radius of the matrix $\tilde{A}$ given in (6). Thus

$$\rho(\tilde{A}) = \rho\left(\begin{bmatrix} 0 & 0.1 & 0.2 & 0.1 & 0.1 & 0 \\ 0 & 0.2 & 0 & 0.1 & 0.2 & 0.1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0.2 & 0 & 0.5 & 0.1 & 0.11 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}\right) = 0.8006,$$

which is less than 1. This means that the above positive 2D state delayed Roesser system is asymptotically stable according to Theorems 2, 3 and 4.

### 5.2. Example 2: the thermal process

This section applies our main result on memory state-feedback of 2D positive systems to the model of thermal process as described in [26], which can be expressed in the following partial differential equation with time delays.

$$\frac{\partial T(x,t)}{\partial x} = -\frac{\partial T(x,t)}{\partial t} - a_0 T(x,t) - a_1 T(x,t-\tau) + bu(x,t), \qquad (19)$$

where $T(x,t)$ is the temperature at $x(\text{space}) \in [0, x_f]$ and $t(\text{time}) \in [0,\infty)$, $u(x,t)$ is the input function, $\tau$ is the time delay and $a_0$, $a_1$ and $b$ are real coefficients.
Taking

$$\begin{cases} T(i,j) = T(i\Delta x, j\Delta t), \ u(i,j) = u(i\Delta x, j\Delta t), \\ \dfrac{\partial T(x,t)}{\partial t} \approx \dfrac{T(i,j+1) - T(i,j)}{\Delta t}, \\ \dfrac{\partial T(x,t)}{\partial x} \approx \dfrac{T(i,j) - T(i-1,j)}{\Delta x}. \end{cases}$$

Then, if we define $x^h(i,j) = T(i-1,j)$ and $x^v(i,j) = T(i,j)$. It is easy to verify that (19) can be transformed into the 2D Roesser model (11) with $q = 1$ and

$$\begin{bmatrix} A^0_{11} & A^0_{21} \\ A^0_{21} & A^0_{22} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \dfrac{\Delta t}{\Delta x} & 1 - \dfrac{\Delta t}{\Delta x} - a_0 \Delta t \end{bmatrix},$$

$$\begin{bmatrix} A^1_{11} & A^1_{21} \\ A^1_{21} & A^1_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -a_1 \Delta t \end{bmatrix}, \quad \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} 0 \\ b\Delta t \end{bmatrix}.$$

To illustrate our result, let set, for example $\Delta t = 0.3$, $\Delta x = 0.5$, $a_0 = 0.5$, $a_1 = -2$ and $b = 1$. The obtained system is given as $A^0_{11} = 0$, $A^0_{12} = 1$, $A^0_{21} = 0.6$, $A^0_{22} = 0.25$, $A^1_{11} = 0$, $A^1_{12} = 0$, $A^1_{21} = 0$, $A^1_{22} = 0.6$, $B_1 = 0$, $B_2 = 1$.

It is easy to see that the autonomous system (when $u \equiv 0$) is positive but not asymptotically stable. Our objective is to design a memory state-feedback controller given by (12) such that the closed-loop system (13) is positive and asymptotically stable. Solving the LP problem (14) in Theorem 5 gives rise to

$$\lambda_1 = 126.3752, \ \lambda_2 = 98.3676, \ Z^0_1 = -114.9760,$$
$$Z^1_1 = 1.2377, \ Z^0_2 = -37.4269, \ Z^1_2 = -88.2134.$$

Then, the gain of stabilizing controller (12) are

$$K^0_1 = -0.9098, \ K^1_1 = 0.0098$$
$$K^0_2 = -0.3805, \ K^1_2 = -0.8968.$$

Fig. 1 shows the state responses of the resulting closed-loop system from random non-negative boundary conditions. It can be observed that the closed-loop system is positive and asymptotically stable, which demonstrates the effectiveness of the proposed method.

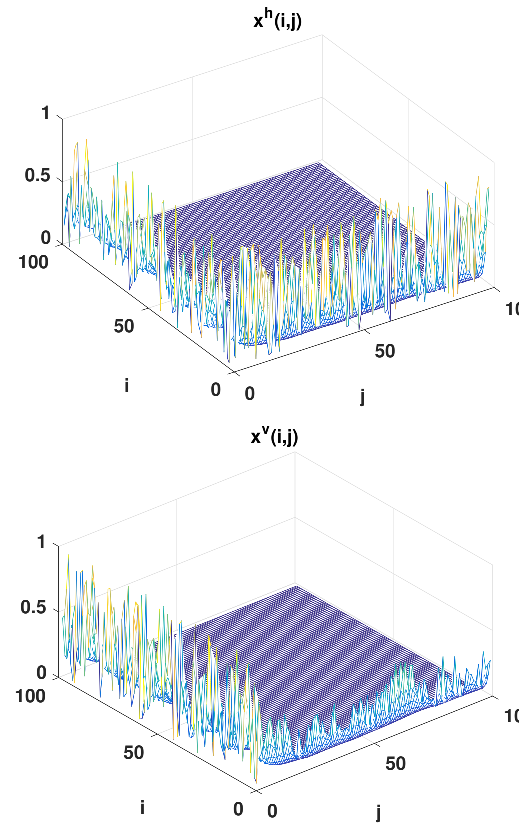


Fig. 1. Horizontal $x^h(i,j)$ and Vertical $x^v(i,j)$ state trajectory.

## 6. CONCLUSIONS

In this paper, we have studied the stability and stabilization problems for positive 2D discrete Roesser model with delays. An LP approach has been provided to checking the asymptotic stability as well as to construct memory, non-negative memory and memoryless state feedback controllers for 2D state-delayed Roesser systems. The stabilizing controllers have been developed to guarantees not only the asymptotic stability of the closed-loop system but also its positivity. Finally, two examples have been included, to demonstrate the application of the obtained results. The results presented in this paper can be extended to positive observer design and controller design by dynamic state-feedback for 2D time-delayed positive 2D Roesser system.

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