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Robust H_∞ Performance of Discrete-time Neural Networks with Uncertainty and Time-varying Delay

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Abstract: In this paper, we are concerned with the robust H_∞ problem for a class of discrete -time neural networks with uncertainties. Under a weak assumption on the activation functional, some novel summation inequality techniques and using a new Lyapunov-Krasovskii (L-K) functional, a delay-dependent condition guaranteeing the robust asymptotically stability of the concerned neural networks is obtained in terms of a Linear Matrix Inequality(LMI). It is shown that this stability condition is less conservative than some previous ones in the literature. The controller gains can be derived by solving a set of LMIs. Finally, two numerical examples result are given to illustrate the effectiveness of the developed theoretical results.

Keywords: H_∞ control, linear matrix inequality, stability, time-varying delay.

1. INTRODUCTION

Neural networks have found a large number of successful applications in various fields of science and engineering. However, it is worth noting that most systems contain digital computers (usually microprocessors or micro controllers) with the necessary input/output hardware to implement the systems. Thus discrete-time system model with time delay plays a significant role in fields of engineering applications. The problems of stability analysis for continue time neural networks and discrete-time neural networks have been extensively studied in recent years and many stability conditions have been reported in the literature [1–5]. Since axonal signal transmission delays often occur in various neural networks, and many also cause undesirable dynamic network behaviors such as oscillation and instability. Therefore, increasing attention has been paid to the problem of stability analysis of neural networks with delays, and a lot of research results have been reported for the neural networks with delays and the references therein see [4–6].

Recently, the stability analysis problems for discrete-time neural networks have received considerable research interests, and many sufficient conditions have been proposed to guarantee the asymptotic and exponential stability of neural networks with various types of time-delay such as constant, time-varying, random and distributed delays, see for example [7–9]. Recently the authors in [10], improved criteria of delay-dependent stability for discrete-time neural networks with leakage delay was studied. Further, it is well known that the connection weights of the neurons are inherently dependent on certain resistance and capacitance values that inevitably bring in uncertainties during the parameter identification process. The deviations and perturbations in parameters have the effect on the performance of neural networks [11, 12]. So, it is important to study the dynamical behaviors of dynamical systems by taking the uncertainties into account. Many scholars have discussed the dynamics of delayed systems with uncertainties, see [13–16].

A great number of control strategies have been proposed for achieving the stability analysis, including adaptive dynamic surface control, sliding mode control [17–20]. In recent years, H_∞ control plays an important role in controller design problems was initially formulated by Zames in 1980s and has found numerous applications in practical engineering systems. More recently, H_∞ control theory has been applied to an actual building in Tokyo, Japan, using a pair of mass dampers to reduce the bending-torsion motion due to earthquakes [21]. In addition, when compared to other control strategies H_∞ control theory is an effective tool to stabilize the uncertain systems with

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time invariant/varying norm-bounded parameter uncertainty in the state and input matrices and applied to minimize the effects of the external disturbance. Naturally, when investigating the discrete time neural networks, an H_∞ performance γ is usually considered [22–24]. In the discrete-time context, there is rapidly growing interest in H_∞ control due to being frequently encountered in many practical engineering systems such as chemical, electronics, process control systems and networked control systems. Meanwhile, the H_∞ control problem for discrete time system has also received much attention and a number of contributions were given in the references, see [25–27]. However, to the best of the authors' knowledge, the robust H_∞ performance of discrete-time neural networks with uncertainty and time-varying delay has not yet received adequate research attention which motivated the recent work.

Motivated by the aforementioned discussion, this paper aims at investigating the robust H_∞ performance of discrete-time neural networks with uncertainty and time-varying delay. The main contributions of this paper are listed as follows: i) The objective of this work is to obtain an H_∞ controller design such that the resulting closed-loop form of neural network is robustly stable with given disturbance attenuation level $\gamma > 0$. ii) The parameter uncertainties are assumed to be norm bounded. By constructing Lyapunov-Krasovskii including the lower and upper delay bound of interval time-varying delay, novel summation inequality approach and LMI technique, we designed robust H_∞ controller such that the resulting closed-loop form of neural network is robustly asymptotically stable with a prescribed H_∞ performance. Moreover the results are formulated in terms of LMIs, which can be easily calculated by MATLAB-LMI control toolbox. (iii) Finally, numerical examples are given to illustrate the effectiveness and applicability of the proposed theories.

Notation: Throughout this paper, N is the set of natural numbers and N^+ stands for the set of nonnegative integers; $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote, respectively, the n dimensional Euclidean space and the set of all $n\times m$ real matrices. The superscript "T" denotes the transpose and $P_1 > 0$ means that P_1 is positive definite. I is the identity matrix with compatible dimension. $diag\{\cdot\}$ stands for a block diagonal matrix. The asterisk * in a matrix is used to denote term that is induced by symmetry.

2. SYSTEM DESCRIPTION AND PRELIMINARIES

Consider the following discrete time-delay neural networks with time varying delays,

$$\begin{aligned} x(k+1) &= -A(k)x(k) + E(k)g(x(k)) \\ &\quad + E_d(k)g(x(k-d(k)) + u(k) + D(k)w(k), \\ z(k) &= Lx(k), \\ x(k) &= \phi(k) \ \text{ for every } k \in [-d_M, 0], \end{aligned} \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the neural state vector, $u(k)$ is the control input, $z(k)$ is the controlled output, and $w(k) \in \mathbb{R}^q$ is the anexogenous disturbance signal which is assumed to belong to $l_2\ [0,\infty)$. $g(x(k)) = [g_1(x_1(k)), g_2(x_2(k)), ..., g_n(x_n(k))]^n$ denotes the neuron activation function, and the positive integer $d(k)$ denotes the time-varying delay satisfying

$$d_m \le d(k) \le d_M \ \text{ for all } \ k \in N, \tag{2}$$

where d_m and d_M are known constant positive scalars. $A(k) = A + \Delta A(k)$, $E(k) = A + \Delta E(k)$, $E_d(k) = E_d + \Delta E_d(k)$ in which $A = diag\{a_1, a_2, ..., a_n\}$represents the state feedback coefficient matrix with $|a_i| < 1, E = (e_{ij})_{n\times n}$, $E_d = (e_{dij})_{n\times n}$, $D = (d_{ij})_{n\times n}$, respectively denotes the connection weights, the delayed connection weights and disturbance weights and L is the known matrix with appropriate dimensions, the initial function $\phi(k)$ is continuous and defined on $[-d_M, 0]$.

$$\begin{aligned} &[\Delta A(k) \quad \Delta E(k) \quad \Delta E_d(k) \quad \Delta D(k)] \\ &= MF(k)[N_1 \quad N_2 \quad N_3 \quad N_4]. \end{aligned}$$

Where M and N_i, $(i = 1,2,3,4)$ are known real constant matrices, and $F(k)$ is the time varying matrix valued function subject to $F^T(k)F(k) \le I, \quad \forall k \in N^+$. where I is the identity matrix with appropriate dimensions.
In this paper, without loss of generality, we make following assumptions for the activation functions:

Assumption 1: Each activation function $g_i(\cdot)$ in (1) is continuous and bounded, and there exist constants F_i^- and F_i^+ such that

$$F_i^- \le \frac{g_i(\ell_1) - g_i(\ell_2)}{\ell_1 - \ell_2} \le F_i^+, \ i = 1,2,...,n \tag{3}$$

where $\ell_1,\ \ell_2 \in \mathbb{R}$, and $\ell_1 \ne \ell_2$.

