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Discrete Time Sliding Mode Controller Using a Disturbance Compensator for Nonlinear Uncertain Systems

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Abstract: In this paper, we propose a new sliding mode control for discrete time nonlinear uncertain systems. The uncertainties include both parametric uncertainties in the state model and external disturbances. To recover the lost invariance and robustness properties of discrete sliding mode control, we develop a disturbance estimation scheme to compensate the system uncertainties without affecting the control law. This control approach ensures the stability of the closed loop system as well as chattering reduction. The performance of the proposed controller is applied to control the motion of a cart-inverted pendulum used as a typical benchmark of nonlinear systems. The stabilization problem of the inverted pendulum system is to design a controller to keep the pendulum in its unstable equilibrium point in the presence of disturbances and parameters variation. The simulation result shows the effectiveness of the control design.

Keywords: Cart-inverted pendulum, discrete sliding mode control, nonlinear system, uncertainty, disturbance estimation.

1. INTRODUCTION

The cart-inverted pendulum is a typical benchmark problem in the control system engineering. The control task of this system is to swing up the pendulum from the stable equilibrium point to the unstable equilibrium point, and balancing the pendulum at the upright position. In general, the main difficulty is to guides the pendulum from any arbitrary initial condition to the upright equilibrium and stabilizes the cart in a desired position. Due to its highly nonlinear structure, some advanced control algorithms for swinging-up and stabilizing the inverted pendulum are developed [1–3].

In recent years, Sliding Mode Control (SMC) was widely studied for continuous-time systems [4–9]. One of the most attractive features of continuous SMC is its invariance and robustness properties to uncertainties including modelling errors and external disturbances. This is achieved by assuming that infinitely fast switching between two different control structures is possible, and that the uncertainties are bounded and matched [10]. For a broad class of systems, this kind of control is particularly appealing due to its ability to deal with nonlinearities, time-variance, as well as uncertainties and disturbances, in a direct manner in the face of modelling imprecisions. The design of SMC consists of two phases: The first phase is to define a sliding surface, along which the process can slide to find its desired final value. Thus, the second phase is to design the control law in such a way that any state outside the sliding surface is driven to reach the surface in finite time and stay here. The condition under which the system states starting from any initial state, move towards the sliding surface and reach it in a finite time is called the reaching condition or reaching law. However, the main drawback of the continuous SMC is the so-called chattering phenomenon. Chattering is a high frequency oscillation around the desired equilibrium point. It is undesirable in practice, because it involves high control activity and can excite high frequency dynamics ignored in the modelling of the system. Several methods to reduce or even eliminate chattering exist in the literature. One approach is to replace the discontinuous control part by a continuous approximation such as for example the saturation function as can be found in [11]. A second approach is to introduce an adaptive switching gain, which adapts the gain according to the circumstances [12]. Another approach to reduce chattering is to create a dynamic reaching condition [13], which ensures smooth reaching of the switching surface, or using an observer to estimate the equivalent control [14]. Using high-order sliding mode controllers is another way to eliminate chattering [15]. This last technique is not only able to resolve the chattering problems but also to ensure the conservation of robustness properties and system performances. Furthermore, the second-

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order sliding mode control is relatively simple to implement and it gives good robustness to external disturbances [16, 17].

On the other hand, finite time stability is a specific property related to the sliding mode control. It is was shown in [18] that finite-time stable systems might enjoy not only faster convergence but also better robustness and disturbance rejection properties. Using the results of finite time Lyapunov stability [19], system trajectories can converge to an equilibrium state in finite time [20]. Moreover, finite time stability of nonlinear systems has been one of the most important area of research due to its importance significance in theory and practice [21–23].

Recently, there has been interest in developing sliding mode control methods for discrete time systems [24–27]. In discrete time sliding mode control (DSMC), the desirable properties of invariance and robustness are not maintained because of the finite sampling time. The limited sampling period makes control input constant between two sampling intervals. This means that when system dynamics cross the sliding surface between sampling intervals, the control input cannot immediately takes measures to make the system remain on the sliding surface. Thus, the system states cannot remain on the sliding surface, but can remain in a neighborhood of the sliding surface. This sliding-like motion in discrete time is called a quasi-sliding mode [28, 29]. Furthermore, chattering phenomenon in DSMC is mainly due to the finite sampling time. Hence, a zigzag motion in the neighborhood of the sliding surface is unavoidable. Reducing the zigzag motion, as in the continuous time is always a main concern of DSMC. So that, many approaches have been proposed to overcome this problem such as a continuous approximation used within the boundary layer of the sliding manifold [30, 31], a discrete second order sliding mode control used in [32] or a non-switching type reaching law presented in [33].

