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A New Parameter Identification Algorithm for a Class of Second Order Nonlinear Systems: An On-line Closed-loop Approach

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Abstract: This paper presents a novel on-line closed-loop parameter identification algorithm for second order nonlinear systems. Parameter convergence of the proposed methodology is ensured by means of a rigorous Lyapunov-based analysis. The estimated parameters are obtained using the actual and an estimation system. Algebraic techniques are applied for estimating velocity and acceleration signals, which are required in the proposed algorithm. A comparative analysis allows assessing the performance of the new parameter identification algorithm with respect to on-line and off-line least squares algorithms. Numerical simulations indicate that the proposed methodology allows estimating different types of non-linearities, converges faster than other methodologies, is robust against disturbances, outperforms on-line techniques, and provides similar estimates as an off-line technique, but without requiring any type of data pre-processing.

Keywords: Algebraic velocity and acceleration estimation, least squares, parameter identification, persistent excitation, second order nonlinear system.

1. INTRODUCTION

1.1. Overview

Parameter identification is a procedure that allows building a mathematical model of a dynamical system by using input-output measurements. The importance of this procedure relies on its applications, which may include filtering, state estimation, and design of more efficient controllers [1–3].

For safety reasons, especially for unstable open-loop systems, parameter identification techniques have to be applied with systems operating in closed-loop, which yields the field of closed-loop parameter identification.

Parameter identification techniques can be split into off-line and on-line techniques. The former collects input-output data, pre-process it, and computes the parameter estimates by using an iterative procedure; the latter process the data at each time instant, and then calculates the parameter estimates in a recursive manner.

Among the diverse parameter identification techniques existing in the literature, the least squares (LS) algorithm remains to be the most widely used approach for parameter identification in a variety of fields [4–7]. However, many of the existing parameter identification techniques identify only a subset of the whole system parameters or identify very specific types of non-linearities. Besides, on-line parameter identification techniques are usually affected by disturbances.

Second order systems are important because they are the mathematical model used to describe a large variety of practical systems such as chemical and biochemical systems, robotic systems, aircraft systems, mechanical systems, and networked systems. Besides, second order systems may appear when dealing with applications involving structural analysis, aerospace control, control of flexible mechanisms, model based fault diagnosis, communications, remote sensing, earthquake engineering, and vibration control [8–12]. Furthermore, the dominant dynamics of many systems (such as mechanical and electrical systems) can be described by using a second order system [13].

An efficient parameter identification stage applied to a class of second order nonlinear systems is important for improving the real-time performance of the system [9], and because of the large variety of applications involving second order systems. Furthermore, a robust parameter identification technique allows reducing model uncertainties, which let us improving the overall system performance. Besides, even if the system is controlled using a robust controller, if the magnitude of system model uncertainties is reduced, the amount of control action demanded by the robust controller will be lesser, reducing both the energy required for controlling the system and the possibility of saturating the actuator. Hence, a parameter identification technique applied to second order systems is an important issue that deserves to be studied.

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Based on the importance of second order nonlinear systems and parameter identification techniques, this work presents a new closed-loop parameter identification algorithm applied to a class of second order nonlinear systems, which allows identifying different types of non-linearities, and is robust against disturbances.

1.2. Literature review

The convenience of using an off-line or an on-line technique depends on the type of process where the system to be identified belongs to. However, in general, off-line techniques provide better parameter estimates because they may include a data pre-processing stage, which allows reducing the negative effect caused by disturbances such as position quantization, noisy measurements, and errors in estimating the time derivatives of a given signal.

Closed-loop parameter identification techniques can be categorized as direct and indirect methods [14]. Direct methods apply a parameter identification technique without taking into account the structure of the controller applied to the system to be identified [15–17]. On the other hand, the structure of the controller may be used for designing the parameter identification algorithm in indirect methods. The parameter identification algorithm proposed in this work falls into the category of indirect methods.

A parameter identification algorithm is reliable if it has conditions guaranteeing parameter convergence. Generally, these conditions are related to the spectral richness of the reference signal used in the identification process [18], and also to the so called persistency of excitation (PE) condition [19, 20]. For linear systems, parameter convergence is equivalent to the fulfillment of the PE condition. This is not the case, however, for nonlinear systems, although some works have presented advances in this direction [21–23].

The usual way to apply an LS algorithm is in its batch form, which generates the parameter estimates using the whole data at the same time. A drawback, however, of this approach is its computational burden, which is of order $O(N^3)$, with N the number of estimated parameters. This shortcoming can be circumvented by using a recursive least squares (RLS) algorithm, which computes each parameter estimate using the previous estimate together with a correction term which depends on the prediction error, instead of using the whole data. As a result, RLS algorithms reduce the computational burden to the order $O(N^2)$. Furthermore, the performance of RLS algorithms can be enhanced if a forgetting factor term is included [4, 24, 25]. Further details about computational complexity can be found in [26,27], where an interesting approach for reducing the computational complexity in control synthesis is addressed.

Position controlled direct current (DC) servomechanisms are an important class of second order nonlinear systems. Their applications are diverse and include hard disc drives, pendulums, and robotics, among others. Hence, in order to design high-performance controllers for these mechanisms, a parameter identification stage must be considered.

References [28–30] identify the parameters of a DC servomechanism using a relay-based technique. However, the main drawback of this approach happens when the reference signal exiting the system becomes constant because no control is being applied at that instant, which implies sensitivity to disturbances. Other works dealing with parameter identification of DC servomechanisms include the use of a batch LS algorithm to identify the load inertia, viscous and Coulomb friction coefficients [15]; an off-line LS algorithm for closed-loop identification [16]; a LS-based algorithm for active noise control [31]; a closed-loop parameter identification technique for velocity controlled servomechanisms [32]; a RLS algorithm applied to a servo drive system [17]; a two step method using a RLS algorithm to identify one type of nonlinearity [33]; a recursive least squares with forgetting factor (RLSFF) algorithm for identifying Coulomb friction and dead zone parameters of a velocity controlled DC servo [34]; a RLS algorithm using a model decomposition approach applied to nonlinear systems [35,36]. Some works present parameter identification methodologies using the gradient approach [37–39], Kalman filter [40], optimization algorithms [41], and modern nonlinear control techniques such as sliding modes [42], support vector machines [43], adaptive techniques [44], and algebraic techniques [45, 46].

The previous literature review shows that the LS approaches are among the most widely used parameter identification techniques. However, many of the existing parameter identification techniques identify only a subset of the whole system parameters or identify very specific types of non-linearities.

1.3. Contribution of the paper

This work presents a new closed-loop parameter identification algorithm applied to second order nonlinear systems. Parameter convergence of the new algorithm is proved using a rigorous Lyapunov-based analysis.

The proposed algorithm considers the actual system and an estimation system. A linear combination of the signals coming from both, the actual and the estimation systems allow designing an estimation output error signal. Then, the proposed algorithm is obtained by means of a minimization procedure based on a cost function, which depends on the output error signal. An algebraic approach is used to estimate the required time derivatives of a given signal. The proposed method is compared to an off-line LS algorithm with data pre-processing, an on-line RLS algorithm, and an on-line parameter identification algorithm reported in [25]. Then, these parameter identification techniques are applied using the model of a second order nonlinear system affected by disturbances due to quantization error, differentiation errors, and constant perturbations.

This paper contributes by:

1) providing a new parameter identification algorithm that allows estimating the whole set of parameters of a class of second order nonlinear system, including different types of non-linearities;
2) proving parameter convergence by means of a rigorous Lyapunov-based analysis;
3) verifying by means of numerical simulations that the proposed algorithm outperforms other existing on-line algorithms; and
4) showing that the proposed algorithm is robust against disturbances, and its performance is similar to that of an off-line algorithm, but without requiring any data pre-processing.

It is important to remark that the proposed algorithm considers an alternative definition of the output error signal. Besides, the regressor vector used in the parameter identification formula relies on signals coming from the estimation system, which is not affected by disturbances. As a consequence, the proposed algorithm turns out to be less affected by signals disturbing the actual system, which is corroborated by numerical simulations.