The following definition and lemmas will be used in the proof of main results.

Definition 1 [1]**:** The discrete time neural networks (1) is said to be robustly stable with given disturbance attenuation level $\hat{\gamma} > 0$, if it is robustly stable under zero initial conditions and satisfies

$$\| z \|_2 \le \hat{\gamma} \| v \|_2 \tag{4}$$

for all every non-zero $v \in l_2(0,\infty)$.

Lemma 1 [2]**:** Given constant matrices $\delta_1\ \delta_2,\ \delta_3$, where $\delta_1 = \delta_1^T > 0$ and $\delta_2 = \delta_2^T > 0$ then $\delta_1 + \delta_3^T \delta_2^{-1} \delta_3 < 0$ if and only if $\begin{bmatrix} \delta_1 & \delta_3^T \\ \delta_3 & -\delta_2 \end{bmatrix} < 0$.

Lemma 2 [6]**:** For any vector $x, y \in \mathbb{R}^n$, matrices A, P, D, E and F are real matrices of appropriate dimensions with $P > 0, F^T F \le I$, and scalar $\lambda > 0$, the following inequalities hold:

(i) $2x^T DFEy \le \lambda^{-1}x^T DD^T x + \lambda y^T E^T Ey$.

(ii) If $P - \lambda DD^T > 0$, then $(A+DFE)^T P^{-1}(A+DFE) \le A^T(P-\lambda DD^T)^{-1}A + \lambda^{-1}E^T E$.

Lemma 3 [4]**:** Consider a given $n\times n$ positive definite matrix $R_2 \ge 0$. Then for all $y_0, y_1, y_2, \ldots, y_n \in \mathbb{R}^n$, the following inequality holds

$$\sum_{k=0}^{n} \Upsilon y_k^T R_2 \Upsilon y_k \ge \frac{1}{n+1}(y_{n+1}-y_0)^T R_2 (y_{n+1}-y_0) + \frac{3}{n+1}\pi_1^T \frac{n+2}{n} R_2 \pi_1,$$

where $\Upsilon y_k = y_{k+1} - y_k$ and $\pi_1 = y_{n+1} + y_0 - \frac{2}{n+2}\sum_{k=0}^{n+1} y_k$.

Lemma 4 [5]**:** For a positive definite symmetric matrix Z_2, any matrix $\tilde{J}$, $d(k) \in [d_1,\ d_2]$ and $r(k) = x(k+1) - x(k)$, the sum term $\Re(k)$ given as $\Re(k) = \sum_{\theta=k-d(k)}^{k-d_1-1} r^T(\theta) Z_2 r(\theta) + \sum_{\theta=k-d_2}^{k-d(k)-1} r^T(\theta) Z_2 r(\theta)$ can be estimated as

$$\begin{aligned} & d_{12}\Re(k) \\ & \ge \hat{\zeta}^T(t)\begin{bmatrix}\hat{\Gamma}_1\\ \hat{\Gamma}_2\end{bmatrix}^T \left(\begin{bmatrix}\check{R} & \tilde{J}\\ * & \check{R}\end{bmatrix} + \begin{bmatrix}\frac{d_2-d(k)}{d_{12}}\hat{M}_1 & 0\\ * & \frac{d(k)-d_1}{d_{12}}\hat{M}_2\end{bmatrix}\right) \\ & \times \hat{\zeta}(t)\begin{bmatrix}\hat{\Gamma}_1\\ \hat{\Gamma}_2\end{bmatrix}, \end{aligned}$$

where $\check{R} = diag\{Z_2,\ 3Z_2\}, \hat{M}_1 = \check{R} - J\check{R}^{-1}J^T, d_{12} = d_2 - d_1, \hat{M}_2 = \check{R} - J^T\check{R}^{-1}J, \zeta(t) = [x^T(k), x^T(k-d_1), x^T(k-d(k)), x^T(k-d_2), \hat{b}_1^T(t), \hat{b}_2^T(t), \hat{b}_3^T(t)]^T$,

$$\hat{b}_1^T(t) = \sum_{\theta=k-d_1}^{k}\frac{x(\theta)}{d_1+1},\quad \hat{b}_2^T(t) = \sum_{\theta=k-d(k)}^{k-d_1}\frac{x(\theta)}{d(k)-d_1+1},$$

$$\hat{b}_3^T(t) = \sum_{\theta=k-d_2}^{k-d(k)}\frac{x(\theta)}{d_2-d(k)+1},\quad \hat{\Gamma}_1 = \begin{bmatrix}\check{e}_2 - \check{e}_3\\ \check{e}_2 + \check{e}_3 - 2\check{e}_6\end{bmatrix},$$

$$\hat{\Gamma}_2 = \begin{bmatrix}\check{e}_3 - \check{e}_4\\ \check{e}_3 + \check{e}_4 - 2\check{e}_7\end{bmatrix},$$

$$\check{e}_s = [0_{n\times(s-1)\times n},\ I_{n\times n},\ 0_{n\times(7-s)n}],\quad s = 1,2,3,\ldots,7.$$

3. MAIN RESULTS

In this section, we study the robust stability results for the DNNs (1) when the disturbance input $w(k) = 0$. Based on Lyapunov technique and the LMI inequality approach, we derive a state feedback controller of the form

$$u(k) = kx(k). \tag{5}$$

For presentation convenience, we denote

$$F_1 = diag\{F_1^- F_1^+,\ F_2^- F_2^+, \ldots,\ F_n^- F_n^+\},$$

$$F_2 = diag\left\{\frac{F_1^- + F_1^+}{2},\ \frac{F_2^- + F_2^+}{2}, \ldots,\ \frac{F_n^- + F_n^+}{2}\right\}$$

$$\Delta_1 = \frac{(d_m+1)}{(d_m-1)},\ \Delta_2 = \frac{(d_m+1)}{d_m(d_m-1)},\ \Delta_3 = \frac{d_M - d(k)}{d_M - d_m},$$

$$\Delta_4 = \frac{d(k) - d_m + 1}{(d(k)-d_1)(d(k)-d_m-1)},\ \Delta_5 = \frac{d(k)-d_m}{d_M - d_m}.$$