The main contribution of the paper is to synthesize a new strategy of discrete time sliding mode control handling a class of nonlinear systems in the presence of parametric uncertainties and external disturbances. Based on a disturbance compensator scheme, the controller ensures robustness against the system uncertainties. It also guarantees the stability of the sliding variables in the quasi-sliding mode as well as chatter mitigation. Compared to the existing schemes proposed in the sliding mode literature [26, 34], the present control design offers some superior properties such as fast response, high tracking precision, strong disturbance rejection and insensitivity to system uncertainty. It provides good performances for stabilizing the motion of a cart inverted pendulum system.

This article is organized as follows: In Section 2, we present the nonlinear model structure in discrete time and we give assumptions to complete its description. Section 3 presents the conception of the proposed discrete sliding mode controller. Firstly, the sliding surface and reaching conditions are addressed and discussed. Secondly, the controller and the disturbance estimation law are developed. Third, the robustness of the closed-loop system in the quasi-sliding mode is studied. In Section 4, the discrete dynamical model of the cart-inverted pendulum is derived. Section 5 presents a numerical example to demonstrate the application of the proposed controller for the inverted pendulum system. Finally, Section 6 gives concluding remarks.

2. PROBLEM FORMULATION

Consider a discrete-time nonlinear single input single output system subjected to external disturbances and parameter variations, defined by the following state model:

$$\begin{aligned} x(k+1) =& f(x(k)) + \Delta f(x(k) \\ &+ (g(x(k)) + \Delta g(x(k)))u(k) + d(k), \end{aligned} \tag{1}$$

where $x(k) = [x_1(k), \ldots, x_n(k)]^T \in \mathbb{R}^n$ is the state vector at the sampling instant k, $u(k) \in \mathbb{R}$ is the system input and $d(k) \in \mathbb{R}^n$ is the external disturbances vector.
$f(\cdot) \in \mathbb{R}^n$ and $g(\cdot) \in \mathbb{R}^n$ are vectors of nonlinear functions while $\Delta f(\cdot) \in \mathbb{R}^n$ and $\Delta g(\cdot) \in \mathbb{R}^n$ represent the parametric uncertainties on $f(\cdot)$ and the control gain uncertainties on $g(\cdot)$ respectively.

In order to satisfy the matching conditions, we consider the following assumption:

Assumption 1: There exist some scalars $\overline{f}$, $\overline{g}$ and $\overline{d}$ such that the conditions

$$\Delta f(x(k)) = g(x(k))\overline{f},\ \Delta g(x(k)) = g(x(k))\overline{g},$$
$$d(k) = g(x(k))\overline{d}$$

holds for system (1).

Therefore, system (1) can be rewritten as:

$$x(k+1) = f(x(k)) + g(x(k))\left(u(k) + \zeta(k)\right), \tag{2}$$

where $\zeta(k) \in \mathbb{R}$ is constructed as:

$$\zeta(k) = \overline{f} + \overline{g}u(k) + \overline{d}. \tag{3}$$

Note that the so-called equivalent disturbance $\zeta(k)$ includes the parametric uncertainties $\Delta f(\cdot)$, the control gain uncertainties $\Delta g(\cdot)$ as well as the external disturbances $d(k)$.

To complete the description of the above system, we consider the following assumptions:

Assumption 2: The external disturbance $d(k)$ is L_2 bounded [35], and satisfy the condition

$$\|d(k)\| \preceq \Delta_d,$$

where Δ_d is a known positive constant and $\|\cdot\|$ represent the 2-norm of a vector or a matrix.

Assumption 3: The equivalent disturbance $\zeta(k)$ is bounded such that

$$|\zeta(k)| \preceq \Delta_\zeta.$$

where Δ_ζ is a positive scalar.

3. MAIN RESULTS

In this section, the objective is to develop a control law such that discrete-time uncertain system (2) is asymptotically stable even in the presence of parametric uncertainties and external disturbances.

