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A New Observer Design for Systems in Presence of Time-varying Delayed Output Measurements

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Abstract: This paper presents a state observer for linear systems and Lipschitz nonlinear systems with delayed output measurements, which are affected by a known and bounded time-varying delay. The structure of the proposed observer is based on a proportional-integral term, which allows to compensate the time-varying delay. The observer gain depends on the maximum bounded delay. This gain is computed by a Linear Matrix Inequality (LMI) approach. The observer exhibits good performance for state estimation of the system despite the presence of significantly long delay. A Lyapunov-Krasovskii functional is used to prove the asymptotical convergence to zero of the observation error. This observer is applied to the case of systems with time-varying delay whose dynamic is described by a piecewise differentiable function. Examples and numerical simulations are provided in order to support the validity of the main results.

Keywords: Lyapunov-Krasovskii and linear matrix inequality, time-varying delay.

1. INTRODUCTION

Time-delayed systems are extensively investigated, since the delay phenomenon is often encountered in several engineering systems such as mechanical and electrical systems, communication networks, among others. A time delay can be produced due to the nature of the system or it can be induced into the system from transmission delays associated to other components interacting with the system. For instance, when the system is controlled or monitored through a remote communication system, or when the measurement process intrinsically causes a non-negligible time-delay. A time-delay may be the origin of instability or undesired oscillations in a system. For this reason, many researchers are devoted to investigate different automatic control approaches for time-delayed systems, such as stability, observability, controllability, system identification and fault detection [7, 9–12, 14, 16, 20].

Observer design for state estimation of linear and nonlinear systems with delayed output measurements has been investigated in previous works [5,19]. In these works the output measurement is affected by a known constant delay. A chained-observer algorithm has been proposed by [6] for a class of nonlinear drift observable systems, where each observer in the chain is used to estimate the state of the system for a suitable fraction of the total delay. A similar approach has been used in [8], where some restrictions of the chain of observers in [6] have been overcome. Also, in [3] a state predictor for nonlinear systems with constant delayed outputs is proposed and a cascade observer is presented. The conditions for the convergence of this state predictor have been derived using linear matrix inequalities. The particular case of piecewise constant delay is presented by [15], where the observer states are given only for the case of linear systems with piecewise time-constant delayed output. In [4], a state observer for drift observable nonlinear systems with time-varying delay in the measured outputs has been presented. The stability condition of this observer depends on the maximum value of the time-varying delay. A Lyapunov-Razumikhin approach was used to prove the asymptotical convergence of the observation error. Similarly, a high-gain observer with time-varying delayed measurements is proposed in [1]. This observer (which has a very restrictive structure) considers that the dynamic of the delay function is unknown but it is considered bounded and piecewise continuous. Two examples are shown: the piecewise constant

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delay case and the sampled-data case. Finally, an observer-based fault detection and diagnosis scheme for two-dimensional discrete time systems with time-varying state delays is studied by [17]. The proposed method is based on a descriptor system approach.

The main contribution of this paper is to propose a new observer design for linear systems and Lipschitz nonlinear systems with time-varying delayed output measurements. It is worth to note that that the class of the nonlinear systems addressed in this work is more general than the one presented in [1, 4]. The structure of the observer is based on a proportional-integral term which allows to compensate the time-varying delayed output. The observer gain is delay-dependent. It is computed by the resolution of a given LMI. The delay is assumed to be known, uniformly bounded but not necessarily time-continuous and the derivative of the delay is not required. The Lyapunov-Krasovskii functional is used to prove the asymptotical convergence to zero of the observation error. In order to illustrate the performance capabilities of the proposed observer, two particular cases are presented: (*i*) when the outputs of the system are subject to time-varying delays represented by piecewise constant functions and (*ii*) when the outputs of the system are subject to delayed non uniformly discrete measurements.

This paper is organized as follows: Section 2 describes the observer synthesis for linear systems with time-varying delayed outputs. The observer design for Lipschitz nonlinear systems with time-varying delayed outputs is presented in Section 3. In Section 4, the proposed observer is applied to the case of Lipschitz nonlinear systems when the output measurements are affected by a piecewise differentiable time-varying delay. Numerical simulations are presented in Section 5 in order to evaluate the performance of the proposed observer. Finally, conclusions are discussed in Section 6.

2. OBSERVER DESIGN FOR LINEAR SYSTEMS WITH DELAYED OUTPUT

Consider the following delayed linear system:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \;\; t \geq 0, \\ \bar{y}(t) = Cx(t - \tau(t)), \;\; t \geq 0, \;\; \tau(t) \in [0 \;\; \tau_M], \\ x(s) = \bar{x}, \; s \in [-\tau_M \quad 0], \end{cases} \tag{1}$$

where $x \in R^n$ is the state vector, $u \in R^m$ is the input, $\bar{y}(t) \in R^p$ is the delayed output. A, B and C are constant matrices of appropriate dimensions. $\tau(t)$ represents the known time-varying delay affecting the output measurements, which is bounded by some $\tau_M > 0$. The undelayed output is $y(t) = Cx(t)$. It is assumed that the system given in (1) is observable, i.e., the pair (A, C) is detectable.

The following assumptions and propositions are taken into account for the observer design:

Assumption 1: The variable delay $\tau(t)$ is known and bounded, e.g., $0 \leq \tau(t) \leq \tau_M$.

Proposition 1: For $a, b \in R^n$ and for any symmetric positive definite matrix P, the following inequality holds: $2a^T b \leq a^T P^{-1} a + b^T P b$.

Proposition 2: From the Newton-Leibniz formula we have for $x(t) \in R^n$:

$$x(t - \tau(t)) = x(t) - \int_{t-\tau(t)}^{t} \dot{x}(s) ds. \tag{2}$$

The proposed observer for system (1) has the following structure:

$$\begin{cases} \dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) - K(C\hat{x}(t) - \bar{y}(t) - C\mu(t)), \\ \mu(t) = \int_{t-\tau(t)}^{t} (A\hat{x}(s) + Bu(s)) ds, \\ \hat{x}(s) = \phi(s), \quad u(s) = \omega(s), \;\; s \in [-\tau_M, 0], \end{cases} \tag{3}$$

where the functions $\phi(s)$ and $\omega(s)$ are known. They are used to initialize system (3) in $[-\tau_M, 0]$. K is a suitable constant gain matrix of appropriate dimensions.