Theorem 1: Under assumption 1, the DNNs (1) with $w(k) = 0$ is robustly asymptotically stable, if there exist symmetric matrices $P_1 > 0,\ Q > 0,\ R > 0,\ S > 0,\ Z_1 > 0,\ T_2 > 0,\ T_4 > 0, W > 0, Z_2 > 0, R_1 > 0$ diagonal matrices $G_i > 0, i = 1,2$, matrix $X,\ Y$ and a scalar $\lambda,\ \varepsilon > 0$, such that the following LMIs hold:

$$\Psi_a = \left[\begin{array}{ccccccc}
\Psi^{11} & 0 & \Psi^{13} & 0 & \Psi^{15} & \Psi^{16} & \Psi^{17}\\
* & \Psi^{22} & 0 & \Psi^{24} & 0 & \Psi^{26} & 0\\
* & * & \Psi^{33} & 0 & 0 & 0 & 0\\
* & * & * & \Psi^{44} & 0 & 0 & 0\\
* & * & * & * & \Psi^{55} & \Psi^{56} & \Psi^{57}\\
* & * & * & * & * & \Psi^{66} & \Psi^{67}\\
* & * & * & * & * & * & \Psi^{77}\\
* & * & * & * & * & * & *\\
* & * & * & * & * & * & *\\
* & * & * & * & * & * & *\\
* & * & * & * & * & * & *\\
* & * & * & * & * & * & *
\end{array}\right.
\left.\begin{array}{ccccc}
\Psi^{18} & 0 & 0 & \Psi^{1,11} & \Psi^{1,12}\\
0 & \Psi^{29} & \Psi^{2,10} & 0 & 0\\
\Psi^{38} & \Psi^{39} & 0 & 0 & 0\\
0 & 0 & \Psi^{4,10} & 0 & 0\\
0 & 0 & 0 & \Psi^{5,11} & \Psi^{5,12}\\
0 & 0 & 0 & \Psi^{6,11} & \Psi^{6,12}\\
0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0\\
\Psi^{88} & 0 & 0 & 0 & 0\\
* & \Psi^{99} & 0 & 0 & 0\\
* & * & \Psi^{10,10} & 0 & 0\\
* & * & * & \Psi^{11,11} & 0\\
* & * & * & * & \Psi^{12,12}
\end{array}\right] < 0, \tag{6}$$

where

$$\begin{aligned}\Psi^{11} &= -\hat{P}_1 + \hat{Q}(1+d_M-d_m) + \hat{R} + \hat{S}\\ &\quad + (1+d_M-d_m)\hat{T}_2 - F_1G_1 + \lambda N_1^T N_1 - \hat{Z}_1\\ &\quad - \frac{R_1}{d_1} - 3\hat{Z}_1\Delta_1 - 3\hat{R}_1\Delta_2,\end{aligned}$$

$$\Psi^{13} = \hat{Z}_1 - \frac{R_1}{d_1} - 3\hat{Z}_1\Delta_1 - 3\hat{R}_1\Delta_2,\ \Psi^{1,11} = XA^T + Y^T,$$

$$\Psi^{15} = (1+d_M-d_m)\hat{W} + F_2G_1 + \lambda N_1^T N_2,$$

$$\Psi^{16} = \lambda N_1^T N_3,\ \ \Psi^{17} = Y^T\beta - X\beta + XA^T\beta,$$

$$\Psi^{18} = 3Z_1\Delta_1 + 3\hat{R}_1\Delta_2,$$

$$\begin{aligned}\Psi^{22} &= -\hat{Q} - \hat{T}_2 - F_1G_2 - \frac{\hat{R}_1}{(d(k)-d_1)} - 8\hat{Z}_2 - 4\Delta_3\hat{Z}_2\\ &\quad - 4\Delta_5\hat{Z}_2,\end{aligned}$$

$$\Psi^{24} = -2\hat{Z}_2 - 2\Delta_5\hat{Z}_2,$$

$$\Psi^{29} = 3\Delta_4\hat{R}_1 + 3\hat{Z}_2 + 3\Delta_3\hat{Z}_2,$$

$$\Psi^{1,12} = XN_1 + Y,\ \ \Psi^{26} = -W + F_2G_2,$$
$$\Psi^{2,10} = 3\hat{Z}_2 + 3\Delta_5\hat{Z}_2,$$
$$\Psi^{33} = -\hat{R} - 3\hat{Z}_1\Delta_1 - 4\hat{Z}_2 - 4\Delta_3\hat{Z}_2 - Z_1 - \frac{\hat{R}_1}{(d(k)-d_1)} - \frac{\hat{R}_1}{d_1} - 3\hat{R}_1\Delta_4 - 3\hat{R}_1\Delta_2,$$
$$\Psi^{38} = 3\hat{Z}_1\Delta_1 + 3\hat{R}_1\Delta_2,\ \ \Psi^{39} = 3\hat{Z}_2 + 3\Delta_3\hat{Z}_2 + 3\hat{R}_1\Delta_4,$$
$$\Psi^{44} = -\hat{S} - 4\hat{Z}_2 - 4\Delta_5\hat{Z}_2,\ \ \Psi^{4,10} = 3\hat{Z}_2 + 3\Delta_5\hat{Z}_2,$$
$$\Psi^{55} = -(1 + d_M - d_m)\hat{T}_4 + \hat{Q}_1 + \lambda N_2^T N_2 - G_1,$$
$$\Psi^{56} = \lambda N_2^T N_3,\ \ \Psi^{57} = E^T\hat{P}_1^T\beta,\ \ \Psi^{5,11} = E_d^T,$$
$$\Psi^{5,12} = N_2,\ \ \Psi^{66} = \lambda N_3^T N_3 - \hat{Q}_1 - \hat{T}_4 - G_2,$$
$$\Psi^{67} = E_d^T\hat{P}_1^T\beta,\ \ \Psi^{6,11} = E_d^T,\ \Psi^{6,12} = N_3,$$
$$\Psi^{77} = -\beta X + \lambda MM^T\beta^2 + d_m^2\hat{Z}_1 + (d_M - d_m)^2\hat{Z}_2 + d_m\hat{R}_1,$$
$$\Psi^{88} = -3\hat{Z}_1\Delta_1 - 3\hat{R}_1\Delta_2,\ \ \Psi^{99} = -3\Delta_4\hat{R}_1,$$
$$\Psi^{10,10} = -3\hat{Z}_2 - 3\Delta_5\hat{Z}_2,\ \ \Psi^{11,11} = -X + \lambda MM^T,$$
$$\Psi^{12,12} = -\varepsilon,\ \ W_4 = diag\{Z_2\ 3Z_2\}.$$

In this case, the appropriate state feedback controller can be chosen as $K = X^{-1}Y$.