The design procedure of a DSMC can be divided in three parts:

1) Define a sliding surface such that in the quasi-sliding mode system response acts like the desired dynamics.
2) Select a reaching law such that the system is pushed onto the predetermined hypersurface in the state space.
3) Determine a control law such that quasi-sliding mode is reached and stayed thereafter.

3.1. Design of sliding surface and reaching law

The sliding mode control comports three modes, namely, reaching mode, sliding mode, and steady-state mode. The idea of discrete quasi-sliding mode is to build a switching function, noted $s(k) \in \mathbb{R}$, in such a way that the controller should steer the system (2) into a finite region around $s(k) = 0$, even in the presence of parametric uncertainties and external disturbances.

To begin with, we consider the discrete switching function defined as:

$$s(x(k)) = C^T(x(k) - x_d(k)) = \sum_{i=1}^{n} c_i(x_i(k) - x_{id}(k)), \tag{4}$$

where $x_d(k) = [x_{1d}(k), \ldots, x_{nd}(k)]^T \in \mathbb{R}^n$ is the desired state vector, with $e(k) = x(k) - x_d(k)$ is the tracking error vector.

$C = [c_1, \ldots, c_n]^T$ is an appropriate vector assumed to be designed such that $C^T g(x(k)) \neq 0$ and that the closed-loop system exhibits the desired performance.

After designing the switching function, we give the following lemma proposed by Sarpturk [28].

Lemma 1: A necessary and sufficient reaching condition for a discrete-time SMC to assure both sliding motion and convergence onto the hyper-plane is:

$$|s(k+1)| \prec |s(k)|. \tag{5}$$

According to the last relation, not only the direction of the closed-loop system is specified (namely towards the sliding surface), but also the norm of the switching function is defined to be strictly decreasing.

Condition (5), can be further decomposed into the following two inequalities:

$$(s(k+1) - s(k))\, sign(s(k)) \prec 0, \tag{6}$$

$$(s(k+1) + s(k))\, sign(s(k)) \succ 0, \tag{7}$$

where (6) and (7) are called sliding condition and convergence condition, respectively.
The sign function is defined as:

$$sign(s(k)) = \begin{cases} +1 \;\; if \;\; s(k) \succeq 0, \\ -1 \;\; if \;\; s(k) \prec 0. \end{cases} \tag{8}$$

Remark 1: The first condition implies that the closed-loop system should be moving in the direction of the sliding surface, whereas the second condition implies that the closed loop system is not allowed to go too far in that direction. In other words, condition (6) results in a lower bound for the control action and condition (7) results in an upper bound.

In sliding mode, the controller push the system onto a predetermined hypersurface in the state space. This can be achieve by applying the reaching law. The original approach proposed by Gao *et al.* in [36] is still very popular due to its simplicity to implement, satisfaction of desired performances such as strong robustness and allows fast convergence.

According to Gao *et al.* [36], the closed loop system in DSMC should possess the following properties:

- Starting from any initial state, the trajectory will move monotonically toward the switching plane and cross it in finite time.
- Once the trajectory has crossed the switching plane, it will cross the plane again in every successive sampling period, resulting in a zigzag motion about the switching plane.
- The size of each successive zigzag step is non increasing and the trajectory stays within a specified band.

In order to achieve the conditions (6)-(7), a reaching law for discrete-time sliding mode control is chosen as [36]:

$$s(k+1) = qs(k) - \eta sign(s(k)), \tag{9}$$

where $0 \prec q \prec 1$ and $\eta \succ 0$ are design parameters.

Remark 2: The reaching law given in (9) results in chattering phenomenon and its magnitude is limited within the so-called quasi-sliding mode bandwidth. So that system (2) converges in finite time into quasi-sliding mode band, the robustness parameter η is chosen appropriately, according to that in [25].

3.2. Design of the controller

In this section, we will further design the controller that ensures the reachability of the specified switching surface.