1.4. Organization

The rest of the paper is organized as follows. Some worthwhile concepts required in the convergence analysis are given in Section 2. The structure of the second order nonlinear system to be identified, its estimation model, definition of the output error signal, and problem formulation are given in Section 3. The proposed parameter identification algorithm and its convergence analysis are developed in Section 4. Some issues related to the implementation of the proposed algorithm are discussed in Section 5. A comparative study that allows assessing the performance of the proposed parameter identification algorithm is presented in Section 6. A discussion about the numerical results is presented in Section 7. Finally, the paper ends up with some concluding remarks in Section 8.

2. BASIC CONCEPTS

Parameter convergence analysis is commonly related to the asymptotic stability of the next differential equation [47, 48]

$$\dot{\tilde{\boldsymbol{\theta}}}(t) = -g\boldsymbol{\phi}(t)\boldsymbol{\phi}^T(t)\tilde{\boldsymbol{\theta}}(t), \tag{1}$$

where $\tilde{\boldsymbol{\theta}}(t), \dot{\tilde{\boldsymbol{\theta}}}(t)$ denote the parameter estimation error and its time derivative, respectively; $\boldsymbol{\phi}(t)$ is the *regressor vector*, and $g > 0$ is the adaptation gain.

The regressor vector $\boldsymbol{\phi}(t)$ is said to be *Persistently Exciting* (PE) if there exist constants $\bar{\alpha}_1$, $\bar{\alpha}_2$, and $\delta > 0$, such that

$$\bar{\alpha}_1 I \leq \int_{t_0}^{t_0+\delta} \boldsymbol{\phi}(\tau)\boldsymbol{\phi}^T(\tau)d\tau \leq \bar{\alpha}_2 I, \tag{2}$$

for all $t_0 \geq 0$, with I being the identity matrix.

The exponential convergence of the parameter estimation error $\tilde{\boldsymbol{\theta}}(t)$ in (1) is related to the PE condition (2), as stated in the next theorem.

Theorem 1 (PE and exponential stability) [48]**:** Let $\boldsymbol{\phi} : \mathbb{R}_+ \to \mathbb{R}^{2n}$ be piecewise continuous. If $\boldsymbol{\phi}$ is PE, then the differential equation (1) is globally exponentially stable.

In this work, the convergence analysis of the proposed parameter identification algorithm employs the Leibniz integral rule [49], which states that, given a function $f(t,\tau)$, whose partial derivative with respect to t exists and is continuous, the expression

$$\frac{d}{dt}\int_{u(t)}^{v(t)} f(t,\tau)d\tau = -f(t,u)\frac{du}{dt} + f(t,v)\frac{dv}{dt} + \int_{u(t)}^{v(t)} \frac{\partial}{\partial t} f(t,\tau)d\tau \tag{3}$$

holds, with $\partial/\partial t$ denoting the partial derivative with respect to time.

Finally, given a time-varying signal $v(t)$, its time derivative is represented by $\dot{v}(t)$.

3. MODEL DESCRIPTION AND PROBLEM FORMULATION

This Section presents the class of nonlinear systems to be identified and defines its estimation model. Then, an error system is obtained, and the problem formulation is stated.

The class of second order nonlinear systems to be identified can be mathematically described by the next state-space equation

$$\begin{aligned} \dot{\xi}_1 &= \xi_2, \\ \dot{\xi}_2 &= -a\xi_2 - c_1 f_1(\xi_1) - c_2 f_2(\xi_2) + \beta + bu, \end{aligned} \tag{4}$$

where a, b, c_1, c_2, β are the system parameters, which are assumed to be constant. Parameter β denotes a constant perturbation, $u(t)$ is the control input, and $f_1(\xi_1)$, $f_2(\dot{\xi}_2)$ are nonlinear functions. Note that given the structure of equation (4), it may represent many different and useful systems. For instance, it may describe a DC servomechanism, with $\xi_1(t), \xi_2(t), \dot{\xi}_2(t)$ denoting the position, velocity and acceleration, respectively.

It is assumed that functions $f_i(\cdot)$, $i = 1, 2$ in (4) are locally Lipschitz, i.e., they satisfy the inequality

$$|f_i(x) - f_i(y)| \leq L_i |x - y|, \forall x, y \in \mathbb{R}, \tag{5}$$

with $L_i > 0$ being the Lipschitz constant, and $|\cdot|$ denoting the absolute value. Note that assumption (5) allows including non-linearities such as gravity torque and friction torque characterized by smooth models [25, 50, 51].

Given the structure of the actual system (4), let us consider the following estimation system

$$\dot{w}_1 = w_2,$$
$$\dot{w}_2 = -\hat{a}w_2 - \hat{c}_1 f_1(w_1) - \hat{c}_2 f_2(w_2) + \hat{\beta} + \hat{b}u_e, \tag{6}$$

where $\hat{a}$, $\hat{b}$, $\hat{c}_1$, $\hat{c}_2$, $\hat{\beta}$ correspond to the estimated values of the parameters a, b, c_1, c_2, β, respectively, u_e is the estimated value of u, and $w_1, w_2, \dot{w}_2$ are the estimated signals corresponding to $\xi_1(t)$, $\xi_2(t)$, $\dot{\xi}_2(t)$, respectively.

Parameter identification methodologies, such as the LS algorithms, usually define the output error signal as the difference between the system output and its estimated value. Then, a cost function depending on this error signal is used to obtain a parameter identification formula. By following this methodology, let us define the output error signal $e(t)$ as

$$e(\boldsymbol{\xi}, \mathbf{w}, \dot{\boldsymbol{\xi}}, \dot{\mathbf{w}}) = g(\boldsymbol{\xi}, \dot{\boldsymbol{\xi}}) - g_e(\mathbf{w}, \dot{\mathbf{w}}), \tag{7}$$

where $\boldsymbol{\xi} = [\xi_1, \xi_2]^T$, $\mathbf{w} = [w_1, w_2]^T$, and signals $g(\boldsymbol{\xi}, \dot{\boldsymbol{\xi}})$, $g_e(\mathbf{w}, \dot{\mathbf{w}})$ are defined as

$$g(\boldsymbol{\xi}, \dot{\boldsymbol{\xi}}) = \mu_1 \xi_1 + \mu_2 \xi_2 + \mu_3 \dot{\xi}_2,$$
$$g_e(\mathbf{w}, \dot{\mathbf{w}}) = \mu_1 w_1 + \mu_2 w_2 + \mu_3 \dot{w}_2, \tag{8}$$

with μ_1, μ_2, μ_3 being positive constants. It is also worth defining the errors $\varepsilon_1, \varepsilon_2, \varepsilon_u$ as

$$\varepsilon_1 = \xi_1 - w_1,$$
$$\varepsilon_2 = \xi_2 - w_2,$$
$$\varepsilon_u = u - u_e. \tag{9}$$

Note from definitions (7)-(9) that the output error (7) consists of a linear combination of the signals coming from the actual system (4) and its estimation model (6). Besides, $u(t), u_e(t)$ correspond to the inputs of the actual and the estimation systems, respectively.

Signals $\mathbf{y} = [\xi_1, \xi_2, \dot{\xi}_2]^T$ and $\mathbf{y}_e = [w_1, w_2, \dot{w}_2]^T$ are considered as the outputs of the actual and the estimation systems, respectively. Using this selection, the identification objective can be formulated as follows:

Identification problem: Given the inputs u, u_e, and the outputs $\mathbf{y}, \mathbf{y}_e$, coming from the actual system (4) and the estimation model (6), respectively, find a parameter updating law that allows obtaining a set of parameter estimates $\hat{\boldsymbol{\theta}} = \left[\hat{a}, \hat{b}, \hat{c}_1, \hat{c}_2, \hat{\beta}\right]^T$, corresponding to the actual parameter vector $\boldsymbol{\theta} = [a, b, c_1, c_2, \beta]^T$, such that the limit

$$\lim_{t\to\infty} \tilde{\boldsymbol{\theta}}(t) = 0, \tag{10}$$

holds, with $\tilde{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}} - \boldsymbol{\theta}$ being the parameter estimation error.