It is worth to mention that the integral term $\mu(t)$ can be computed numerically by using an integration numerical method (e.g., trapezoidal, Simpson's rules, etc.). The implementation of $\mu(t)$ needs a careful choice of the numerical method [2, 13].

For system (1), it is assumed that Assumption 1 is fulfilled. In the sequel, $\bigstar$ is used for the blocks induced by symmetry.

Theorem 1: If there exist two symmetric and positive definite matrices S and P and a matrix R of adequate dimensions such that the following LMI holds:

$$\begin{bmatrix} M_2 & \tau_M R^T CA \\ \bigstar & -\tau_M P \end{bmatrix} < 0. \tag{4}$$

$M_2 = SA + A^T S - R^T C - C^T R + \tau_M P$, then for a given delay $0 \leq \tau(t) \leq \tau_M$, system (3) is an asymptotic observer for system (1). The observation error $\tilde{x} = \hat{x}(t) - x(t)$ converges asymptotically to zero and $\lim_{t\to\infty} ||\tilde{x}|| = 0$. The observer gain is computed by: $K = S^{-1} R^T$.

Proof: Consider the observation error $\tilde{x}(t) = \hat{x}(t) - x(t)$. By combining equations (1) and (3), the dynamics of the observation error is:

$$\begin{aligned} \dot{\tilde{x}}(t) =& A\tilde{x}(t) - KC(\hat{x}(t) - x(t - \tau(t))) \\ &+ KC \int_{t-\tau(t)}^{t} (A\hat{x}(s) + Bu(s)) ds. \end{aligned} \tag{5}$$

According to Proposition 2:

$$\dot{\tilde{x}}(t) = (A - KC)\tilde{x}(t) + KCA \int_{t-\tau(t)}^{t} \tilde{x}(s) ds. \tag{6}$$

Now, consider the following Lyapunov-Krasovskii functional:

$$V = \tilde{x}^T(t)S\tilde{x}(t) + \int_{-\tau_M}^{0}\int_{\nu}^{0} \tilde{x}^T(t+s)P\tilde{x}(t+s)dsd\nu, \tag{7}$$

where τ_M is the constant introduced in Assumption 1.
The time derivative of V is:

$$\begin{aligned}\dot{V} =& 2\tilde{x}^T(t)S(A-KC)\tilde{x}(t) + 2\tilde{x}^T(t)SKCA\int_{t-\tau(t)}^{t}\tilde{x}(s)ds \\ &+ \tau_M\tilde{x}^T(t)P\tilde{x}(t) - \int_{t-\tau_M}^{t-\tau(t)} \tilde{x}(s)^T P\tilde{x}(s)ds \\ &- \int_{t-\tau(t)}^{t} \tilde{x}^T(s)P\tilde{x}(s)ds. \end{aligned}\tag{8}$$

According to Proposition 1, equation (8) can be rewritten as

$$\begin{aligned}\dot{V} \leq& \tilde{x}^T(t)(S(A-KC)+(A-KC)^TS+\tau_M P)\tilde{x}(t) \\ &+ \int_{t-\tau(t)}^{t} 2\tilde{x}^T(t)SKCA\tilde{x}(s)ds \\ &- \int_{t-\tau(t)}^{t} \tilde{x}^T(s)P\tilde{x}(s)ds. \end{aligned}\tag{9}$$

Now, consider the expression $2\tilde{x}^T(t)SKCA\tilde{x}(s) = 2((SKCA)^T\tilde{x}(t))^T\tilde{x}(s)$. Then, according to Proposition 1, with $a = (SKCA)^T\tilde{x}(t)$ and $b = \tilde{x}(s)$, for $P > 0$, it yields:

$$\begin{aligned}&\int_{t-\tau(t)}^{t} 2\tilde{x}^T(t)SKCA\tilde{x}(s)ds \\ &\leq \int_{t-\tau(t)}^{t} \tilde{x}^T(s)P\tilde{x}(s)ds \\ &\quad + \tau_M\tilde{x}(t)^T(SKCA)P^{-1}(SKCA)^T\tilde{x}(t). \end{aligned}\tag{10}$$

By considering (10) into (9):

$$\begin{aligned}\dot{V} \leq& \tilde{x}^T(t)(S(A-KC)+(A-KC)^TS+\tau_M P \\ &+ \tau_M(SKCA)P^{-1}(SKCA)^T)\tilde{x}(t) = \tilde{x}^T(t)\Lambda_1\tilde{x}(t),\end{aligned}$$

where $\Lambda_1 = S(A-KC)+(A-KC)^TS+\tau_M SKCAP^{-1}A^TC^T K^TS+\tau_M P$.

Therefore, if $\Lambda_1 < 0$, then $\dot{V} < 0$ and system (6) is asymptotically stable, i.e., $\lim_{t\to+\infty}||\tilde{x}(t)|| \to 0$. Hence, system (3) is an asymptotic observer for system (1). According to the Schur Lemma, inequality $\Lambda_1 < 0$ is equivalent to:

$$\begin{bmatrix} M_1 & \tau_M SKCA \\ \star & -\tau_M P \end{bmatrix} < 0, \tag{11}$$

where $M_1 = S(A-KC)+(A-KC)^TS+\tau_M P$. By considering $SK = R^T$, inequality (11) has the form of inequality (4) and $K = S^{-1}R^T$. This completes the proof. □

3. OBSERVER DESIGN FOR NONLINEAR LIPSCHITZ SYSTEMS WITH DELAYED OUTPUT

Consider the following Lipschitz nonlinear system with delayed outputs:

$$\begin{cases} \dot{x}(t) = Ax(t) + \varphi(u(t),x(t)), \ \ t \geq 0, \\ \bar{y}(t) = Cx(t-\tau(t)), \ \ t \geq 0, \ \ \tau(t) \in [0 \ \ \tau_M], \\ x(s) = \bar{x}, s \in [-\tau_M \quad 0], \end{cases}\tag{12}$$

where $x(t) \in R^n$ is the state vector, $u(t) \in R^m$ is the input, $\bar{y}(t) \in R^p$ is the measured delayed output. A and C are constant matrices of appropriate dimensions. $\tau(t)$ is a known time-varying delay, which satisfies Assumption 1.