According to system equation (2), the switching function $s(k+1)$ at instant $k+1$ can be rewritten as:

$$s(k+1) = F(x) + G(x)u(k) + G(x)\zeta(k) - X_d(k+1) \tag{10}$$

with

$$F(x) = C^T f(x(k)),\ G(x) = C^T g(x(k)),$$
$$X_d(k+1) = C^T x_d(k+1).$$

Assume that the equivalent disturbance $\zeta(k)$ is slowly varying and is compensated using the following disturbance estimation law:

$$\hat{\zeta}(k+1) - \hat{\zeta}(k) = \lambda\left(\zeta(k) - \hat{\zeta}(k)\right), \tag{11}$$

where $\hat{\zeta}(k)$ is the estimate of $\zeta(k)$ and λ is a weighting factor.
According to the reaching law (9) and relation (10), the control law for discrete-time uncertain system (2) is expressed as:

$$\begin{aligned} u(k) = &-\hat{\zeta}(k) + G(x)^{-1} \\ &\times [-F(x) + X_d(k+1) + qs(k) - \eta sign(s(k))]. \end{aligned} \tag{12}$$

From relation (10), we can write:

$$G(x)\zeta(k) = s(k+1) - F(x) - G(x)u(k) + X_d(k+1). \tag{13}$$

Substitute (12) into (13), we obtain:

$$\begin{aligned} \zeta(k) =& \hat{\zeta}(k) + G(x)^{-1} \\ &\times [s(k+1) - qs(k) + \eta sign(s(k))]. \end{aligned} \tag{14}$$

Substitute (14) into (11), the estimation law of the equivalent disturbance can be written at instant $k+1$ as:

$$\begin{aligned} \hat{\zeta}(k+1) =& \hat{\zeta}(k) + \lambda G(x)^{-1} \\ &\times [s(k+1) - qs(k) + \eta sign(s(k))]. \end{aligned} \tag{15}$$

In order to verify the reachability condition (5) when the discrete sliding mode control law (12) is applied to the system (2), we consider the next theorem.

Theorem 1: If the discrete sliding mode control law (12), with the sliding function (4) and the disturbance estimation law (15) is applied to the nonlinear system (2), the reachability condition $|s(k+1)| \prec |s(k)|$ can be satisfied under the following condition:

$$|s(k)| \succeq \frac{\eta}{1+q}. \tag{16}$$

Proof: It is well known that a control input using the concept of discrete time sliding mode control is chosen such that $V(k+1) \prec V(k)$ for all instant k, where $V(k)$ is the candidate Lyapunov function given as [30]:

$$V(k) = \frac{1}{2}s^2(k). \tag{17}$$

Using expression (17), we obtain:

$$\frac{1}{2}\left(s(k+1) - s(k)\right)\left(s(k+1) + s(k)\right) \prec 0, \tag{18}$$

which can be further decomposed into the following two inequalities:

$$(s(k+1) - s(k))\, s(k) \prec 0, \tag{19}$$
$$(s(k+1) + s(k))\, s(k) \succ 0. \tag{20}$$

Replace $\zeta(k)$ by its estimate $\hat{\zeta}(k)$ in system equation (2), the difference between $s(k+1)$ and $s(k)$ can be expressed as:

$$s(k+1) - s(k) = (q-1)s(k) - \eta sign(s(k)). \tag{21}$$

Pre-multiplying (21) by $s(k)$, we obtain:

$$s(k)\left(s(k+1) - s(k)\right) = |s(k)|\left[(q-1)|s(k)| - \eta\right]. \tag{22}$$

Then, the sliding condition (19) will be satisfied if:

$$|s(k)| \succeq -\frac{\eta}{1-q}. \tag{23}$$

On the other hand, the sum of $s(k+1)$ and $s(k)$ is given by:

$$s(k+1) + s(k) = (1+q)s(k) - \eta sign(s(k)). \tag{24}$$

Pre-multiplying (24) by $s(k)$, we obtain:

$$s(k)\left(s(k+1) + s(k)\right) = |s(k)|\left[(1+q)|s(k)| - \eta\right]. \tag{25}$$

The convergence condition (20) will be satisfied if:

$$|s(k)| \succeq \frac{\eta}{1+q}. \tag{26}$$

From (23) and (26), we conclude that both sliding condition and convergence condition will be satisfied if the condition (16) is achieved. □

Note that the reaching law (9) cannot asymptotically converge to zero, even if system uncertainties are not present [37]. It can be shown that for a nominal system, using a saturation function instead of the sign function in (9) can achieve asymptotic convergence of the switching function to zero, as well as chatter reduction.