Proof: Lyapunov-Krasovskii Functional is defined as follows

$$V(k) = V_1(k) + V_2(k) + V_3(k) + V_4(k) + V_5(k) + V_6(k), \tag{7}$$

where

$$V_1(k) = x^T(k)P_1x(k),$$
$$V_2(k) = \sum_{i=k-d(k)}^{k-1} x^T(i)Qx(i) + \sum_{j=-d_M+1}^{-d_m}\sum_{i=k+j}^{k-1} x^T(i)Qx(i),$$
$$V_3(k) = \sum_{i=k-d_m}^{k-1} x^T(i)Rx(i) + \sum_{i=k-d_M}^{k-1} x^T(i)Sx(i),$$
$$V_4(k) = \sum_{j=-d_M+1}^{-d_m+1}\sum_{i=k-1+j}^{k-1} \begin{bmatrix} x(i) \\ g(x(i)) \end{bmatrix}^T U \begin{bmatrix} x(i) \\ g(x(i)) \end{bmatrix},$$
$$V_5(k) = d_m\sum_{\beta=-d_m}^{-1}\sum_{\theta=k+\beta}^{k-1} r^T(\theta)Z_1r(\theta) + (d_M - d_m)\sum_{i=-d_M+1}^{-d_m}\sum_{\theta=k+i-1}^{k-1} r^T(\theta)Z_2r(\theta),$$
$$V_6(k) = \sum_{i=-d(k)}^{-1}\sum_{\theta=k+i}^{k-1} r^T(\theta)R_1r(\theta),$$

where $r(k) = x(k+1) - x(k)$. Calculating the difference of $V(k)$ by defining $\Delta V(k) = V(k+1) - V(k)$ along the system with $w(k) = 0$ and taking the mathematical expectation we obtain,

$$\mathbb{E}[\Delta V(k)] = \mathbb{E}[V_1(k) + V_2(k) + V_3(k) + V_4(k) + V_5(k) + V_6(k)], \tag{8}$$

where

$$\begin{aligned}
&\mathbb{E}[\Delta V_1(k)] \\
&= \mathbb{E}\{V_1(k+1) - V_1(k)\}, \\
&= \mathbb{E}[x^T(k+1)P_1x(k+1) - x^T(k)P_1x(k)], \\
&= \mathbb{E}[[(A + \Delta A(k))x(k) + (E + \Delta E(k))g(x(k)) \\
&\quad + (E_d + \Delta E_d(k))g(x(k - d(k))) + u(k)]^T \\
&\quad \times P[(A + \Delta A(k))x(k) + (E + \Delta E(k))g(x(k)) \\
&\quad + (E_d + \Delta E_d(k))g(x(k - d(k))) + u(k)]], \\
&= \mathbb{E}[[Ax(k) + Eg(x(k)) + E_dg(x(k - d(k))) \\
&\quad + u(k)]^T(P_1^{-1} - \lambda^{-1}MM^T)^{-1} \\
&\quad \times [Ax(k) + Eg(x(k)) + E_dg(x(k - d(k))) + u(k)]] \\
&\quad + \lambda[N_1x(k) + N_2g(x(k)) + N_3g(x(t - d(k)))]^T \\
&\quad \times [N_1x(k) + N_2g(x(k)) + N_3g(x(t - d(k)))],
\end{aligned} \tag{9}$$

$$\begin{aligned}
&\mathbb{E}[\Delta V_2(k)] \\
&= \mathbb{E}[\{V_2(k+1) - V_2(k)\}], \\
&= \sum_{i=k+1-d(k+1)}^{k} x^T(i)Qx(i) + \sum_{j=-d_M+1}^{-d_m}\sum_{i=k+1+j}^{k} x^T(i)Qx(i) \\
&\quad - \sum_{i=k-d(k)}^{k-1} x^T(i)Qx(i) - \sum_{j=-d_M+1}^{-d_m}\sum_{i=k+j}^{k-1} x^T(i)Qx(i), \\
&= \{(d_M - d_m + 1)x^T(k)Qx(k) \\
&\quad - x^T(k - d(k))Qx(k - d(k))\},
\end{aligned}$$

$$\begin{aligned}
&\mathbb{E}[\Delta V_3(k)] \\
&= \mathbb{E}\{V_3(k+1) - V_3(k)\}, \\
&= \{x^T(k)Rx(k) + x^T(k)Sx(k) - x^T(t - d_m)Rx(t - d_m) \\
&\quad - x^T(t - d_M)Sx(t - d_M)\},
\end{aligned}$$

$$\begin{aligned}
&\mathbb{E}[\Delta V_4(k)] \\
&= \mathbb{E}\{V_4(k+1) - V_4(k)\}, \\
&= h\begin{bmatrix} x(k) \\ g(x(k)) \end{bmatrix}^T \begin{bmatrix} T_2 & W \\ * & T_4 \end{bmatrix} \begin{bmatrix} x(k) \\ g(x(k)) \end{bmatrix} \\
&\quad - \begin{bmatrix} x(k - d(k)) \\ g(x(k - d(k))) \end{bmatrix}^T \begin{bmatrix} T_2 & W \\ * & T_4 \end{bmatrix} \begin{bmatrix} x(k - d(k)) \\ g(x(k - d(k))) \end{bmatrix},
\end{aligned}$$

$$\begin{aligned}
&\mathbb{E}[\Delta V_5(k)] \\
&= d_m\sum_{\beta=-d_m}^{-1}\left\{\sum_{\theta=k+\beta+1}^{k} r^T(\theta)Z_1r(\theta) - \sum_{\theta=k+\beta}^{k-1} r^T(\theta)Z_1r(\theta)\right\} \\
&\quad + (d_M - d_m)\sum_{i=d_M+1}^{-d_m}\left\{\sum_{\theta=k+i}^{k} r^T(\theta)Z_2r(\theta) - \sum_{\theta=k+i-1}^{k-1} r^T(\theta)Z_2r(\theta)\right\}, \\
&= d_m^2r^T(k)Z_1r(k) - d_m\sum_{\theta=k-d_m}^{k-1} r^T(\theta)Z_1r(\theta) \\
&\quad + (d_M - d_m)^2r^T(k)Z_2r(k)
\end{aligned}$$

$$-(d_M-d_m)\sum_{\theta=k-d_M}^{k-d_m-1} r^T(\theta)Z_2r(\theta), \tag{10}$$

$$\begin{aligned}&\mathbb{E}[\Delta V_6(k)]\\&=d(k)r^T(k)R_1r(k)-\sum_{\theta=k-d(k)}^{k-1} r^T(\theta)R_1r(\theta).\end{aligned} \tag{11}$$