Assumption 2: The function $\varphi(u,x)$ is globally Lipschitz in x, uniformly in u. Then there exists a matrix G of appropriate dimensions such that for all x and $z \in R^n$:

$$||\varphi(u,x) - \varphi(u,z)|| \leq ||G(x-z)||.$$

An observer for system (12) is given by:

$$\begin{cases} \dot{\hat{x}}(t) = A\hat{x}(t) + \varphi(u(t),\hat{x}(t)) - K(C\hat{x}(t) \\ \qquad\qquad - \bar{y}(t) - C\mu(t)), \ \ t \geq 0, \\ \mu(t) = \int_{t-\tau(t)}^{t} (A\hat{x}(s) + \varphi(u(s),\hat{x}(s)))ds, \\ \hat{x}(s) = \phi(s), \ \ u(s) = \omega(s), s \in [-\tau_M, 0], \end{cases}\tag{13}$$

where $\phi(s)$ and $\omega(s)$ are known functions. They are used to initialize the system (13) in $[-\tau_M, 0]$. K is a suitable constant gain matrix of appropriate dimensions. In order to implement the observer (13), the integral term $\mu(t)$ should be computed by using a numerical integration method (e.g., trapezoidal, Simpson's rules, etc.).

Assume that Assumptions 1 and 2 are fulfilled for system (12). In the sequel I is the identity matrix of appropriate dimensions.

Theorem 2: If there exist three scalars $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ and $\varepsilon_3 > 0$, a symmetric positive definite matrix S and a matrix R with appropriate dimensions such that:

$$\begin{bmatrix} M_2 & S & \tau_M R^T CA & \tau_M R^T C \\ \star & -\varepsilon_1 I & 0 & 0 \\ \star & \star & -\tau_M\varepsilon_2 I & 0 \\ \star & \star & \star & -\tau_M\varepsilon_3 I \end{bmatrix} < 0 \tag{14}$$

$M_2 = SA + A^TS - R^TC - C^TR + \varepsilon_1 G^TG + \tau_M\varepsilon_2 I + \tau_M\varepsilon_3 G^TG$, then for a given delay $0 \leq \tau(t) \leq \tau_M$, system (13) is an asymptotic observer for system (12), i.e., the observation error $\tilde{x} = \hat{x}(t) - x(t)$ converges asymptotically to zero: $\lim_{t\to\infty}||\tilde{x}|| = 0$. The observer gain can be computed as: $K = S^{-1}R^T$.

It is worth to note that, in contrast with the work presented in [1, 4] where only the triangular case was considered, Theorem 2 deals a more general class of nonlinear Lipschitz systems with time-varying delayed output, since the matrix A is any known constant matrix and the nonlinear function $\varphi(u(t),x(t))$ is any Lipschitz nonlinear function.

Proof: By considering (12) and (13), the dynamics of the observation error $\tilde{x}(t)$ is:

$$\begin{aligned}\dot{\tilde{x}}(t) =& A\tilde{x}(t)+\tilde{\varphi}(u(t),\hat{x}(t),x(t))\\ &-KC(\hat{x}(t)-x(t-\tau(t)))\\ &+KC\int_{t-\tau(t)}^{t}(A\hat{x}(s)+\varphi(u(s),\hat{x}(s))ds,\end{aligned} \quad (15)$$

where $\tilde{\varphi}(u(t),\hat{x}(t),x(t))=\varphi(u(t),\hat{x}(t))-\varphi(u(t),x(t))$.

According to Proposition 2:

$$\begin{aligned}\dot{\tilde{x}}(t) =& (A-KC)\tilde{x}(t)+\tilde{\varphi}(u(t),\hat{x}(t),x(t))\\ &+KC\int_{t-\tau(t)}^{t}(A\tilde{x}(s)+\tilde{\varphi}(u(s),\hat{x}(s),x(s)))ds.\end{aligned} \quad (16)$$

Consider the following Lyapunov-Krasovskii functional:

$$\begin{aligned}V =& \tilde{x}^T(t)S\tilde{x}(t)\\ &+\int_{-\tau_M}^{0}\int_{\nu}^{0}\tilde{x}^T(t+s)(\varepsilon_2 I+\varepsilon_3 G^T G)\tilde{x}(t+s)dsd\nu,\end{aligned}$$

where G is the constant matrix introduced in Assumption 2, τ_M is the positive constant denoted in Assumption 1; $\varepsilon_2>0$ and $\varepsilon_3>0$ are positive constants.

The time derivative of V is:

$$\begin{aligned}\dot{V} =& 2\tilde{x}^T(t)S(A-KC)\tilde{x}(t)+2\tilde{x}^T(t)S\tilde{\varphi}(u(t),\hat{x}(t),x(t))\\ &+\tau_M\tilde{x}^T(t)(\varepsilon_2 I+\varepsilon_3 G^T G)\tilde{x}(t)\\ &+2\tilde{x}^T(t)SKC\int_{t-\tau(t)}^{t}(A\tilde{x}(s)+\tilde{\varphi}(u(s),\hat{x}(s),x(s)))ds\\ &-\int_{t-\tau_M}^{t}\tilde{x}^T(s)(\varepsilon_2 I+\varepsilon_3 G^T G)\tilde{x}(s)ds,\end{aligned}$$

or equivalently:

$$\begin{aligned}\dot{V} =& 2\tilde{x}^T(t)S(A-KC)\tilde{x}(t)+2\tilde{x}^T(t)S\tilde{\varphi}(u(t),\hat{x}(t),x(t))\\ &+\tau_M\tilde{x}(t)^T(\varepsilon_2 I+\varepsilon_3 G^T G)\tilde{x}(t)\\ &+2\tilde{x}^T(t)SKC\int_{t-\tau(t)}^{t}(A\tilde{x}(s)+\tilde{\varphi}(u(t),\hat{x}(t),x(t)))ds\\ &-\int_{t-\tau_M}^{t-\tau(t)}\tilde{x}(s)^T(\varepsilon_2 I+\varepsilon_3 G^T G)\tilde{x}(s)ds\\ &-\int_{t-\tau(t)}^{t}\tilde{x}^T(s)(\varepsilon_2 I+\varepsilon_3 G^T G)\tilde{x}(s)ds\end{aligned} \quad (17)$$