4. IDENTIFICATION ALGORITHM

The block diagram of the proposed parameter identification algorithm is depicted in Fig. 1, and is described as follows. Both, the actual second order nonlinear system and the estimation system have the same structure. These systems are operated in closed-loop by means of the stabilizing controllers $u(t)$ and $u_e(t)$, respectively. These controllers also have the same structure, but each of them uses signals coming from the actual and the estimation systems, respectively. It is assumed that outputs $\mathbf{y}(t), \mathbf{y}_e(t)$ are measurable. Using the input-output measurements from both, the actual and the estimation systems, a regressor vector $\phi_2(t)$, a matrix $P(t)$, and the output error signal $e(t)$ are computed. These values are utilized in a parameter identification algorithm, which yields the vector of parameter estimates $\hat{\boldsymbol{\theta}}(t)$. The actual and the estimation systems are driven by the same reference signal $x_d(t)$, which is assumed bounded, with bounded first-time derivative. Furthermore, the parameters of the estimation system are continuously updated by the new parameters $\hat{\boldsymbol{\theta}}(t)$.

In order to obtain the proposed parameter identification algorithm, we follow the next steps:

- define a cost function;
- obtain a parameter identification algorithm, using the proposed cost function; and
- perform a parameter convergence analysis.

4.1. Cost function

Let us consider the cost function

$$J(\boldsymbol{\xi}, \mathbf{w}, \dot{\boldsymbol{\xi}}, \dot{\mathbf{w}}, t) = \int_0^t \lambda^{(t-\tau)/T} e^2(\tau) d\tau, \tag{11}$$

where λ, T are positive constants.

The cost function (11) is similar to the integral cost function presented in [47, 52]. The design parameter λ acts as a forgetting factor, i.e., as time t increases, the effect of the old data at time $\tau < t$ is discarded "λ-fast". For instance, data will be exponentially discarded for $\lambda = 2.7182$.

Using (8)-(9), the output error signal $e(t)$ and the cost function J can be rewritten as follows:

$$e = \mu_1 \varepsilon_1 + \mu_2 \varepsilon_2 + \mu_3 \dot{\varepsilon}_2, \tag{12}$$
$$J = \int_0^t \lambda^{(t-\tau)/T} \left[\mu_1 \varepsilon_1 + \mu_2 \varepsilon_2 + \mu_3 \dot{\varepsilon}_2\right]^2 d\tau. \tag{13}$$

4.2. Parameter identification algorithm

Let us define the vectors

$$\phi_1 = \begin{bmatrix} -\xi_2 & u & -f_1(\xi_1) & -f_2(\xi_2) & 1 \end{bmatrix}^T,$$
$$\phi_2 = \begin{bmatrix} -w_2 & u_e & -f_1(w_1) & -f_2(w_2) & 1 \end{bmatrix}^T,$$
$$\phi_3 = \phi_1 - \phi_2 = \begin{bmatrix} -\varepsilon_2 & \varepsilon_u & -\tilde{f}_1 & -\tilde{f}_2 & 0 \end{bmatrix}^T,$$

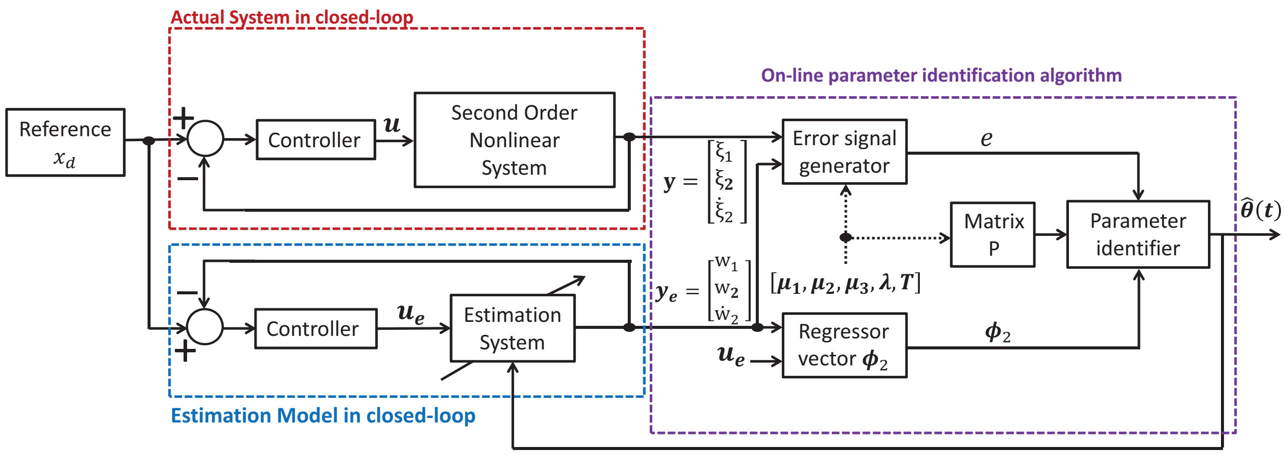

Fig. 1. Block-diagram of the proposed on-line parameter identification algorithm.

$$
\begin{aligned}
\boldsymbol{\theta} &= \begin{bmatrix} a & b & c_1 & c_2 & \beta \end{bmatrix}^T, \\
\hat{\boldsymbol{\theta}} &= \begin{bmatrix} \hat{a} & \hat{b} & \hat{c}_1 & \hat{c}_2 & \hat{\beta} \end{bmatrix}^T, \\
\tilde{\boldsymbol{\theta}} &= \boldsymbol{\theta} - \hat{\boldsymbol{\theta}},
\end{aligned} \tag{14}
$$

where $\tilde{f}_1 = [f_1(\xi_1) - f_1(w_1)]$, $\tilde{f}_2 = [f_2(\xi_2) - f_2(w_2)]$, $\tilde{\boldsymbol{\theta}}$ denotes the parameter estimation error, $\boldsymbol{\theta}$ the nominal parameter vector corresponding to (4), and $\hat{\boldsymbol{\theta}}$ the estimated parameter vector corresponding to the system (6).

Using definitions (14), the signal $\dot{\varepsilon}_2$ can be expressed as

$$
\begin{aligned}
\dot{\varepsilon}_2 &= \tilde{\boldsymbol{\theta}}^T \phi_1 + \hat{\boldsymbol{\theta}}^T \phi_3 \\
&= \tilde{\boldsymbol{\theta}}^T \phi_2 + \boldsymbol{\theta}^T \phi_3 \\
&= \tilde{\boldsymbol{\theta}}^T \phi_2 - a\varepsilon_2 + b\varepsilon_u - c_1 \tilde{f}_1 - c_2 \tilde{f}_2.
\end{aligned} \tag{15}
$$

The solution of the parameter identification problem is obtained by minimizing the cost function J in (11) with respect to $\hat{\boldsymbol{\theta}}$. To this end, let us compute the partial derivative of J with respect to $\hat{\boldsymbol{\theta}}$ as follows:

$$
\frac{\partial J}{\partial \hat{\boldsymbol{\theta}}} = 2 \int_0^t \lambda^{(t-\tau)/T} e \frac{\partial e}{\partial \hat{\boldsymbol{\theta}}} d\tau. \tag{16}
$$

Note that the partial derivative $\partial e / \partial \hat{\boldsymbol{\theta}}$ depends on signals w_1, w_2, and these signals have an implicit dependence on the parameters $\hat{\boldsymbol{\theta}}$. However, this partial derivative can be obtained by considering only the explicit derivatives and neglecting the implicit dependencies on the parameters [52]. Under this assumption, using (13) and (15), the partial derivative of J with respect to $\hat{\boldsymbol{\theta}}$ is given by

$$
\frac{\partial J}{\partial \hat{\boldsymbol{\theta}}} = -2\mu_3 \int_0^t \lambda^{(t-\tau)/T} e(\tau) \phi_2(\tau) d\tau. \tag{17}
$$

The validity of equation (17) can be verified by numerical simulations [52]. Then, by substituting (12) into (17), and equating to zero, the vector $\hat{\boldsymbol{\theta}}(t)$ can be separated as follows

$$
\begin{aligned}
&\int_0^t \lambda^{(t-\tau)/T} \left[\mu_1 \varepsilon_1 + \mu_2 \varepsilon_2 + \mu_3 \dot{\xi}_2 \right] \phi_2 d\tau \\
&= \mu_3 \left[\int_0^t \lambda^{(t-\tau)/T} \phi_2 \phi_2{}^T d\tau \right] \hat{\boldsymbol{\theta}}.
\end{aligned} \tag{18}
$$

Let us define the symmetric matrix $P^{-1}(t)$ as

$$
P^{-1}(t) = \int_0^t \lambda^{(t-\tau)/T} \phi_2(\tau) \phi_2{}^T(\tau) d\tau, \tag{19}
$$

with initial value $P(t_0) = P_0 = P_0^T > 0$.