Thus, a stable reaching law can be rewritten as:

$$s(k+1) = qs(k) - \eta sat(s(k)). \tag{27}$$

The saturation function is given by:

$$sat(s(k)) = \begin{cases} \frac{1}{\phi}s(k) & \text{if } |s(k)| \prec \phi, \\ sign(s(k)) & \text{if } |s(k)| \succeq \phi, \end{cases} \tag{28}$$

where ϕ is the boundary layer thickness [38].

Hence, the expression of the control law (12) became:

$$u(k) = -\hat{\zeta}(k) + G(x)^{-1} \times [-F(x) + X_d(k+1) + qs(k) - \eta sat(s(k))]. \tag{29}$$

The disturbance estimation law (15) can be rewritten as:

$$\hat{\zeta}(k+1) = \hat{\zeta}(k) + \lambda G(x)^{-1} \times [s(k+1) - qs(k) + \eta sat(s(k))]. \tag{30}$$

3.3. Robustness and stability analysis

In this section, we study the system robustness and stability when the controller (29) is enforced on the system (2) and the reachability of the specified surface $s(k) = 0$ can be obtained, even in the presence of the equivalent disturbance $\zeta(k)$.

Substitute (29) into (10), we obtain:

$$s(k+1) = qs(k) - \eta sat(s(k)) + G(x)\left(\zeta(k) - \hat{\zeta}(k)\right). \tag{31}$$

Let

$$\beta(k) = G(x)\left(\zeta(k) - \hat{\zeta}(k)\right), \ \alpha = q - \frac{\eta}{\phi}.$$

The relation (31) can be rewritten as:

$$s(k+1) = qs(k) - \eta sat(s(k)) + \beta(k). \tag{32}$$

According to (32), we consider the next theorem.

Theorem 2: For the discrete-time nonlinear uncertain system (2) with control law (29) and disturbance estimation law (30), the closed loop sliding mode dynamics described by (32) is asymptotically stable if the condition (33) holds for all instant k, with $|\alpha| \prec 1$ and for some constant $\mu \succ 0$.

$$|\beta(k)| \preceq \mu. \tag{33}$$

Proof: Consider the case $|s(k)| \prec \phi$.

In this case, the difference between $s(k+1)$ and $s(k)$ is given by:

$$s(k+1) - s(k) = (\alpha - 1)s(k) + \beta(k). \tag{34}$$

Pre-multiplying (34) by $sign(s(k))$, we obtain:

$$(s(k+1) - s(k))\, sign(s(k)) = (\alpha - 1)|s(k)| + \beta(k) sign(s(k)). \tag{35}$$

The sliding condition (6) implies

$$\beta(k) sign(s(k)) \prec (1 - \alpha)\phi. \tag{36}$$

The sum of $s(k+1)$ and $s(k)$ can be written as follows:

$$s(k+1) + s(k) = (1 + \alpha)s(k) + \beta(k). \tag{37}$$

Pre-multiplying (37) by $sign(s(k))$, we obtain:

$$(s(k+1) + s(k))\, sign(s(k)) = (1 + \alpha)|s(k)| + \beta(k) sign(s(k)). \tag{38}$$

The convergence condition (7) implies

$$\beta(k) sign(s(k)) \succ -(1 + \alpha)\phi. \tag{39}$$

According to relations (36) and (39), we obtain the following inequality:

$$-(1 + \alpha)\phi \prec \beta(k) sign(s(k)) \prec (1 - \alpha)\phi. \tag{40}$$

Consider the case $|s(k)| \succ \phi$.

The difference between $s(k+1)$ and $s(k)$ is given by:

$$s(k+1) - s(k) = (q - 1)s(k) - \eta sign(s(k)) + \beta(k). \tag{41}$$

Pre-multiplying (41) by $sign(s(k))$, we have:

$$(s(k+1) - s(k))\, sign(s(k)) = (q - 1)|s(k)| - \eta + \beta(k) sign(s(k)). \tag{42}$$

The sliding condition (6) implies

$$\beta(k) sign(s(k)) \prec \eta + (1 - q)|s(k)|. \tag{43}$$

The sum of $s(k+1)$ and $s(k)$ is expressed as:

$$s(k+1) + s(k) = (q + 1)s(k) - \eta sign(s(k)) + \beta(k). \tag{44}$$

Pre-multiplying (44) by $sign(s(k))$, we obtain:

$$(s(k+1) + s(k))\, sign(s(k)) = (q + 1)|s(k)| - \eta + \beta(k) sign(s(k)). \tag{45}$$

The convergence condition (7) implies

$$\beta(k) sign(s(k)) \succ \eta - (1 + q)|s(k)|. \tag{46}$$

Note that when $|s(k)| \succ \phi$, $|s(k)|$ decreases until $|s(k)| \preceq \phi$. In this case, we obtain the same condition as in (40).