The following inequality can be deduced :

$$\begin{aligned}\dot{V} \leq& \tilde{x}^T(t)(S(A-KC)+(A-KC)^T S)\tilde{x}(t)\\ &+2\tilde{x}^T(t)S\tilde{\varphi}(u(t),\hat{x}(t),x(t))\\ &+\int_{t-\tau(t)}^{t}2\tilde{x}^T(t)SKCA\tilde{x}(s)ds\\ &+\int_{t-\tau(t)}^{t}2\tilde{x}^T(t)SKC(\tilde{\varphi}(u(t),\hat{x}(t),x(t)))ds\\ &+\tau_M\tilde{x}^T(t)(\varepsilon_2 I+\varepsilon_3 G^T G)\tilde{x}(t)\\ &-\int_{t-\tau(t)}^{t}\tilde{x}^T(s)(\varepsilon_2 I+\varepsilon_3 G^T G)\tilde{x}(s)ds.\end{aligned} \quad (18)$$

According to Proposition 1 with $a=||S\tilde{x}(t)||$, $b=||G\tilde{x}(t)||$ and $\varepsilon_1>0$:

$$\begin{aligned}&2\tilde{x}^T(t)S(\tilde{\varphi}(u(t),\hat{x}(t),x(t)))\\ &\leq \varepsilon_1^{-1}\tilde{x}^T(t)SS\tilde{x}(t)+\varepsilon_1\tilde{x}^T(t)G^T G\tilde{x}(t).\end{aligned} \quad (19)$$

Consider the expression $2\tilde{x}^T(t)SKCA\tilde{x}(s)=2((SKCA)^T\tilde{x}(t))^T\tilde{x}(s)$. Then, by considering Proposition 1 with $a=(SKCA)^T\tilde{x}(t)$, $b=\tilde{x}(s)$ and $\varepsilon_2>0$:

$$\begin{aligned}&\int_{t-\tau(t)}^{t}2\tilde{x}^T(t)SKCA\tilde{x}(s)ds\\ &\leq\int_{t-\tau(t)}^{t}\varepsilon_2\tilde{x}^T(s)\tilde{x}(s)ds\\ &\quad+\tau_M\varepsilon_2^{-1}\tilde{x}^T(t)(SKCA)(SKCA)^T\tilde{x}(t).\end{aligned} \quad (20)$$

Likewise, by considering Proposition 2:

$$\begin{aligned}&\int_{t-\tau(t)}^{t}2\tilde{x}^T(t)SKC(\varphi(u(s),\hat{x}(s),x(s)))ds\\ &\leq\tau_M\varepsilon_3^{-1}\tilde{x}^T(t)(SKC)(SKC)^T\tilde{x}(t)\\ &\quad+\int_{t-\tau(t)}^{t}\varepsilon_3\tilde{x}^T(s)G^T G\tilde{x}(s)ds.\end{aligned} \quad (21)$$

By combining equations (19), (20) and (21) into (18):

$$\begin{aligned}\dot{V} \leq& \tilde{x}^T(t)(S(A-KC)+(A-KC)^T S+\varepsilon_1^{-1}SS\\ &+\varepsilon_1 G^T G+\tau_M\varepsilon_2^{-1}(SKCA)(SKCA)^T\\ &+\tau_M\varepsilon_3^{-1}(SKC)(SKC)^T+\tau_M\varepsilon_2 I+\tau_M\varepsilon_3 G^T G)\tilde{x}(t)\\ =& \tilde{x}^T(t)\Lambda_2\tilde{x}(t),\end{aligned} \quad (22)$$

where $\Lambda_2=S(A-KC)+(A-KC)^T S+\varepsilon_1^{-1}SS+\varepsilon_1 G^T G+\tau_M\varepsilon_2 I+\tau_M\varepsilon_3^{-1}(SKC)(SKC)^T+\tau_M\varepsilon_3 G^T G+\tau_M\varepsilon_2^{-1}(SKCA)(SKCA)^T$.

Therefore if $\Lambda_2<0$, then $\dot{V}<0$ and system (16) is asymptotically stable, i.e., $\lim_{t\to+\infty}||\tilde{x}(t)||\to 0$. In other words, system (13) is an asymptotic observer for system (12). According to the Schur's lemma, inequality $\Lambda_2<0$ is equivalent to:

$$\begin{bmatrix} M_1 & S & \tau_M SKCA & \tau_M SKC\\ \star & -\varepsilon_1 I & 0 & 0\\ \star & \star & -\tau_M\varepsilon_2 I & 0\\ \star & \star & \star & -\tau_M\varepsilon_3 I\end{bmatrix}<0, \quad (23)$$

where $M_1=S(A-KC)+(A-KC)^T S+\varepsilon_1 G^T G+\tau_M\varepsilon_2 I+\tau_M\varepsilon_3 G^T G$. By considering $SK=R^T$, inequality (23) becomes inequality (14). This completes the proof. □

Remark 1: It is worth to note that the initial conditions play an important role in the stability analysis of the observer error dynamics. In order to clarify this issue consider the stability condition of the observer (3) given in the proof of Theorem (1). This condition is: $\dot{V} = \tilde{x}^T(t)\Lambda_1\tilde{x}(t)$, where $\Lambda_1 = S(A-KC)+(A-KC)^TS+\tau_M SKCAP^{-1}A^TC^TK^TS+\tau_M P$.