Using the Leibniz rule (3), the time derivative of matrix $P^{-1}(t)$ is given by

$$
\frac{d}{dt}[P^{-1}(t)] = \phi_2(t)\phi_2^T(t) + \frac{ln\lambda}{T} P^{-1}(t).
$$

Furthermore,

$$
\frac{d}{dt}\left[PP^{-1}\right] = \dot{P}P^{-1} + P\frac{d}{dt}\left[P^{-1}\right] = 0. \tag{20}
$$

Using (20), the time derivative of matrix $P(t)$ can be written in the next form

$$
\dot{P} = \frac{d}{dt}P = -P\phi_2\phi_2^T P - \frac{ln\lambda}{T} P. \tag{21}
$$

From equation (18) and definition (19), the estimated parameter vector $\hat{\boldsymbol{\theta}}$ can be expressed as

$$
\hat{\boldsymbol{\theta}}(t) = \frac{1}{\mu_3} P(t) F(t), \tag{22}
$$

with

$$
F(t) = \int_0^t \lambda^{(t-\tau)/T} \left[\mu_1 \varepsilon_1 + \mu_2 \varepsilon_2 + \mu_3 \dot{\xi}_2 \right] \phi_2 d\tau,
$$

and the time derivative of $\hat{\boldsymbol{\theta}}(t)$ is given by

$$
\begin{aligned}
\dot{\hat{\boldsymbol{\theta}}} &= \frac{1}{\mu_3} \dot{P}F + \frac{1}{\mu_3} P\dot{F} \\
&= \frac{1}{\mu_3} P \left[\mu_1 \varepsilon_1 + \mu_2 \varepsilon_2 + \mu_3 \dot{\varepsilon}_2 \right] \phi_2.
\end{aligned} \tag{23}
$$

Using (12) and (23), the differential equation that governs the dynamics of the estimated parameters can be written as

$$\dot{\hat{\theta}}(t) = \frac{1}{\mu_3} P(t)\phi_2(t)e(t), \tag{24}$$

and, from definition of $\tilde{\theta}$ in (14) and using (24), the dynamics corresponding to the parameter error $\tilde{\theta}$ can be expressed as follows

$$\dot{\tilde{\theta}}(t) = -\frac{1}{\mu_3} P(t)\phi_2(t)e(t). \tag{25}$$

Equation (25) allows computing the estimated parameter vector $\hat{\theta}(t)$. This equation is valid if matrix $P^{-1}(t)$ in (19) is nonsingular. Then, the forthcoming stability analysis is based on the next assumption.

Assumption 1: Matrix $P^{-1}(t)$ fulfills the next condition

$$\lambda_{min}\{P^{-1}(t)\} > 0, \mathit{for\ all}\ t \geq 0, \tag{26}$$

with $\lambda_{min}(\cdot)$ denoting the minimum eigenvalue of the corresponding matrix.

Note that if vector $\phi_2(t)$ is PE, it also fulfils the next inequality

$$\alpha_3 I \leq \int_{t_0}^{t_0+\delta_1} \lambda^{(t-\tau)/T} \phi_2(\tau)\phi_2^T(\tau)d\tau \leq \alpha_4 I, \tag{27}$$

for all $t_0 \geq 0$, with $\alpha_3, \alpha_4, \delta_1$ being positive constants. Thus, if the regressor vector $\phi_2(t)$ is PE, then $P^{-1}(t)$ is positive definite, i.e., the PE condition on $\phi_2(t)$ implies the fulfillment of Assumption 1 and the validity of the parameter estimation formula (25). This fact is important because $\phi_2(t)$ depends on signals coming from the estimation system (6), which is not affected by disturbances such as those due to time derivative estimations, quantization error, or noisy measurements. This is not the case, however, for the vector $\phi_1(t)$, which is the one used in standard LS parameter identification techniques. In the simulations presented in this work, this feature on $\phi_2(t)$ seems to make the proposed algorithm less sensitive to disturbances.

Finally, note that equation (25) is similar to (1), with $\phi_2(t)$ being the corresponding regressor vector, and matrix $P^{-1}(t)$ may be interpreted as the covariance matrix used in the RLS algorithms. In the following, equation (27) is referred to as the λ-persistently exciting or λ-PE condition, and it is equivalent to the PE condition (2) when $\lambda = 1$.

4.3. Parameter convergence analysis

In general, the actual system (4) may not be open-loop stable. Then, for safety reasons, the parameter identification procedure must be implemented with the system working in closed-loop. Besides, for parameter identification purposes, we only require a stabilizing controller with relative low tracking error [53], which must not depend on the system parameters. Thus, in order to stabilize the system (4), let us consider the next PD controller for the actual system

$$u(t) = k_1\tilde{\xi}_1(t) + k_2\tilde{\xi}_2(t), \tag{28}$$

and, for the estimation system, the estimated controller

$$u_e(t) = k_1\tilde{w}_1(t) + k_2\tilde{w}_2(t), \tag{29}$$

with

$$\tilde{\xi}_1(t) = x_d(t) - \xi_1(t),\ \tilde{\xi}_2(t) = \dot{x}_d(t) - \xi_2(t),$$
$$\tilde{w}_1(t) = x_d(t) - w_1(t),\ \tilde{w}_2(t) = \dot{x}_d(t) - w_2(t),$$

and $k_1, k_2 > 0$ being the controller gains, $x_d(t)$ the desired reference signal, and $\dot{x}_d(t)$ its time derivative. Note that both, the control law (28) and (29) use the same gains.

Controller (28) was selected because the system (4), in closed-loop with (28), may be seen as a stable linear system, affected by a bounded disturbance. Therefore, the PD control (28) ensures boundedness of ξ_1, ξ_2 with positive PD gains k_1 and k_2.

From (28)-(29), the error signal ε_u in (9) can be expressed as

$$\varepsilon_u = -k_1\varepsilon_1 - k_2\varepsilon_2, \tag{30}$$

which allows expressing (15) as

$$\dot{\varepsilon}_2 = \tilde{\theta}^T\phi_2 - bk_1\varepsilon_1 - [a + bk_2]\varepsilon_2 - c_1\tilde{f}_1 - c_2\tilde{f}_2. \tag{31}$$

Let us define

$$\begin{aligned} A &= \begin{bmatrix} 0 & 1 \\ -bk_1 & -[a+bk_2] \end{bmatrix}, \\ B &= \begin{bmatrix} 0 & 1 \end{bmatrix}^T, \\ \epsilon &= \begin{bmatrix} \varepsilon_1 & \varepsilon_2 \end{bmatrix}^T, \\ r &= -c_1\tilde{f}_1 - c_2\tilde{f}_2. \end{aligned} \tag{32}$$

Using (31) and the definition of r in (32), the output error $e(t)$ in (12) can be rewritten as follows

$$\begin{aligned} e = &-[\mu_3 bk_1 - \mu_1]\varepsilon_1 - [\mu_3[a+bk_2] - \mu_2]\varepsilon_2 \\ &+ \mu_3\left[\tilde{\theta}^T\phi_2 + r\right]. \end{aligned} \tag{33}$$

Then, from (9), (25), (32), and (33), the state-space equation describing the closed-loop error and parameter error dynamics is given by

$$\frac{d}{dt}\begin{bmatrix} \epsilon \\ \tilde{\theta} \end{bmatrix} = \begin{bmatrix} A\epsilon + B\left[\tilde{\theta}^T\phi_2 + r\right] \\ -\frac{1}{\mu_3}P(t)\phi_2(t)e(t) \end{bmatrix}, \tag{34}$$

where the origin $[\epsilon^T, \tilde{\theta}^T]^T = \mathbf{0} \in \mathbb{R}^7$ is an equilibrium of (34).

In order to prove the stability of the origin of (34), let us consider the next positive definite Lyapunov candidate function

$$V(\boldsymbol{\epsilon},\tilde{\boldsymbol{\theta}}) = V_1(\boldsymbol{\epsilon}) + V_2(\tilde{\boldsymbol{\theta}}), \tag{35}$$

with

$$V_1(\boldsymbol{\epsilon}) = \frac{\alpha_1}{2}\boldsymbol{\epsilon}^T P_1 \boldsymbol{\epsilon},$$
$$V_2(\tilde{\boldsymbol{\theta}}) = \frac{\alpha_2\mu_3}{2}\tilde{\boldsymbol{\theta}}^T P^{-1}\tilde{\boldsymbol{\theta}},$$

where α_1, α_2 are positive constants, and $P_1 \in \mathbb{R}^{2\times 2}$ is a symmetric positive definite matrix, which is solution of the Lyapunov equation

$$\frac{1}{2}\left[A^T P_1 + P_1 A\right] = -Q, \tag{36}$$

with $Q \in \mathbb{R}^{2\times 2}$ being a positive definite matrix.