Utilizing Lemma 3, we get

$$\begin{aligned}&-d_m\sum_{i=k-d_m}^{k-1}x^T(i)Z_1x(i)\\&\le -[x(k)-x(k-d_m)]^TZ_1[x(k)-x(k-d_m)]\\&\quad -3[x(k)-x(k-d_m)\\&\quad -\frac{2}{d_m+1}\sum_{i=k-d_m}^{k}x(i)]^TZ_1\frac{d_m+1}{d_m-1}[x(k)-x(k-d_m)\\&\quad -\frac{2}{d_m+1}\sum_{i=k-d_m}^{k}x(i)].\end{aligned}$$

Also,

$$\begin{aligned}&-\sum_{j=k-d(k)}^{k-1}x^T(j)R_1x(j)\\&=-\sum_{j=k-d_m}^{k-1}x^T(j)R_1x(j)-\sum_{j=k-d(k)}^{k-d_m-1}x^T(j)R_1x(j)\end{aligned}$$

since

$$\begin{aligned}&-\sum_{j=k-d_m}^{k-1}x^T(j)R_1x(j)\\&\le\begin{bmatrix}x(k)-x(k-d_m)\\x(k)+x(k-d_m)-\frac{2}{d_m+1}\sum_{i=k-d_1}^{k}u(i)\end{bmatrix}^T\\&\quad\times\begin{bmatrix}-\frac{R_1}{d_m}&0\\0&-3\frac{(d_m+1)}{d_m(d_m-1)}R_1\end{bmatrix}\\&\quad\times\begin{bmatrix}x(k)-x(k-d_m)\\x(k)+x(k-d_m)-\frac{2}{d_m+1}\sum_{i=k-d_1}^{k}u(i)\end{bmatrix},\\&\quad -\sum_{j=k-d(k)}^{k-d_m-1}x^T(j)R_1x(j)\le\frac{-1}{d(k)-d_m}[x(k-d_m)\\&\quad -x(k-d(k))]^TR_1[x(k-d_m)-x(k-d(k))]\\&\quad -3\frac{(d(k)-d_m+1)}{(d(k)-d_m)(d(k)-d_m-1)}[x(k-d_m)\\&\quad +x(k-d(k))-\frac{2}{d(k)-d_m+1}\sum_{j=k-d(k)}^{k-d_m}x(j)]^TR_1\\&\quad\times[x(k-d_m)+x(k-d(k))\\&\quad -\frac{2}{d(k)-d_m+1}\sum_{j=k-d(k)}^{k-d_m}x(j)].\end{aligned}$$

Then, for any matrix V_2, the improved summation inequality in Lemma 4 is employed to estimate other sum terms possessed time-varying delay $d(k)$ in $\Delta V_3(k,u(k))$, we have

$$\begin{aligned}&(d_M-d_m)\sum_{i=k-d_M}^{k-d_m-1}r^T(i)Z_2r(i)\\&=\left\{\sum_{i=k-d_M}^{k-d(k)-1}r^T(i)Z_2r(i)+\sum_{i=k-d(k)}^{k-d_m-1}r^T(i)Z_2r(i)\right\}\\&\le\Lambda_2(k),\end{aligned}$$

where

$$\begin{aligned}\Lambda_2(k)=&\begin{bmatrix}\hat\Gamma_1\\\hat\Gamma_2\end{bmatrix}^T\left(\begin{bmatrix}W_4&V_2\\ *&W_4\end{bmatrix}\right.\\&+\begin{bmatrix}\frac{d_M-d(k)}{d_M-d_m}(W_4-V_2W_4^{-1}V_2^T)\\ *\end{bmatrix.\\&\qquad\left.\begin{matrix}0\\\frac{d(k)-d_m}{d_M-d_m}(W_4-V_2^TW_4^{-1}V_2)\end{bmatrix}\right)\begin{bmatrix}\hat\Gamma_1\\\hat\Gamma_2\end{bmatrix},\end{aligned}$$

Moreover, for any matrices $P_2>0$, we have

$$\begin{aligned}&2r(k)P_2[((A+\Delta A(t))-I)x(k)+(E+\Delta E(t))g(x(k))\\&\quad+(E_d+\Delta E_d(t))g(x(k-d(k)))\\&\quad+(D+\Delta D(k))w(k)+u(k)-r(k)]\\&=0.\end{aligned}$$

From Assumption 1, we have

$$(g_i(x_i(k))-F_i^-x_i(k))(g_i(x_i(k))-F_i^+x_i(k))\le 0,$$

which is equivalent to

$$\begin{aligned}&\sum_{i=1}^n d_i\begin{bmatrix}x(k)\\g(x(k))\end{bmatrix}^T\begin{bmatrix}F_i^-F_i^+\bar e_i\bar e_i^T&\frac{-F_i^-+F_i^+}{2}\bar e_i\bar e_i^T\\\frac{-F_i^-+F_i^+}{2}\bar e_i\bar e_i^T&\bar e_i\bar e_i^T\end{bmatrix}^T\\&\quad\times\begin{bmatrix}x(k)\\g(x(k))\end{bmatrix}\\&\le 0,\\&\begin{bmatrix}x(k)\\g(x(k))\end{bmatrix}^T\begin{bmatrix}F_1G_1&-F_2G_1\\-F_2G_1&G_1\end{bmatrix}\begin{bmatrix}x(k)\\g(x(k))\end{bmatrix}\\&\le 0,\\&\begin{bmatrix}x(k-d(k))\\g(x(k-d(k)))\end{bmatrix}^T\begin{bmatrix}F_1G_2&-F_2G_2\\-F_2G_2&G_2\end{bmatrix}\\&\quad\times\begin{bmatrix}x(k-d(k))\\g(x(k-d(k)))\end{bmatrix}\\&\le 0.\end{aligned} \tag{12}$$