Therefore, if $\Lambda_1 < 0$, then $\dot{V} \leq -|\lambda_{min}(\Lambda_1)|||\tilde{x}(t)||^2$ where $\lambda_{min}(\Lambda_1)$ is the small eigenvalue of Λ_1 and $\dot{V} \leq 0$. This means that $V(t) \leq V(0)$. It yields:

$$|\lambda_{min}(\Lambda_1)| \lim_{t->\infty} \int_0^t ||\tilde{x}(t)||^2 \leq V(0) - \lim_{t->\infty} V(t). \quad (24)$$

Inequality (24) involves indirectly the initial conditions, i.e., from (7), $V(0)$ is:

$$V(0) = \tilde{x}^T(0)S\tilde{x}(0) + \int_{-\tau_M}^{0}\int_{v}^{0} \tilde{x}^T(s)P\tilde{x}(s)dsdv. \quad (25)$$

By considering the initial condition $\bar{x}$ in (1), and $\phi(s)$ in (3), then $\tilde{x}(0) = \hat{x}(0) - x(0) = \phi(0) - \bar{x}$ and $\tilde{x}(s) = \hat{x}(s) - x(s) = \phi(s) - \bar{x}$. According to (24), it can be concluded that from any finite initial condition $\tilde{x}(0)$, the observation error $\tilde{x}(t)$ tends to zero when $t \to \infty$.

Remark 2: An example is provided in order to prove that a solution exists for LMI (4). Consider the following system:

$$\left\{ \dot{x}(t) = -ax + bu, \;\; \bar{y}(t) = cx(t-\tau(t)), \right. \quad (26)$$

where $a > 0$ and b are real scalars, c is a non-zero real scalar and $\tau(t)$ fulfills Assumption 2. Note that system (26) has the form (1) with $A = -a$, $B = b$ and $C = c$. Therefore, LMI (4) can be written as follows

$$\begin{bmatrix} -2sa+2ac+\tau_M p & \tau_M ca^2 \\ \bigstar & -\tau_M p \end{bmatrix} < 0, \quad (27)$$

here $R = -a$, $P = p > 0$ and $S = s > 0$. The matrix in (27) is negative definite if there exist $s > 0$ and $p > 0$ such that the following conditions are satisfied: $s > c + \frac{\tau_M p}{2a}$ and $s > c + \frac{\tau_M p}{2a} + \frac{\tau_M c^2 a^3}{2p}$.

By considering $p = a$, the matrix in (27) becomes negative definite: $s > c + \frac{\tau_M}{2} + \frac{\tau_M c^2 a^2}{2}$. Therefore, $s = 2c + \tau_M + \tau_M c^2 a^2$. According to Theorem 1, the observer gain is $K = \dfrac{-a}{2c+\tau_M+\tau_M c^2 a^2}$. It can be concluded that the LMI (4) is feasible.

4. APPLICATION TO SYSTEMS WITH OUTPUT SUBJECTS TO A PIECEWISE DIFFERENTIABLE DELAY

In many practical applications, the time-varying delay is given as a piecewise differentiable function as follows:

$$\tau(t) = w_k(t), \;\; t \in [t_k \;\; t_{k+1}), \;\; k \in N, \quad (28)$$

where $w_k(t)$ is a known bounded differentiable function for $t \in [t_k \;\; t_{k+1})$ and $(t_k)_{k\in N}$ is a known sequence of positive real numbers representing time instants satisfying $0 \leq t_0 < t_1 < \ldots < t_k < t_{k+1} < \ldots$ with $\lim_{k\to\infty} t_k = \infty$ and time-varying intervals $T_k = t_{k+1} - t_k$. Furthermore, it is assumed that $w_k(t)$ is differentiable for $t \in [t_k \;\; t_{k+1})$, $k \in N$ and $\dot{w}_k(t)$ is available. Finally consider that $w_{M,k} = \max\{w_k(t)\}$, $w_M = \max\{w_{M,k}\}$, where $w_{M,k} > 0$ and $w_M > 0$ are two scalars.

4.1. Lipschitz nonlinear systems with piecewise differentiable delay

When the delay has the form (28), then system (12) can be written as:

$$\begin{cases} \dot{x}(t) = Ax(t) + \varphi(u(t), x(t)), \\ \bar{y}(t) = Cx(t - w_k(t)), \;\; t \in [t_k \;\; t_{k+1}) \end{cases} \quad (29)$$

for $t \in [t_k \;\; t_{k+1})$. Therefore, an observer for system (29) is given by:

$$\begin{aligned} \dot{\hat{x}}(t) &= A\hat{x}(t) + \varphi(u(t), \hat{x}(t)) \\ &\quad - K(C\hat{x}(t) - \bar{y}(t) - C\mu(t)), \\ \mu(t) &= \int_{t-w_k(t)}^{t} (A\hat{x}(s) + \varphi(u(s), \hat{x}(s)))ds, \;\; t \in [t_k, t_{k+1}), \\ \hat{x}(s) &= \phi(s), \;\; u(s) = \omega(s), \;\; s \in [-w_M, 0]. \end{aligned} \quad (30)$$

Since $w_k(t)$ is differentiable, system (30) can be considered as follows:

$$\begin{cases} \dot{\hat{x}}(t) = A\hat{x}(t) + \varphi(u(t), \hat{x}(t)) - K(C\hat{x}(t) \\ \qquad - \bar{y}(t) - C\mu(t)) \\ \dot{\mu}(t) = A\hat{x}(t) + \varphi(u(t), \hat{x}(t)) \\ \qquad - (1 - \dot{w}_k(t))(A\hat{x}(t - w_k(t)) \\ \qquad + \varphi(u(t - w_k(t)), \hat{x}(t - w_k(t)))) \\ \qquad t \in [t_k \;\; t_{k+1}) \end{cases} \quad (31)$$

with initial conditions $\mu(t_k) = \int_{t_k - w_k(t_k)}^{t_k} (A\hat{x}(s) + \varphi(u(s), \hat{x}(s)))ds$ and $\hat{x}(s) = \phi(s)$, $u(s) = \omega(s)$ for $s \in [-w_M, 0]$.