The time derivative of V_1 and V_2 along the trajectories of (34) is given by

$$\dot{V}_1 = -\alpha_1 \boldsymbol{\epsilon}^T Q \boldsymbol{\epsilon} + \alpha_1 \boldsymbol{\epsilon}^T P_1 B\left[\tilde{\boldsymbol{\theta}}^T \phi_2 + r\right], \tag{37}$$

$$\begin{aligned}\dot{V}_2 = &-\frac{\alpha_2\mu_3}{4}\tilde{\boldsymbol{\theta}}^T\left[\phi_2\phi_2^T - \frac{2\ln\lambda}{T}P^{-1}\right]\tilde{\boldsymbol{\theta}} \\ &+ \alpha_2\tilde{\boldsymbol{\theta}}^T\phi_2\left[\left[\mu_3 bk_1 - \mu_1\right]\varepsilon_1 + \left[\mu_3\left[a+bk_2\right] - \mu_2\right]\varepsilon_2\right. \\ &\left.-\mu_3 r\right] - \frac{\alpha_2\mu_3}{4}\left[\tilde{\boldsymbol{\theta}}^T\phi_2\right]^2.\end{aligned} \tag{38}$$

Let p_{ij} denote the ij-th element of matrix P_1. Then, we can obtain the relations

$$\begin{aligned}\alpha_1\boldsymbol{\epsilon}^T P_1 B &= \alpha_1\left[p_{12}\varepsilon_1 + p_{22}\varepsilon_2\right], \\ -\alpha_2\mu_3\tilde{\boldsymbol{\theta}}^T\phi_2 r &= \mu_3\alpha_2\tilde{\boldsymbol{\theta}}^T\phi_2\left[c_1\tilde{f}_1 + c_2\tilde{f}_2\right],\end{aligned} \tag{39}$$

and

$$-\alpha_1\left[p_{12}\varepsilon_1 + p_{22}\varepsilon_2\right]\left[c_1\tilde{f}_1 + c_2\tilde{f}_2\right] \le \frac{3\alpha_1\beta_1\beta_2}{2}\|\boldsymbol{\epsilon}\|^2, \tag{40}$$

with

$$\beta_1 = \max\left\{|p_{12}|, |p_{22}|\right\}, \tag{41}$$
$$\beta_2 = \max\left\{L_1 c_1, L_2 c_2\right\}. \tag{42}$$

Using $\dot{V}_1, \dot{V}_2$, and relations (39)-(40), the time derivative of $V(\boldsymbol{\epsilon},\tilde{\boldsymbol{\theta}})$ can be written as

$$\begin{aligned}\dot{V} = &-\alpha_1\boldsymbol{\epsilon}^T Q\boldsymbol{\epsilon} + \alpha_1\tilde{\boldsymbol{\theta}}^T\phi_2\left[p_{12}\varepsilon_1 + p_{22}\varepsilon_2\right] \\ &- \alpha_1\left[p_{12}\varepsilon_1 + p_{22}\varepsilon_2\right]\left[c_1\tilde{f}_1 + c_2\tilde{f}_2\right] \\ &- \frac{\alpha_2\mu_3}{4}\tilde{\boldsymbol{\theta}}^T\left[\phi_2\phi_2^T - \frac{2\ln\lambda}{T}P^{-1}\right]\tilde{\boldsymbol{\theta}} \\ &+ \alpha_2\tilde{\boldsymbol{\theta}}^T\phi_2\left[\left[\mu_3 bk_1 - \mu_1\right]\varepsilon_1 + \left[\mu_3\left[a+bk_2\right] - \mu_2\right]\varepsilon_2\right. \\ &\left.+\mu_3\left[c_1\tilde{f}_1 + c_2\tilde{f}_2\right]\right] - \frac{\alpha_2\mu_3}{4}\left[\tilde{\boldsymbol{\theta}}^T\phi_2\right]^2,\end{aligned} \tag{43}$$

which allows bounding $\dot{V}$ as follows:

$$\begin{aligned}\dot{V} \le &-\alpha_1\left[\lambda_{\min}(Q) - \frac{3}{2}\beta_1\beta_2\right]\|\boldsymbol{\epsilon}\|^2 \\ &- \frac{\alpha_2\mu_3}{4}\tilde{\boldsymbol{\theta}}^T\left[\phi_2\phi_2^T - \frac{2\ln\lambda}{T}P^{-1}\right]\tilde{\boldsymbol{\theta}} \\ &+ \alpha_2\tilde{\boldsymbol{\theta}}^T\phi_2\left[\left[\mu_3 bk_1 - \mu_1\right]\varepsilon_1 + \left[\mu_3\left[a+bk_2\right] - \mu_2\right]\varepsilon_2\right. \\ &\left.+\mu_3\left[c_1\tilde{f}_1 + c_2\tilde{f}_2\right]\right] - \frac{\alpha_2\mu_3}{4}\left[\tilde{\boldsymbol{\theta}}^T\phi_2\right]^2 \\ &+ \alpha_1\tilde{\boldsymbol{\theta}}^T\phi_2\left[p_{12}\varepsilon_1 + p_{22}\varepsilon_2\right] \\ \le &-\alpha_1\left[\lambda_{\min}(Q) - \frac{3}{2}\beta_1\beta_2\right]\|\boldsymbol{\epsilon}\|^2 \\ &- \frac{\alpha_2\mu_3}{4}\tilde{\boldsymbol{\theta}}^T\left[\phi_2\phi_2^T - \frac{2\ln\lambda}{T}P^{-1}\right]\tilde{\boldsymbol{\theta}} \\ &- \left|\tilde{\boldsymbol{\theta}}^T\phi_2\right|\left[\frac{\alpha_2}{4}\left|\tilde{\boldsymbol{\theta}}^T\phi_2\right| - \left[\alpha_1\beta_1 + \mu_3\beta_2 + \alpha_2\beta_3\right]\right. \\ &\left.\left[|\varepsilon_1| + |\varepsilon_2|\right]\right],\end{aligned} \tag{44}$$

where

$$\beta_3 = \max\left\{\mu_3 bk_1 - \mu_1, \mu_3\left[a+bk_2\right] - \mu_2\right\}. \tag{45}$$

Let us define the set Ω_1, matrix R, and the element s_1 as follows

$$\begin{aligned}\Omega_1 &= \left\{\tilde{\boldsymbol{\theta}}^T\phi_2 : \left|\tilde{\boldsymbol{\theta}}^T\phi_2\right| > \right. \\ &\qquad\left.\frac{4}{\alpha_2}\left[\alpha_1\beta_1 + \mu_3\beta_2 + \alpha_2\beta_3\right]\left[|\varepsilon_1| + |\varepsilon_2|\right]\right\}, \\ R &= \phi_2\phi_2^T - \frac{2\ln\lambda}{T}P^{-1}, \\ s_1 &= \lambda_{\min}(Q) - \frac{3}{2}\beta_1\beta_2.\end{aligned} \tag{46}$$

From inequality (44) and definitions (46), if $\lambda_{\min}(R) > 0$, $s_1 > 0$, and $\tilde{\boldsymbol{\theta}}^T\phi_2 \in \Omega_1$, then

$$\dot{V} \le -\alpha_1 s_1\|\boldsymbol{\epsilon}\|^2 - \frac{\alpha_2\mu_3}{4}\lambda_{\min}(R)\left\|\tilde{\boldsymbol{\theta}}\right\|^2 < 0, \tag{47}$$

which implies that both the error vector $\boldsymbol{\epsilon}$ and the parameter error $\tilde{\boldsymbol{\theta}}$ converge to zero. Besides, the convergence to zero of $\varepsilon_1, \varepsilon_2$ implies that all the signals corresponding to the estimated model (6) remain bounded.

The previous development is summarized in the next theorem.