Combining (9)-(12) and using Schur complement Lemma

$$\mathbb{E}\{\Delta V(k)\}\le\mathbb{E}\{\xi^T(k)\Pi_a\xi(k)\}, \tag{13}$$

$$\Pi_a = \left[\begin{array}{ccccccc} \Psi^{11} & 0 & \Psi^{13} & 0 & \Psi^{15} & \Psi^{16} & \Psi^{17} \\ * & \Psi^{22} & 0 & \Psi^{24} & 0 & \Psi^{26} & 0 \\ * & * & \Psi^{33} & 0 & 0 & 0 & 0 \\ * & * & * & \Psi^{44} & 0 & 0 & 0 \\ * & * & * & * & \Psi^{55} & \Psi^{56} & \Psi^{57} \\ * & * & * & * & * & \Psi^{66} & \Psi^{67} \\ * & * & * & * & * & * & \Psi^{77} \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \end{array}\right.$$
$$\left.\begin{array}{ccccc} \Psi^{18} & 0 & 0 & \Psi^{1,11} & \Psi^{1,12} \\ 0 & \Psi^{29} & \Psi^{2,10} & 0 & 0 \\ \Psi^{38} & \Psi^{39} & 0 & 0 & 0 \\ 0 & 0 & \Psi^{410} & 0 & 0 \\ 0 & 0 & 0 & \Psi^{511} & \Psi^{512} \\ 0 & 0 & 0 & \Psi^{611} & \Psi^{612} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \Psi^{88} & 0 & 0 & 0 & 0 \\ * & \Psi^{99} & 0 & 0 & 0 \\ * & * & \Psi^{10,10} & 0 & 0 \\ * & * & * & \Psi^{11,11} & 0 \\ * & * & * & * & \Psi^{12,12} \end{array}\right] < 0, \quad (14)$$

where

$$\Psi^{17} = P_2^T + P_2^T K^T - P_2^T,\ \ \Psi^{1,11} = A^T + K^T,$$
$$\Psi^{1,12} = N_1 + K,\ \ \Psi^{57} = E^T P_2,\ \ \Psi^{5,11} = E^T,$$
$$\Psi^{5,12} = N_2,\ \Psi^{67} = E_d^T P^T \beta,\ \Psi^{6,11} = E_d^T,\ \Psi^{6,12} = N_3,$$
$$\Psi^{77} = -P_2 + \lambda P_2 M M^T P_2 + d_m^2 Z_1 + (d_M - d_m)^2 Z_2 + d_m R_1,$$
$$\Psi^{11,11} = -(P^{-1} - \lambda M M^T),\ \ \Psi^{12,12} = -\varepsilon,$$
$$\xi^T(k) = \Big[x^T(k)\ \ x^T(k-d(k))\ \ x^T(k-d_m)\ \ x^T(k-d_M)\ \ g^T(x(k))\ \ g^T(x(k-d(k)))\ \ r^T(k)\frac{2}{d_1+1}\sum_{j=k-d_1}^{k} x^T(j)\ \ \frac{2}{d(k)-d_1+1}\sum_{j=k-d(k)}^{k-d_1} x^T(j)\ \ \frac{2}{d_2-d(k)+1}\sum_{j=k-d_2}^{k-d(k)} x^T(j)\Big].$$

In order to obtain the feedback controller gain matrices, let us define $P_2 = \beta P_1$, here β is the design parameter. Pre and post multiply by (14) by $diag\{X, I, I, I, I, X, I, I, I, I, I\}$, we can obtain (6). where $X = P_1^{-1}$ and letting $\hat{P}_1 = XP_1X$, $\hat{Q} = XQX$, $\hat{R} = XRX$, $\hat{S} = XSX$, $\hat{T}_1 = XT_1X$, $\hat{Z}_1 = XZ_1X$, $\hat{Z}_2 = XZ_2X$, $\hat{R}_1 = XR_1X$. Thus we conclude that

$$\mathbb{E}\{\Delta V(k)\} \le -\hat{v}\mathbb{E}|\xi(k)|^2.$$

where $\hat{v}$ is the positive scalar and by using controller gain matrix $K = X^{-1}Y$ in (14). This indicates that the closed-loop system (1) with $w(k) = 0$ is robustly asymptotically stable in the mean square. The proof is complete. Next, we will analyze the H_∞ performance of the closed-loop system. □

4. ROBUST H_∞ PERFORMANCE ANALYSIS

This section is devoted to focus a state feedback controller $u(k) = X^{-1}Yx(k)$ that stabilizes system (1) and guarantees that the closed-loop system reaches the disturbance attenuation level $\hat{\gamma} > 0$.
In order to deal the H_∞ performance of the DNNs (1), we introduce

$$\tilde{J}(n) = \mathbb{E}\{\sum_{k=0}^{n} z^T(k)z(k) - \hat{\gamma}^2 w^T(k)w(k)\}, \quad (15)$$

where n is a non-negative integer.

Theorem 2: Under assumption (1), the DNNs (1) hold, if there exist symmetric matrices $P > 0, Q > 0, R > 0, S > 0, Z_1 > 0, T_2 > 0, T_4 > 0, W > 0, Z_2 > 0, R_1 > 0$ diagonal matrices $G_i > 0, i = 1, 2$, matrix X, Y and a scalar $\lambda > 0, \varepsilon > 0$, such that the following LMIs hold:

$$\Psi_a = \left[\begin{array}{ccccccc} \Psi^{11} & 0 & \Psi^{13} & 0 & \Psi^{15} & \Psi^{16} & \Psi^{17} \\ * & \Psi^{22} & 0 & \Psi^{24} & 0 & \Psi^{26} & 0 \\ * & * & \Psi^{33} & 0 & 0 & 0 & 0 \\ * & * & * & \Psi^{44} & 0 & 0 & 0 \\ * & * & * & * & \Psi^{55} & \Psi^{56} & \Psi^{57} \\ * & * & * & * & * & \Psi^{66} & \Psi^{67} \\ * & * & * & * & * & * & \Psi^{77} \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \end{array}\right.$$
$$\left.\begin{array}{cccccc} \Psi^{18} & \Psi^{19} & 0 & 0 & \Psi^{1,12} & \Psi^{1,13} \\ 0 & 0 & \Psi^{2,10} & \Psi^{2,11} & 0 & 0 \\ 0 & \Psi^{39} & \Psi^{310} & 0 & 0 & 0 \\ 0 & 0 & 0 & \Psi^{411} & 0 & 0 \\ \Psi^{58} & 0 & 0 & 0 & \Psi^{5,12} & \Psi^{5,13} \\ \Psi^{68} & 0 & 0 & 0 & \Psi^{6,12} & \Psi^{6,13} \\ \Psi^{78} & 0 & 0 & 0 & 0 & 0 \\ \Psi^{88} & 0 & 0 & 0 & 0 & 0 \\ * & \Psi^{99} & 0 & 0 & 0 & 0 \\ * & * & \Psi^{10,10} & 0 & 0 & 0 \\ * & * & * & \Psi^{11,11} & 0 & 0 \\ * & * & * & * & \Psi^{12,12} & 0 \\ * & * & * & * & * & \Psi^{13,13} \end{array}\right] < 0, \quad (16)$$