The following result is derived from Theorem 2:

Remark 3: Consider system (29) with $\tau_M = w_M$ and Assumptions 1 and 2 are fulfilled. If there exist a symmetric and positive definite constant matrix S, three constants $\varepsilon_1 > 0, \varepsilon_2 > 0$ and $\varepsilon_3 > 0$ and a matrix R with appropriate dimensions such that the following LMI holds:

$$\begin{bmatrix} M_2 & S & w_M R^T CA & w_M R^T C \\ \bigstar & -\varepsilon_1 I & 0 & 0 \\ \bigstar & \bigstar & -w_M \varepsilon_2 I & 0 \\ \bigstar & \bigstar & \bigstar & -w_M \varepsilon_3 I \end{bmatrix} < 0 \quad (32)$$

$M_2 = SA + A^TS - R^TC - C^TR + \varepsilon_1 G^TG + w_M\varepsilon_2 I + w_M\varepsilon_3 G^TG$, then system (31) is an asymptotic observer for system (29). The observation error $\tilde{x}$ converges asymptotically to zero: $\lim_{t\to\infty} ||\tilde{x}|| = 0$. The observer gain is $K = S^{-1}R^T$.

Now, two practical cases of $w_k(t)$ are considered.

Case 1: Lipschitz nonlinear systems with outputs subject to time delay described by piecewise constant function

In this case, $w_k(t) = d_k,\ \ t \in [t_k\ \ t_{k+1}),\ \ k \in N$, and $d_k \geq 0$ is a sequence of known positive numbers. System (29) is:

$$\begin{cases} \dot{x}(t) = Ax(t) + \varphi(u(t), x(t)), \\ \bar{y}(t) = Cx(t - d_k),\ \ t \in [t_k\ \ t_{k+1}). \end{cases} \tag{33}$$

Then, observer (31) can be applied with $\dot{w}_k(t) = 0$ for $t \in [t_k\ \ t_{k+1})$ and $w_M = \max\{d_k\}$, with initial conditions $\hat{x}(s) = \phi(s),\ \ u(s) = \omega(s),\ \ s \in [-w_M, 0]$ and $\mu(t_k) = \int_{t_k - d_k}^{t_k} (A\hat{x}(s) + \varphi(u(s), \hat{x}(s)))ds$. The gain K matrix is computed as in Theorem 2.

Case 2: Lipschitz nonlinear systems with delayed nonuniformly discrete measurements

When $w_k(t) = t - t_k,\ \ t \in [t_k + d_k\ \ t_{k+1} + d_{k+1})$. System (29) becomes:

$$\begin{cases} \dot{x}(t) = Ax(t) + \varphi(u(t), x(t)), \\ \bar{y}(t) = y_k = Cx(t_k), \\ t \in [t_k + d_k\ \ t_{k+1} + d_{k+1}),\ \ k \in N. \end{cases} \tag{34}$$

Therefore, observer (31) can be applied with $\dot{w}_k(t) = 1$ for $t \in [t_k + d_k\ \ t_{k+1} + d_{k+1})$ and $w_M = \max\{t_{k+1} + d_{k+1} - t_k\} = T + d$, where $T = \max\{t_{k+1} - t_k\}$ and $d = \max\{d_k\}$, with initial conditions $\hat{x}(s) = \phi(s),\ \ u(s) = \omega(s),\ \ s \in [-w_M, 0]$ and $\mu(t_k + d_k) = \int_{t_k}^{t_k + d_k} (A\hat{x}(s) + \varphi(u(s), \hat{x}(s)))ds$.

Remark 4: Linear systems can be addressed in the same way as described above. Consider that Assumption 1 is fulfilled for system (1) with a bounded delay $\tau_M = w_M$. If there exist symmetric and positive definite constant matrices S and P, and a matrix R of appropriate dimensions such that:

$$\begin{bmatrix} M_2 & w_M R^T CA \\ \star & -w_M P \end{bmatrix} < 0. \tag{35}$$

$M_2 = SA + A^T S - R^T C - C^T R + w_M P$, then an asymptotic observer for the linear system (1) with a given delay $0 \leq \tau(t) \leq w_M$ can be designed. By considering an observer gain $K = S^{-1}R^T$, the observation error $\tilde{x}$ converges asymptotically to zero.

5. EXAMPLES

5.1. Linear Systems with outputs subject to piecewise-constant time delay.

Consider the same example as in [15], which is given by the following state-space representation:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -3 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u,$$

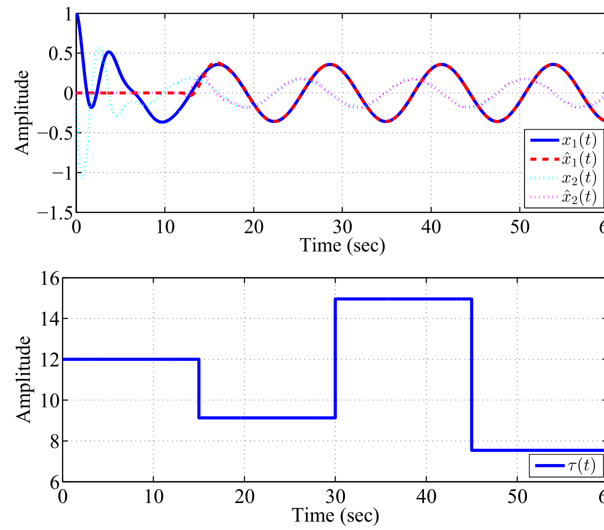

Fig. 1. Estimation of the linear system state and the time evolution of the delay, in Example 1.

$$\bar{y} = x_1(t - \tau(t)), \tag{36}$$

where the input $u(t) = \sin(t)$ and the initial conditions are $x(0) = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$ and $\hat{x}(\tau(0)) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$.

A simulation is carried out with a time-delayed output, with a time-varying delay $\tau(t)$. The value of $\tau(t)$ varies randomly in the range $\tau \in (5, 15)$s, $\Delta t = 15$s. This delay is displayed in Fig. 1. Note that the range of variation of the delay considered in [15] is $\tau \in (0.1, 1)$s which is smaller compared with the range considered in the present simulation. By using the Theorem 1 for $\tau_M = 15s$, the matrix gain K is: $K = \begin{bmatrix} -0.0020 & -0.0085 \end{bmatrix}^T$. Simulation results are plotted in Fig. 1. It can be appreciated that in counterpart with the work presented in [15] (where larger peak overshoot are presented in the transient response of the observer), there are not peak overshoots in the transient response of the observer.