Theorem 2: Let us consider the closed-loop dynamic system (34), and the set Ω_1, matrix R, and s_1 defined as in (46). Given the parameter updating law

$$\dot{\hat{\boldsymbol{\theta}}}(t) = -\frac{1}{\mu_3}P(t)\phi_2(t)e(t), \tag{48}$$

with

$$e = \mu_1\varepsilon_1 + \mu_2\varepsilon_2 + \mu_3\dot{\varepsilon}_2, \tag{49}$$
$$\phi_2 = \left[-w_2, u_e, -f_1(w_1), -f_2(w_2), 1\right]^T, \tag{50}$$

$$P^{-1}(t) = \int_0^t \lambda^{(t-\tau)/T}\phi_2(\tau)\phi_2{}^T(\tau)d\tau, \tag{51}$$

if the regressor vector $\phi_2(t)$ is λ-PE, $\lambda_{\min}(R) > 0$, $s_1 > 0$, and $\tilde{\theta}^T\phi_2 \in \Omega_1$, then

$$\lim_{t\to\infty}\begin{bmatrix}\varepsilon_1(t)\\ \varepsilon_2(t)\\ \tilde{\theta}(t)\end{bmatrix} = \mathbf{0} \in \mathbb{R}^7, \tag{52}$$

i.e., the vector of parameter estimates $\hat{\theta}$ converges asymptotically to the nominal parameter vector θ, and all the signals from the actual system (4) and the estimation system (6) remain bounded.

5. ALGORITHM IMPLEMENTATION

In order to apply equation (48), the values corresponding to $\xi_2(t)$ and $\dot{\xi}_2(t)$ are needed. In practice, the time derivatives of the variable of interest are seldom available and if so, these measurements may be noisy and biased. Furthermore, if noisy estimates were used from bad quality sensors, biased estimations would be obtained if a LS algorithm were used. Then, a trustworthy procedure to estimate these values is required. In addition, it must be verified that the reference signal $x_d(t)$ satisfies Assumption (27).

5.1. Velocity and acceleration estimation

A possibility for computing the estimates of $\xi_2(t), \dot{\xi}_2(t)$ consists of using an algebraic approach, which is based on differential algebra theory [54]. This methodology allows obtaining a general formula for computing the time derivatives of a measurable signal. The procedure used in this work for estimating these values follows the same lines of reference [25].

Computation of $\xi_2, \dot{\xi}_2$ is performed using two estimators for each signal. Each estimator provides the estimated signals $\tilde{\xi}_2, \dot{\tilde{\xi}}_2$, corresponding to $\xi_2, \dot{\xi}_2$ respectively, using the next formulas [25]

$$\begin{aligned}\tilde{\xi}_2(t) =& \frac{1}{[t-t_r]^6}\Bigg[-720\int^{(5)}\xi_1 + 4320\int^{(4)}[t-t_r]\,\xi_1\\ &- 5400\int^{(3)}[t-t_r]^2\,\xi_1 + 2400\int^{(2)}[t-t_r]^3\,\xi_1\\ &-450\int[t-t_r]^4\,\xi_1 + 30\,[t-t_r]^5\,\xi_1\Bigg],\end{aligned} \tag{53}$$

$$\begin{aligned}\dot{\tilde{\xi}}_2(t) =& \frac{1}{[t-t_r]^6}\Bigg[-720\int^{(4)}\xi_1 + 4320\int^{(3)}[t-t_r]\xi_1\\ &- 5400\int^{(2)}[t-t_r]^2\xi_1 + 2400\int[t-t_r]^3\xi_1\\ &-300[t-t_r]^4\xi_1 + 24[t-t_r]^5\tilde{\xi}_2\Bigg],\end{aligned} \tag{54}$$

where the term $\int^{(j)} h$ denotes an iterated integral of the form $\int^{(j)} h = \int^t\int^{\sigma_1}\ldots\int^{\sigma_{j-1}} h(\sigma_j)d\sigma_j\ldots d\sigma_1$; and $t_r > 0$ is the resetting time. Then, by combining two estimators, signals $\tilde{\xi}_2, \dot{\tilde{\xi}}_2$ are computed as follows

$$\tilde{\xi}_2 = \begin{cases}\tilde{\xi}_{2a}, & \text{for } 0 \le t \mod t_r < \frac{\kappa_v}{2}\\ \tilde{\xi}_{2b}, & \text{for } \frac{\kappa_v}{2} \le t \mod t_r < \kappa_v,\end{cases} \tag{55}$$

$$\dot{\tilde{\xi}}_2 = \begin{cases}\dot{\tilde{\xi}}_{2a}, & \text{for } 0 \le t \mod t_r < \frac{\kappa_a}{2}\\ \dot{\tilde{\xi}}_{2b}, & \text{for } \frac{\kappa_a}{2} \le t \mod t_r < \kappa_a,\end{cases} \tag{56}$$

where κ_v, κ_a are positive constants. Signals $\tilde{\xi}_{2a}(t)$, $\dot{\tilde{\xi}}_{2a}$ and $\tilde{\xi}_{2b}$, $\dot{\tilde{\xi}}_{2b}$ correspond to the estimations provided by the first and second estimator, respectively.

The previous algebraic estimation procedure was selected because it provides more accurate and less noisy estimates when compared, for instance, to the estimates obtained using the dirty derivative [25,55].

5.2. Tuning of the algorithm gains

A reliable set of parameter estimates is obtained by a proper tuning of the parameter identification gains. This tuning stage requires setting the values of gains λ, μ_1, μ_2 and μ_3. The structure of the parameter identification algorithm given in Section 4, together with the parameter convergence analysis, permit obtaining some guidelines for tuning the gains of the proposed parameter identification algorithm.

There are some elements that allow us to know how to select the value of parameter λ. First, note that matrix $-ln(\lambda)P/T$ in equation (21) is alike to the positive definite matrix Q in the Kalman filter (see equation (2.3.12) in [48]), which allows concluding that the value of λ has to be less than one. On the other hand, from definitions given in equation (46) in the parameter convergence analysis, note that condition $\lambda_{\min}(R) > 0$ is trivially ensured by selecting $\lambda < 1$. Furthermore, an LS with forgetting factor algorithm is the usual way to avoid the covariance wind up, a problem that is present in ordinary LS algorithms. The usual values for the forgetting factor are within the interval $(0,1)$ [48]. Therefore, the values of λ are selected as $0 < \lambda < 1$. In this work, it was verified that a good set of parameter estimates is obtained by selecting λ close to one.

According to the parameter convergence analysis, gains μ_1, μ_2 can be selected having large values. However, simulations results show that this usually yields overshoots in the parameter estimates. For the parameter μ_3, it can be verified that the element $1/\mu_3$ in (25) plays the role of the adaptation gain, which implies that a faster convergence is obtained if the value of $1/\mu_3$ is large. Besides, small values for μ_3 allow ensuring that condition $\tilde{\theta}^T\phi_2 \in \Omega_1$ is satisfied. As a consequence, the value of μ_3 must be small. By considering the previous analysis, a usual way to select the values of μ_1, μ_2, μ_3 is as follows. Select small values for μ_1, μ_2 (for instance $\mu_1 = \mu_2 = 0.1$), and select the value of μ_3 to be about ten times smaller than the value of μ_2. This selection usually allows the parameter identification algorithm to converge, but with a slow rate of convergence. Then, better estimates are obtained by increasing the value of μ_1, although with overshoots in the parameter estimates. These overshoots can be reduced by first increasing the value of μ_2, and then by slightly varying the value of μ_3.

5.3. Persistency of excitation

Another important issue in parameter identification is associated with the PE condition, which is also related to the spectral richness of the reference signal $x_d(t)$ [14, 48]. An example of this is given in the next.

Five numerical simulations were performed using the model (4), with $f_1(\xi_1) = \sin(\xi_1)$, and $f_2(\xi_2) = \tanh(\beta_4\xi_2)$, $\beta_4 > 0$. It is assumed that this nonlinear system corresponds to a position controlled DC servomechanism. Each simulation corresponds to one of the reference signals x_{di}, $i = 0,1,2,3,4$, described by

$$x_{di} = \begin{cases} 1 & ,i=0 \\ \sum_{j=1}^{i} m_i \sin(\omega_i t) & ,\text{otherwise} \end{cases} \quad [\text{rad}], \qquad (57)$$

with $m_1 = 1.2$, $m_2 = 2$, $m_3 = 0.5$, $m_4 = 0.5$, $\omega_1 = 1$ rad/s, $\omega_2 = 2$ rad/s, $\omega_3 = 2.5$ rad/s and $\omega_4 = 3$ rad/s.