where

$$
\begin{aligned}
\Psi^{11} &= -\hat{P}_1 + \hat{Q}(1+d_M-d_m) + \hat{R} + \hat{S} \\
&\quad + (1+d_M-d_m)\hat{T}_2 - F_1G_1 + \lambda N_1^T N_1 + L^T L \\
&\quad - \hat{Z}_1 - \frac{\hat{R}_1}{d_1} - 3\hat{Z}_1\Delta_1 - 3\hat{R}_1\Delta_2, \\
\Psi^{17} &= \lambda N_1^T N_4, \ \ \Psi^{57} = \lambda N_2^T N_4, \ \ \Psi^{67} = \lambda N_3^T N_4, \\
\Psi^{77} &= \lambda N_4^T N_4 - \hat{\gamma}^2 I, \ \ \Psi^{1,11} = XA^T + Y^T, \\
\Psi^{18} &= Y^T\beta - X\beta + XA^T\beta, \ \ \Psi^{1,12} = XN_1 + Y, \\
\Psi^{58} &= E^T\hat{P}_1^T\beta, \ \ \Psi^{5,11} = E_d^T, \ \ \Psi^{5,12} = N_2, \\
\Psi^{68} &= E_d^T\hat{P}_1^T\beta, \ \ \Psi^{6,11} = E_d^T, \ \ \Psi^{6,12} = N_3, \\
\Psi^{78} &= D^T P^T\beta, \ \ \Psi^{7,11} = D^T, \ \ \Psi^{7,12} = N_4, \\
\Psi^{88} &= -\beta X + \lambda MM^T\beta^2 + d_m^2\hat{Z}_1 + (d_M-d_m)^2\hat{Z}_2 \\
&\quad + d_m\hat{R}_1, \\
\Psi^{11,11} &= -X + \lambda MM^T, \ \ \Psi^{12,12} = -\varepsilon,
\end{aligned}
$$

other terms are same as defined in Theorem 1. Then, a stabilizing feedback controller to provide $\hat{\gamma} > 0$-disturbance attenuation can be constructed as $u(k) = X^{-1}Yx(k)$.

Proof: Under the zero initial condition, (15) becomes

$$
\begin{aligned}
\tilde{J}(n) =& \mathbb{E}\left[\sum_{k=0}^{n}[x^T(k)L^T Lx(k) \right. \\
&\left. - \hat{\gamma}^2 w^T(k)w(k) + \Delta V(k)] - V(n+1)\right], \\
\le& \mathbb{E}\left[\sum_{k=0}^{n}[x^T(k)L^T Lx(k) - \hat{\gamma}^2 w^T(k)w(k) + \Delta V(k)\right], \\
\le& \mathbb{E}\left[\sum_{k=0}^{n}\xi^T(k)\Psi_b\xi(k)\right].
\end{aligned}
$$

By using $\Delta V(k)$ given in (8), thus we have Ψ_b by using the same procedure as in Theorem 1. If LMI (16) holds, then we obtain $\tilde{J}(n) < 0$ and by letting $n \to \infty$, we have $\| z \|_2 \le \hat{\gamma} \| w \|_2$. Therefore by Definition 1 the DNNs (1) is robustly asymptotically stable with a disturbance attenuation level $\hat{\gamma} > 0$. □

Remark 1: In Theorems 1 and 2, the criteria that ensure the asymptotic stability of discrete time neural networks with time-varying delay are established in terms of LMIs. If there is no parameter uncertainties then the DNNs (1) is reduced to the following neural network model (17).

$$
\begin{aligned}
x(k+1) &= -Ax(k) + Eg(x(k)) + E_d g(x(k-d(k))), \\
z(k) &= Lx(k).
\end{aligned} \tag{17}
$$

According to Theorem 1, we have the following Corollary 1 for the asymptotic stability of discrete time neural networks (17).

Corollary 1: Under assumption (1), the neural networks (17) with $w(k) = 0$ are robustly asymptotically stable, if there exist symmetric matrices $P > 0, Q > 0, R > 0, S > 0, Z_1 > 0, T_2 > 0, T_4 > 0, W > 0, Z_2 > 0, R_1 > 0$, and diagonal matrices $G_i > 0, i = 1,2$, such that the following LMIs hold:

$$
\Psi_a = \begin{bmatrix}
\Psi^{11} & 0 & \Psi^{13} & 0 & \Psi^{15} & 0 & \Psi^{17} & \Psi^{18} & 0 & 0 & A^T P \\
* & \Psi^{22} & 0 & 0 & 0 & \Psi^{26} & 0 & 0 & \Psi^{29} & \Psi^{2,10} & 0 \\
* & * & \Psi^{33} & 0 & 0 & 0 & 0 & \Psi^{38} & \Psi^{39} & 0 & 0 \\
* & * & * & \Psi^{44} & 0 & 0 & 0 & 0 & 0 & \Psi^{4,10} & 0 \\
* & * & * & * & \Psi^{55} & 0 & \Psi^{57} & 0 & 0 & 0 & E^T P \\
* & * & * & * & * & \Psi^{66} & \Psi^{67} & 0 & 0 & 0 & E_d^T P \\
* & * & * & * & * & * & \Psi^{77} & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & \Psi^{88} & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & \Psi^{99} & 0 & 0 \\
* & * & * & * & * & * & * & * & * & \Psi^{10,10} & 0 \\
* & * & * & * & * & * & * & * & * & * & -P
\end{bmatrix}
< 0, \tag{18}
$$

where

$$
\begin{aligned}
\Psi^{11} =& -P_1 + Q(1+d_M-d_m) + R + S \\
&+ (1+d_M-d_m)T_2 - F_1G_1 - Z_1 - \frac{R_1}{d_1} \\
&- 3Z_1\Delta_1 - 3R_1\Delta_2, \\
\Psi^{17} =& -P_1^T\beta + P_1^T A^T\beta, \ \ \Psi^{57} = P_1^T E^T\beta, \\
\Psi^{67} =& P_1^T E_d^T\beta, \\
\Psi^{77} =& -P_1\beta + d_m^2 Z_1 + (d_M-d_m)^2 Z_2 + d_m R_1,
\end{aligned}
$$

and the other terms are same as defined in Theorem 1.

Proof: Consider the same Lyapunov function as defined in Theorem 1. The proof immediately follows from the similar way of proof of Theorem 1, hence it is omitted. □

Remark 2: One can use different L-K functional and free weighting matrix techniques to obtain much better performance. However it is noted that the more complex L-K functional together with free weighting matrix technique brings more number of decision variables, consequently it leads to the computational burdens. So that in this paper, we have chosen an appropriate L-K functional of the form (7), without using any free weighting matrix technique (to reduce the computational complexity), we got comparatively less conservative results than some existing results. This has been proved through the numerical examples.