From relation (40), we can write the following inequalities:

$$\begin{cases} -(1 + \alpha)\phi \prec \beta(k) \prec (1 - \alpha)\phi & \text{if } s(k) \succ 0, \\ (-1 + \alpha)\phi \prec \beta(k) \prec (1 + \alpha)\phi & \text{if } s(k) \prec 0. \end{cases} \tag{47}$$

According to relation (47), we can choose the constant parameter μ as:

$$\mu = min((1 + \alpha)\phi, (1 - \alpha)\phi). \tag{48}$$

We conclude that inequality (33) is proved and the proof of the theorem is completed. □

4. DYNAMICAL MODEL OF THE INVERTED PENDULUM

A cart-inverted pendulum is widely used in the control literature of nonlinear uncertain systems. The system consists of a cart and a pendulum attached to it. Applying a force $\kappa(t)$ to the cart, for instance through a built-in electrical motor, and thus moving it forwards and backwards will cause the pendulum to swing. A simple schematic of the cart-inverted pendulum is shown in Fig. 1 and the parameter values are listed in Table 1.

The cart-inverted pendulum is assumed to have two degrees of freedom and can therefore be fully represented using two generalized coordinates chosen as the position x of the cart, which is the moving horizontally, and the angular deviation θ of the pendulum.

The dynamical behaviour of the cart-inverted pendulum can be described by the two following differential equations [39, 40]:

$$(M+m)\ddot{x}+ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta = u, \tag{49}$$

$$\ddot{x}\cos\theta + l\ddot{\theta} - g\sin\theta = 0. \tag{50}$$

Using the Euler discretisation with a sampling rate T and taking into account of uncertainties on the mass and the external disturbances, a discrete time state space model for the cart-inverted pendulum system can be derived by arranging the above equations (49)-(50) as:

$$\begin{cases} x_1(k+1) = x_1(k)+Tx_2(k) \\ x_2(k+1) = f_2(x(k))+g_2(x(k))\,(u(k)+\zeta(k)) \\ x_3(k+1) = x_3(k)+Tx_4(k) \\ x_4(k+1) = f_4(x(k))+g_4(x(k))\,(u(k)+\zeta(k)) \end{cases} \tag{51}$$

with:

$$f_2(x) = x_2(k)+T\frac{\left(lx_4{}^2(k)-g\cos x_3(k)\right)m\sin x_3(k)}{M+m\sin^2 x_3(k)},$$

$$f_4(x) = x_4(k) + T\frac{\left((m+M)g-mlx_4{}^2(k)\cos x_3(k)\right)\sin x_3(k)}{l\left(M+m\sin^2 x_3(k)\right)},$$

$$g_2(x) = \frac{T}{M+m\sin^2 x_3(k)},$$

$$g_4(x) = -\frac{T\cos x_3(k)}{l\left(M+m\sin^2 x_3(k)\right)},$$

where the state variables $x_1(k)$ denotes the cart position, $x_2(k)$ is the velocity of the cart, $x_3(k)$ is the angular deviation of the pendulum and $x_4(k)$ is the angular velocity. $u(k)\in\mathbb{R}$ is the system input and $\zeta(k)$ is the equivalent disturbance.

5. SIMULATION RESULTS

To validate the present controller, we consider the discrete uncertain model (51) describing the motion of the cart-inverted pendulum.

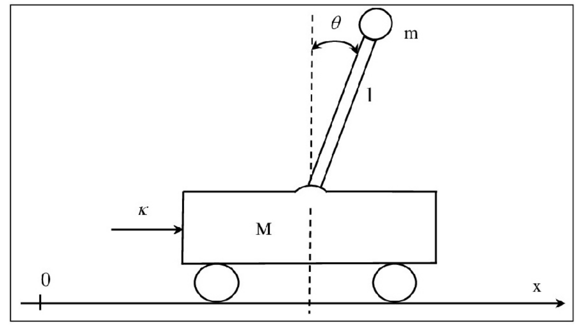

Fig. 1. Schematic of the cart-inverted pendulum

Table 1. Parameter values of the cart-inverted pendulum.