5.2. Lipschitz nonlinear systems with outputs subject to time-varying delays described by piecewise constant functions

Consider the example presented in [1]

$$\dot{x} = \begin{bmatrix} -l_1 & c_1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ c_2 \sin(x_2) + c_3 \cos(x_2) + c_4 u \end{bmatrix},$$
$$\bar{y} = x_1(t - \tau(t)), \tag{37}$$

where $u(t) = \sin(0.35t)$, $c_1 = 1$, $c_2 = c_3 = 0.02$, $c_4 = 8$, $l_1 = l_2 = 1$; the initial conditions are: $x(0) = [-50, -50]$ and $\hat{x}(\tau(0)) = [0, 0]$.

The observer (13) is simulated for a time-delayed output, considering a random time-varying delay $\tau(t) \in (0.2s, 1s)$, which remains constant on the time interval $\Delta_t = 2s$. This delay is displayed in Fig. 2. By consid-

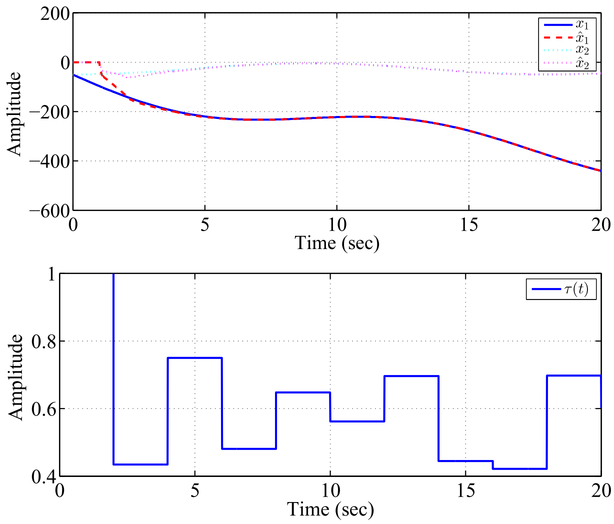

Fig. 2. Estimation of the states and evolution of the delay, in Example 2.

ering the Theorem 2 for $\tau_M = 1$, the observer gain is: $K = [24.49, 15.77]^T$.

The simulation results are displayed in Fig. 2. It can be seen that the state estimation $\hat{x}(t)$ is continuous but its derivative is not continuous, due to the change Δ_t in the function $\tau(t)$. In this case, the derivative of the delay function is defined by $\dot{\tau}(t) = 0$. This consideration is used to compute $\mu(t)$. Thus, for each interval Δ_t, the observer is initialized by the integral term $\mu(0)$ in $[0, -\tau(0)]$. It is worth to note that the proposed observer can tackle arbitrarily long delays, which are significantly higher than those reported in [1], where the maximum delay is $\tau_M = 0.01s$.

5.3. Nonlinear Lipschitz systems with delayed non-uniformly discrete-time measurements

Consider the nonlinear Lipschitz system presented in [18]:

$$\dot{x} = \begin{bmatrix} x_2 \\ -48.6x_1 - 1.25x_2 + 48.6x_3 + 21.6u \\ 0.1x_1 - 0.1x_3 \\ 1.95x_1 - 1.95x_3 - x_4 + 0.33\sin x_4 \end{bmatrix}, \quad (38)$$

$$\bar{y} = \begin{bmatrix} x_1(t-\tau(t)) & x_2(t-\tau(t)) & x_3(t-\tau(t)) \end{bmatrix}^T,$$

where $u(t) = \sin(t)$, and the initial conditions are: $x(0) = [4, 0, 4, 0]$ and $\hat{x}(\tau(0)) = [3, 1, 3.5, 1]$.

The observer is simulated for a discrete-time delayed output with $\tau(t) = t - t_k + d$, $t_k = 1$s and $d = 0.5$s. By using the results presented in subsection 4.1 for $w_M = 1$, the observer gain is:

$$K = \begin{bmatrix} -0.0398 & 0.0182 & 7.5331 \\ -0.2406 & -0.0049 & -4.4244 \\ -0.0385 & 0.0169 & 7.0886 \\ -0.0217 & -0.0004 & -0.1110 \end{bmatrix}.$$

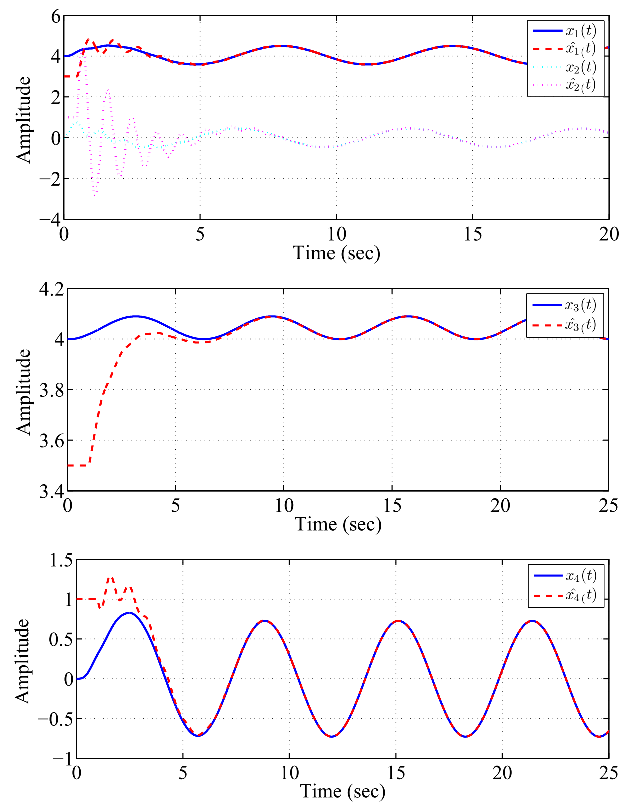

Fig. 3. Estimation of the states in Example 3.