Reference signals (57) range from a constant reference x_{d0} (low excitation), up to a combination of four sinusoids (corresponding to eight spectral lines, i.e., high excitation). In addition, the robustness of the proposed parameter identification algorithm was assessed by adding a quantizer before measuring $\xi_1(t)$. This quantizer simulates an optical encoder with a resolution of 2000 pulses per revolution. The velocity and the acceleration estimates were obtained using formulas (55)-(56), with gains $\kappa_v = \kappa_a = 1.2$. The proposed parameter identification algorithm was applied using the gains $\mu_1 = 4, \mu_2 = 1.3, \mu_3 = 0.04, \lambda = 0.9999$, and $T = 0.1$. Table 1 shows the final estimates obtained with each simulation after 20 seconds. The nominal values of the system parameters used in the simulation are also provided.

Table 1 shows that the performance of the proposed parameter identification algorithm enhances when the excitation level corresponding to the reference signal $x_{di}(t)$ increases. In linear systems, parameter convergence is guaranteed if the number of spectral components of the reference signal equals the number of unknown parameters [48]. However, for nonlinear systems, a less amount of spectral lines may be enough for identifying the same number of parameters [22].

The results of the previous simulations show that good estimates are obtained with a reference signal having two sinusoids, and these estimates get closer to the real ones when three sinusoids are employed. However, adding more frequencies to $x_{di}(t)$ did not reflect an important improvement in the final estimates. Therefore, it can be expected that, when applying the proposed method to a second order system as that given in (4), parameter convergence will be assured by selecting a reference signal having three sinusoids.

Table 1. Parameter estimates obtained using the proposed parameter identification algorithm (48), and the reference signals given in (57).

Parameters	Nominal	x_{d0}	x_{d1}	x_{d2}
$\hat{a}$	0.35	1.8090	0.1090	0.3560
$\hat{b}$	87	97.14	44.46	85.27
$\hat{c}_1$	13	9.618	7.216	12.710
$\hat{c}_2$	1	1.833	0.5534	0.9766
$\hat{\beta}$	0.5	-3.095	0.2592	0.4452
Parameters	Nominal	x_{d3}	x_{d4}	-
$\hat{a}$	0.35	0.3591	0.3590	-
$\hat{b}$	87	86.22	87.56	-
$\hat{c}_1$	13	12.800	12.800	-
$\hat{c}_2$	1	0.9418	0.9596	-
$\hat{\beta}$	0.5	0.4738	0.4809	-

It is also worth mentioning that, despite the disturbances due to the position quantization error, velocity and acceleration estimations, the performance of the proposed parameter identification algorithm is remarkable. This may be due to the fact that the regressor vector $\phi_2(t)$ does not depend on signals coming from the actual system (4), but on signals coming from the estimated system (6), where the effect due to disturbances is absent.

We conclude this Section by providing the steps required to implement the proposed parameter identification algorithm. These steps are listed below:

1) create an estimation model with the state-space description given in (6);
2) design the stabilizing controllers (28)-(29) for the actual system (4) and the estimation system (6), respectively;
3) at each time instant, get the values of u ,u_e, ξ_1, ξ_2, w_1, w_2, $\dot{\xi}_2$, $\dot{w}_2$;
4) obtain the output error signal e using (49), the regressor vector ϕ_2 using (50), and compute the matrix P using (51);
5) compute the estimate $\hat{\theta}$ using (48),
6) update the parameters of the estimation model (6) using the new value of $\hat{\theta}$,
7) go to Step 3.

6. COMPARATIVE STUDY

This Section presents numerical simulations that validate the performance of the proposed on-line parameter identification algorithm.

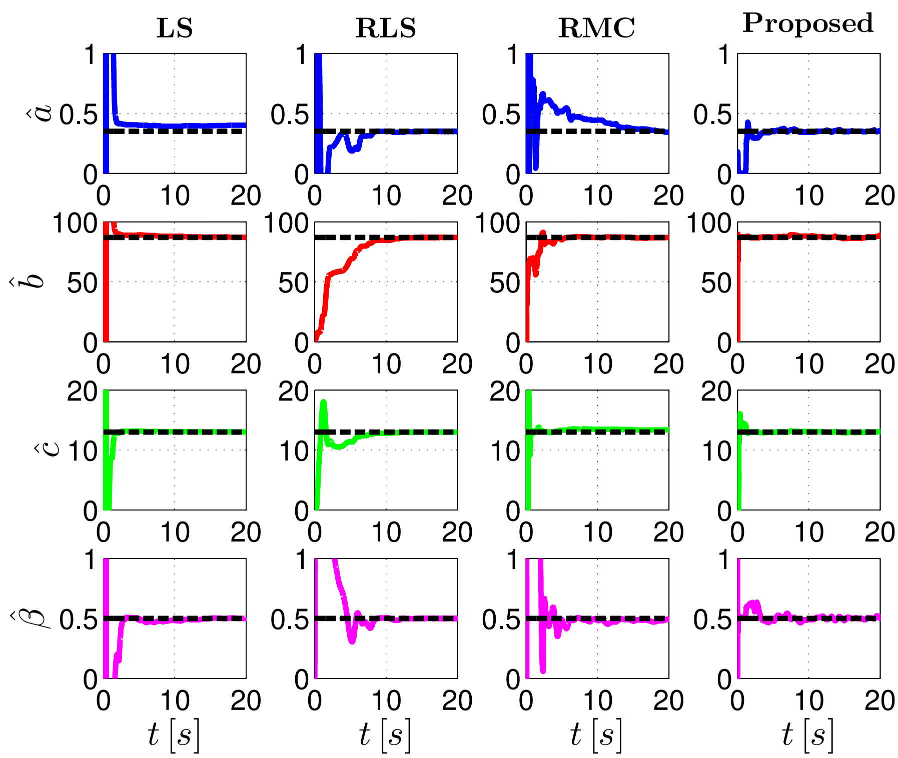

Fig. 2. Parameter estimates obtained using the off-line batch LS algorithm (LS), on-line RLS with forgetting factor algorithm (RLS), the on-line parameter identification method from [25] (RMC), and the proposed parameter identification method (Proposed). Dashed lines correspond to the nominal values.

Table 2. Parameter estimates obtained using the off-line LS algorithm, on-line RLS algorithm, the RMC algorithm, and the proposed method.

	$\hat{a}$	$\hat{b}$	$\hat{c}$	$\hat{\beta}$
Nominal	0.35	87	13	0.5
LS	0.3995	86.8605	12.9863	0.5089
RLS	0.3499	87	13	0.4996
RMC	0.3425	86.93	13.35	0.4913
Proposed	0.3566	87.1600	13.0400	0.5054

Simulations were performed using the system

$$\dot{\xi}_1 = \xi_2,$$
$$\dot{\xi}_2 = -a\xi_2 - c\sin(\xi_1) + \beta + bu, \tag{58}$$

where a, b, c are positive constants, and β is a constant perturbation affecting the system. It is assumed that model (58) corresponds to a DC servomotor attached to a pendulum, which moves in the vertical plane.

A quantizer was added after the output corresponding to signal $\xi_1(t)$ for simulating the effect of an optical encoder of 2000 pulses per revolution, and the velocity and acceleration estimations $\xi_2(t), \dot{\xi}_2(t)$ were obtained using the algebraic state estimators (55)-(56).

Table 3. RMS parameter error of each parameter identification scheme.

	$\hat{a}$	$\hat{b}$	$\hat{c}$	$\hat{\beta}$
LS	4.3353	107.9816	79.2632	10.9808
RLS	0.0051	0.1732	0.0161	0.0136
RMC	0.0031	0.0684	0.0281	0.0038
Proposed	**0.0004**	**0.0306**	**0.0111**	**0.0002**

The estimation system corresponding to (58) is given by

$$\dot{w}_1 = w_2,$$
$$\dot{w}_2 = -\hat{a}w_2 - \hat{c}\sin(w_1) + \hat{\beta} + \hat{b}u_e. \tag{59}$$

The actual system (58) and its estimation model (59) were operated in closed-loop, using the PD controllers (28)-(29) with gains $k_1 = 0.1$ and $k_2 = 0.001$. The reference signal for this simulation was $x_{d2}(t)$ from (57), and the proposed parameter identification algorithm was implemented using the gains $\mu_1 = 3.4$, $\mu_2 = 1.3$, $\mu_3 = 0.02$, $\lambda = 0.9999$, and $T = 0.1$.