Designation	Symbol	Value	Unit
Mass of the cart	M	20	kg
Mass of the pendulum	m	1.8	kg
Length of the pendulum	l	0.3	m
Gravity acceleration	g	9.8	ms^{-2}

The system robustness was verified by choosing $\hat{\zeta}(0) = 0.1$. As design parameters, we take $q = 0.85$, $\eta = 0.2$, $\lambda = 0.25$, $\phi = 0.75$, and $T = 0.1$ s. For the coefficients c_i, we select $c_1 = 0.65$, $c_2 = 0.85$, $c_3 = 4.25$ and $c_4 = 1$. As desired state trajectories, we take $x_{1d}(k) = 2$ for the cart position and $x_{3d}(k) = 0$ for the angular deviation of the pendulum. As initial states, we take $x_1(0) = 0$, $x_2(0) = 0$, $x_3(0) = 0.1$ and $x_4(0) = 0$.
As performance index, we define

$$\rho_s = \frac{1}{N}\|S\|,$$

where $S = [s(1), \ldots, s(N)]^T$ and $N = 300$ is the number of iterations used in simulation.

Using (48) for calculating the parameter μ, we have

$$|\beta(k)| \preceq 0.3125.$$

The behaviours of the state $x_1(k)$ and the desired state $x_{1d}(k)$ are presented in Fig. 2, while Fig. 3 shows the evolutions of the state $x_3(k)$ and the desired state $x_{3d}(k)$. Fig. 4 shows the control input profile of the proposed controller, whereas the switching function is depicted in Fig. 5. The evolution of the equivalent disturbance is shown in Fig. 6, while Fig. 7 presented the tracking errors.

In order to verify the robustness of the closed loop system, simulation results are carried out for different numerical values of the robustness parameter η when $q = 0.85$. Table 2 illustrated the design parameters obtained in simulation.

As can be seen in Figs. 2-3, the system states follow perfectly their desired state trajectories. Thus, the present controller has allowed the stabilization and the

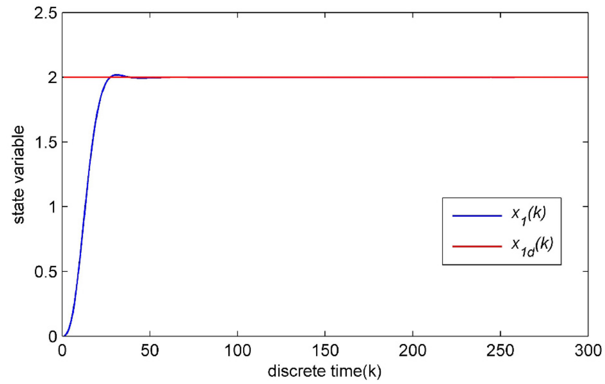

Fig. 2. Evolutions of the states $x_1(k)$ and $x_{1d}(k)$.

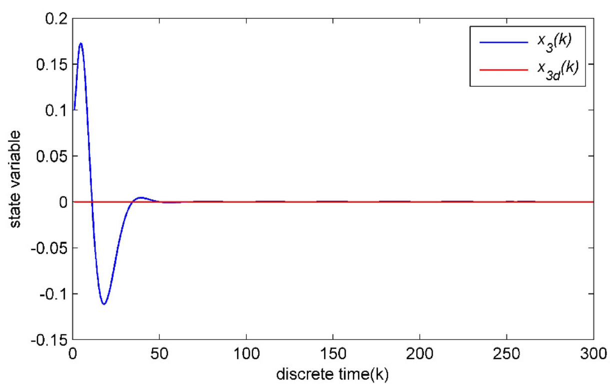

Fig. 3. Evolutions of the states $x_3(k)$ and $x_{3d}(k)$.