The difference between the delayed output with the estimated output, is used for the correction term of the observer. In this case, the derivative of the delay function $\tau(t) = w_k = t - t_k + d$ is defined by $\dot{\tau}(t) = 1$. This consideration is used to compute $\mu(t)$. Thus, for each interval $\Delta_t = 1$s (since d is constant), the observer is initialized at each sampled output, through the integral function $\mu(t_k)$. It is worth to note that if interval $w_k = t - t_k + d$ is not known, it will not be possible to provide an accurate estimation of the states at that instant of time, therefore, new interval w_{k+1} must be adapted at new instant of time. In order to evaluate the performance and versatility of the proposed observer, some simulations are carried out. In Fig. 3, it can be appreciated the asymptotic convergence of the observation error even if $t_k > d$.

6. CONCLUSIONS

The main contribution of this paper is to propose an innovative approach to design observers for linear systems and Lipschitz nonlinear systems with time-varying delayed measurements. It is important to highlight that this class of systems is more general than the triangular case treated in other works, e.g., [1,4]. A Lyapunov-Krasovskii functional is used to prove the asymptotical convergence of the observation error. An advantage of the proposed observer is that the delay can be dynamically compensated by the observer gain. This observer has been applied to the case of systems with delays described by piecewise differentiable functions and also to the case of discrete-time measurements with time-varying constant delay. It is important to highlight, that there are no stronger restrictions for the time-varying delay. It is only necessary to know the dynamics of the delay and its boundedness.

REFERENCES

[1] V. V. Assche, T. Ahmed-Ali, C.A.B. Hann, and F. Lamnabhi-Lagarrigue, "High gain observer design for nonlinear systems with time varying delayed measurements," *Proc. of the 18th World Congress*, pp. 692-696, Aug. 2011.

[2] N. Bekiaris-Liberis and M. Krstic, "Compensation of State-Dependent Input Delay for Nonlinear Systems," *IEEE Trans. on Automatic Control*, vol. 58, no. 2, pp. 275-289, Feb. 2013.

[3] G. Besancon, D. Georges, and Z. Benayache, "Asymptotic state prediction for continuous-time systems with delayed input and application to control," *Proc. of European Control Conference*, pp. 1786-1791, Kos, Greece, July 2007.

[4] F. Cacace, A. Germani and C. Manes, "An observer for a class of nonlinear systems with time-varying observation delay," *Systems & Control Letters*, vol. 59, no. 5, pp. 305-312, May 2010.

[5] M. Darouach, "Linear functional observers for systems with delays in state variables," *IEEE Trans. on Automatic Control*, vol. 46, no. 3, pp. 491-496, Mar. 2001.

[6] A. Germani, C. Manes, and P. Pepe, "A new approach to state observation of nonlinear systems with delayed output," *IEEE Trans. on Automatic Control*, vol. 47, no. 1, pp. 96-101, Aug. 2002.

[7] R. Houimli, N. Bedioui, and M. Besbes, "An improved polytopic adaptive LPV observer design under actuator fault," *International Journal of Control, Automation and Systems*, vol. 16, no. 1, pp. 168-180, Mar. 2018.

[8] N. Kazantzis and R. A. Wright, "Nonlinear observer design in the presence of delayed output measurements," *Systems & Control Letters*, vol. 54, no. 9, pp. 877-886, Sep. 2005.

[9] M. Kazerooni, A. Khayatian, and A. A. Safavi, "Robust delay dependent fault estimation for a class of interconnected nonlinear time delay systems," *International Journal of Control, Automation, and Systems*, vol. 14, no. 2, pp.569-578, April 2016.

[10] J. Lei and H.-K. Khalil, "High-gain-predictor-based output feedback control for time-delay nonlinear systems," *Automatica*, vol. 71, no. 1, pp. 324333, Sep. 2016.

[11] J. Lu, C. Feng, S. Xu, and Y. Chu, "Observer design for a class of uncertain state-delayed nonlinear systems," *International Journal of Control, Automation, and Systems*, vol. 4, no. 4, pp. 448-455, Aug. 2006.

[12] S. Mobayen, "Robust tracking controller for multivariable delayed systems with input saturation via composite nonlinear feedback," *Nonlinear Dynamics*, vol. 76, no. 1, pp 827-838, Dec. 2013.

[13] S. Mondie and W. Michiels, "Finite spectrum assignment of unstable time-delay systems with a safe implementation," *IEEE Trans. on Automatic Control*, vol. 48, no. 12, pp. 2207-2212, Dec. 2003.

[14] S. Santra, H. R. Karimi, R. Sakthivel, and S. Marshal Anthoni, "Dissipative based adaptive reliable sampled-data control of time-varying delay Systems," *International Journal of Control, Automation, and Systems*, vol. 14, no. 1, pp. 39-50, Feb. 2016.

[15] K. Subbarao and P. C. Muralidhar, "A state observer for LTI systems with delayed outputs: time-varying delay," *Proc. of American Control Conference*, pp. 3029-3033, Seattle, USA, June 2008.

[16] Y. Wang, D. Zhao, Y. Li, and S. X. Ding, "Unbiased minimum variance fault and state estimation for linear discrete time-varying two-dimensional systems," *IEEE Trans. on Automatic Control*, vol. 62, no. 10, pp. 5463-5469, April 2017.

[17] D. Zhao, D. Shen, and Y. Wang, "Fault diagnosis and compensation for two-dimensional discrete time systems with sensor faults and time-varying delays," *International Journal of Robust and Nonlinear Control*, vol. 27, no. 16, pp. 3296-3320, Dec. 2016.

[18] Y. Zhao, J. Tao, and N.-Z. Shi, "A note on observer design for one-sided Lipschitz nonlinear systems," *Systems & Control Letters*, vol. 59, no. 1, pp. 66-71, Jan. 2010.

[19] G. Zheng and F.-J. Bejaranoc, "Observer design for linear singular time-delay systems," *Automatica*, vol 80, no. 1, pp. 1-9, June 2017.

[20] M. Zhong, D. Zhou, and S. X. Ding, "On designing H_∞ fault detection filter for linear discrete time-varying systems," *IEEE Trans. on Automatic Control*, vol. 55, no. 7, pp. 1689-1695, July 2010.

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