5. NUMERICAL EXAMPLES

In this section, two numerical examples are provided to illustrate the effectiveness of proposed method.

Example 1: We consider the discrete neural networks system (1) when $w(k) \neq 0$ and with the following parameters:

$$A = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.9 \end{bmatrix}, \; E = \begin{bmatrix} 0.17 & 0.3 \\ 0.14 & 0.34 \end{bmatrix},$$

$$E_d = \begin{bmatrix} 0.18 & -0.5 \\ 0.2 & 0.15 \end{bmatrix}, \; D = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.5 \end{bmatrix},$$

$$L = \begin{bmatrix} 0.7 & 0 \\ 0 & 0.2 \end{bmatrix},$$

$$N_1 = N_2 = N_3 = N_4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; M = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}.$$

The nonlinear function are given as $g(x(k)) = \begin{bmatrix} \tanh(0.2x_1(k)) \\ \tanh(-0.2x_2(k)) \end{bmatrix}$. The activation functions satisfy Assumption 1 with the following parameters:

$$F_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \; F_2 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}.$$

The time-varying delay are taken as $d(k) = \frac{1}{2}(9+5\sin\frac{k\pi}{2})$ the corresponding lower and upper bounds are $d_m = 2$ and $d_M = 7$. Solving the LMIs stated in Theorem 1, a set of feasible solution is obtained as

$$P_1 = \begin{bmatrix} 337.5552 & -25.6620 \\ -25.6620 & 313.5850 \end{bmatrix},$$

$$Q = \begin{bmatrix} 5.5461 & -0.3195 \\ -0.3195 & 1.8030 \end{bmatrix},$$

$$R = \begin{bmatrix} 30.3321 & -2.8137 \\ -2.8137 & 10.7932 \end{bmatrix},$$

$$S = \begin{bmatrix} 29.0316 & -0.1711 \\ -0.1711 & 17.5520 \end{bmatrix},$$

$$G_1 = \begin{bmatrix} 485.7276 & 0 \\ 0 & 385.6017 \end{bmatrix},$$

$$G_2 = \begin{bmatrix} 373.5129 & 0 \\ 0 & 166.1809 \end{bmatrix},$$

$$Y = \begin{bmatrix} 9.5152 & -0.8959 \\ -0.8959 & 2.7325 \end{bmatrix}.$$

with a stabilising state feedback controller having the gain matrix as

$$K = X^{-1}Y = \begin{bmatrix} 0.0281 & -0.0020 \\ -0.0006 & 0.0085 \end{bmatrix}.$$

Example 2: Consider the following discrete neural networks :

$$x(k+1) = -Ax(k) + Eg(x(k)) + E_d g(x(k-d(k)),$$

with the following parameters:

$$A = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.9 \end{bmatrix}, \; E = \begin{bmatrix} 0.001 & 0 \\ 0 & 0.005 \end{bmatrix},$$

$$E_d = \begin{bmatrix} -0.1 & 0.01 \\ -0.2 & -0.1 \end{bmatrix}.$$

and the activation function are taken as $g_1(s) = \tanh(s)$, $g_2(s) = \tanh(s)$. For different values of d_m, the upper bounds d_M are obtained by various approaches which guarantee the asymptotic stability of the considered neural networks (17) are listed in Table 1. From Table 1, it is clear that the proposed condition in Corollary 1 is less conservative than those results in [28–33].

Remark 3: In [28–30], the authors discussed with the problem of stability criterion with time-varying delays using some free weight matrix technique. But in this paper, without using free weighting matrix technique, less number of decision variables are obtained than some existing ones in the literature.

Table 1. Calculated maximum d_M for given d_m for Example 2.

d_m	2	4	6	8
[28]	11	11	12	13
[29]	11	12	13	14
[30]	13	15	17	19
[31]	20	21	23	23
[32]	-	20	20	21
[33]	-	20	25	28
Corollary 1	16	21	27	29

Table 2. Number of decision variables involved in various papers.

	No of decision variables
[29]	$15n^2+5n$
[28]	$17.5n^2+4.5n$
[30]	$28.5n^2+7.5n$
This paper	$5n^2+7n$

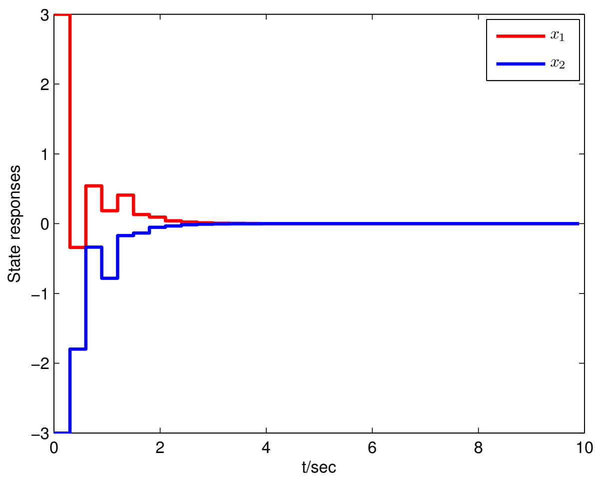

Fig. 1. State trajectory of the system (1) in Example 1.

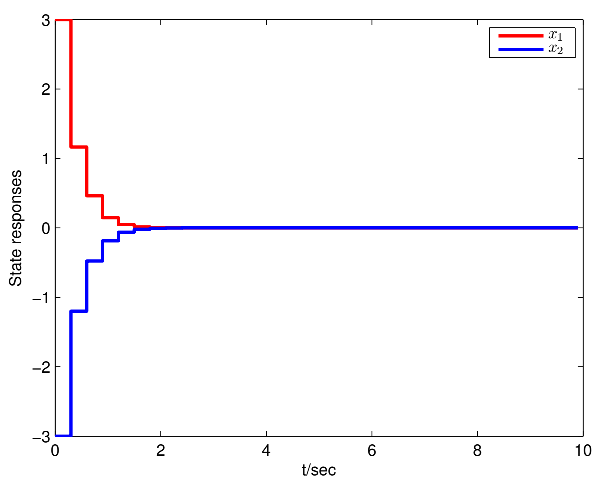

Fig. 2. State trajectory of the system (17) in Example 2.

6. CONCLUSION

In this paper, we have investigated the problem of Robust H_∞ performance of discrete-time neural networks with uncertainty and time-varying delay. By employing Lyapunov technique and LMI approach, we designed robust H_∞ controller such that the resulting closed-loop neural network is robustly asymptotically stable with a prescribed H_∞ performance. The obtained results are all in the form of an effective linear matrix inequality (LMI), which can be easily optimized by MATLAB-LMI control toolbox. Finally, two numerical examples are given to show the superiority of our proposed stability conditions.

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