The new algorithm is compared to an off-line batch LS algorithm, an on-line RLS with forgetting factor algorithm, and the parameter identification algorithm proposed in [25]. The on-line RLS algorithm and the on-

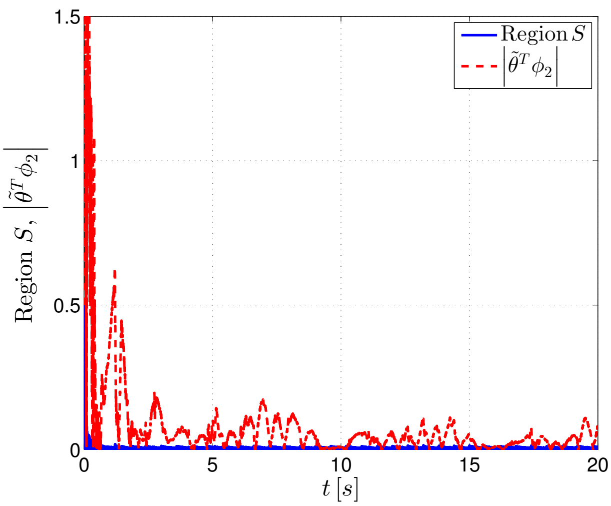

Fig. 3. Behavior of the region Ω_1 defined in (46). The red dashed line corresponds to $\left|\tilde{\theta}^T\phi_2\right|$, and the solid blue line shows the behavior of region $S(t)$ defined in (62).

line parameter identification algorithm given in [25] were used in order to compare the proposed method with other existing on-line methodologies. Particularly, the on-line algorithm addressed in [25] is robust against disturbances and has some similarities to that presented in this work. The off-line LS algorithm was selected because it has a pre-processing stage, which allows reducing the effects of disturbances; in this simulation, the off-line LS algorithm was implemented using a low-pass non-causal zero-phase digital filter for reducing the effect of position quantization error, and velocity estimates were computed without phase-shift using a central difference algorithm. A complete description of these three additional algorithms can be found in [25]. By using the same notation from [25], the on-line RLS algorithm was implemented using the values $\delta = 0.5$, $r_1 = 20$, $r_2 = 400$, and $P_0 = diag\{150, 200, 50, 50\}$; the off-line LS algorithm was implemented using the same values given in [25]. Finally, the on-line parameter identification algorithm introduced in [25] was implemented using the gains $\mu_1 = 0.35$, $\mu_2 = 0.5$, $\mu_3 = 0.06$ and $\lambda = 10^7$.

6.1. Parameter estimation results

In the following, the off-line LS algorithm and the on-line RLS algorithm will be referred to as the LS and RLS algorithms, respectively. The parameter identification algorithm given in [25] will be referred to as the RMC algorithm. The parameter estimates obtained with the four aforementioned algorithms are depicted in Fig. 2, where the dashed line corresponds to the nominal value. The numerical value of the parameter estimates obtained with each algorithm after 20 seconds are given in Table 2. The nominal values used in the simulation are also included

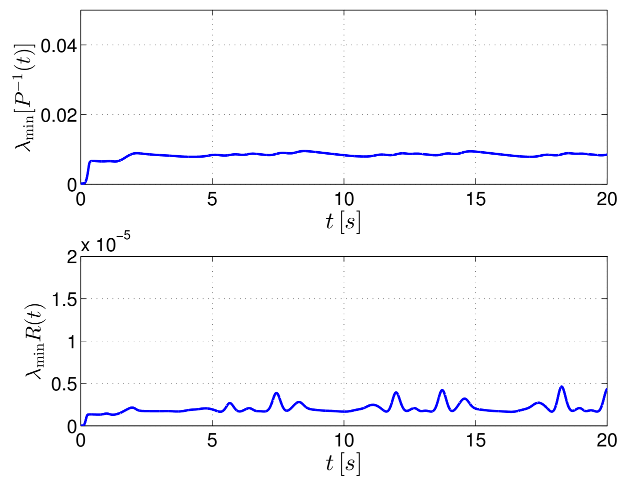

Fig. 4. Time evolution of the minimum eigenvalues corresponding to matrices $P^{-1}(t)$ and $R(t)$, ensuring the convergence of the parameter estimation error.

there.

Note that the values obtained with each algorithm are almost the same. However, the convergence of the proposed methodology is faster than that obtained with the other algorithms. This fact can be verified by computing the RMS values

$$RMS\{e_{\tilde{\theta}_i}\} = \sqrt{\frac{1}{m}\sum_{i=1}^{k} e^2_{\tilde{\theta}_i}(i)}, \tag{60}$$

where

$$e_{\tilde{\theta}_i} = \theta_i - \hat{\theta}_i, \tag{61}$$

is the i-th parameter estimation error, m is the number of samples, θ_i is the i-th actual parameter, and $\hat{\theta}_i$ is the i-th estimated parameter obtained with each parameter identification scheme. Table 3 shows the corresponding RMS errors. The lowest RMS values are highlighted in bold. Note that the proposed parameter identification method has the smallest RMS errors.

7. DISCUSSION

In the previous Section, the simulation results using the on-line RLS algorithm, off-line batch LS algorithm, RMC algorithm and the proposed parameter identification scheme were presented. It was observed that, despite the disturbances due to position quantization error, velocity estimation, and acceleration estimation, the final estimates obtained with the four algorithms were similar. However, the proposed methodology behaves faster than the other ones. Specifically, the accuracy of the proposed algorithm is similar to that obtained using the off-line LS algorithm, although the proposed scheme does not require any data pre-processing and operates on-line.

It is also important to verify if the conditions given in Theorem 2 are satisfied. To this end, the λ-PE condition given in (27), the set Ω_1, matrix R and the element s_1 defined in (46), were computed. Fig. 3 shows the behavior of Ω_1. The red dashed line corresponds to $\left|\tilde{\theta}^T \phi_2\right|$, and the blue line shows the behavior of $S(t)$ defined by

$$S(t) = \frac{4\left[\alpha_1\beta_1 + \mu_3\beta_2 + \alpha_2\beta_3\right]\left[|\varepsilon_1| + |\varepsilon_2|\right]}{\alpha_2}. \tag{62}$$

Condition (46) on the set Ω_1 is satisfied almost everywhere, and those points not satisfying this condition does not imply parameter drift, but time instants where the convergence is not asymptotic, which was also verified by the numerical simulations.

Fig. 4 shows the time evolution of the minimum eigenvalues of matrices $P^{-1}(t)$ and $R(t)$. Note that the minimum eigenvalue of matrix R is always positive, as required in Theorem 2. On the other hand, the positiveness of the minimum eigenvalue of matrix P^{-1} implies the λ-PE condition (27) on the regressor vector $\phi_2(t)$. Thus, all the conditions of Theorem 2 are satisfied.

Finally, regarding the computational complexity of the proposed parameter identification algorithm, it can be verified that the number of operations required for the parameter updating law (48) is the same as that required for implementing the on-line RLS algorithm (see equation (39) in [25]). However, the number of operations required for implementing the off-line LS algorithm and the proposed method in [25] are higher than those required for the method introduced in this paper. As a consequence, the proposed parameter identification algorithm turns out to be an appealing method that can be used for on-line parameter identification of second order nonlinear systems.

8. CONCLUSION

In this paper, a methodology for on-line closed-loop parameter identification of second order nonlinear systems was presented. The proposed methodology was applied to second order nonlinear systems working in closed-loop with a PD controller. The new parameter identification methodology considered an estimation model of the actual system, a cost function, and an output error signal based on a linear combination of signals from the actual and the estimation systems. The stability analysis of the proposed parameter identification scheme was accomplished by using a Lyapunov-based approach.

The proposed parameter identification algorithm was compared to an on-line RLS with forgetting factor algorithm, an off-line batch LS algorithm, and a previously reported on-line parameter identification algorithm. Numerical simulations proved that the proposed methodology converges faster than other methodologies, but without requiring any data preprocessing. Besides, different types of non-linearities can be identified using the proposed scheme.

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