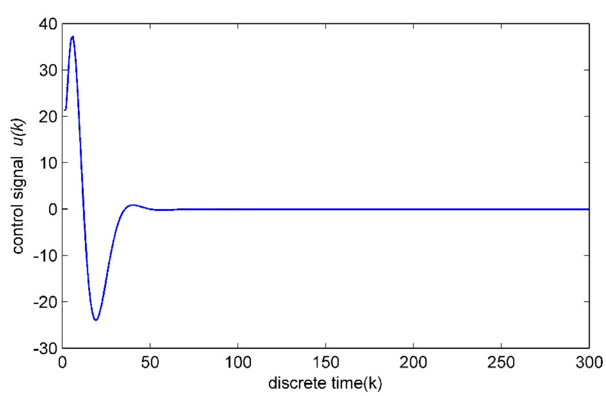

Fig. 4. Evolution of the control signal $u(k)$.

tracking of the desired trajectories. In addition, it attenuates the effect of both parameter uncertainties and external disturbances. From Fig. 4-5, one can see that the controller is able to compensate effectively the chattering phenomenon. Moreover, it can be observed from Fig. 6 that the system uncertainty is completely compensated using the estimation law (30) and the dynamics of the closed loop sliding mode dynamics (32) are effectively improved in the presence of uncertainties. Therefore, the discrete

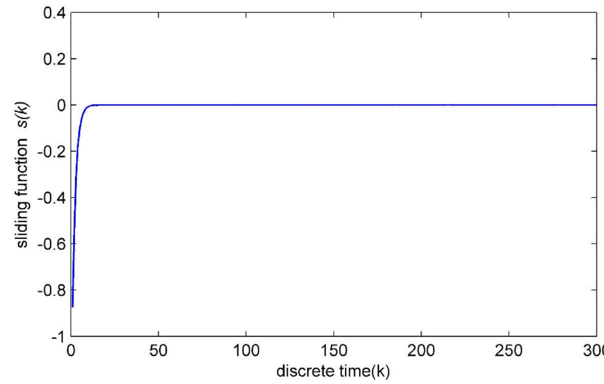

Fig. 5. Evolution of the sliding function $s(k)$.

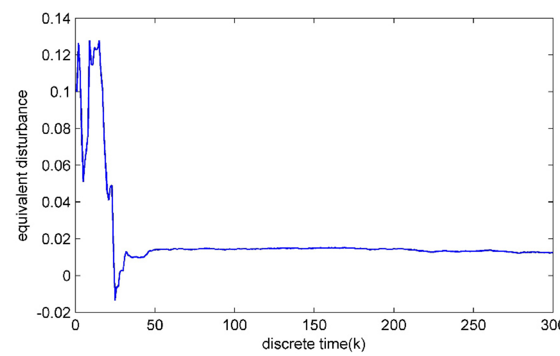

Fig. 6. Evolution of the equivalent disturbance.

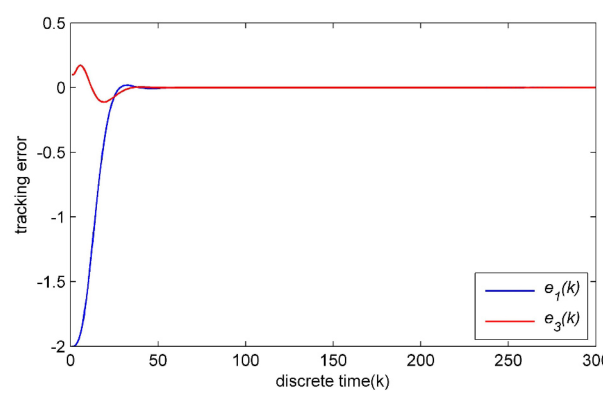

Fig. 7. Evolutions of the tracking errors.

quasi-sliding mode is reachable in finite time and the controller guarantees the reachability of the sliding motion, with $\rho_s = 3.7 \times 10^{-3}$.

From Table 2, when $0 \prec \eta \preceq 1.38$, the closed loop system is robust and the disturbance estimation error in (33) must become smaller than some bond $\mu = 0.7475$.

We conclude that the design parameters q and η determine the convergence rate and precision of the system states. As a result, a suitable choice of these parameters will force the system states to converge to the quasi-sliding

Table 2. Variation of the robustness parameter.

η	0.1	0.5	0.9	1.1	1.4
α	0.7167	0.1833	−0.35	−0.6167	−1.067
μ	0.2125	0.6125	0.4875	0.2875	−0.0125
ρ_s	0.0042	0.0030	0.0030	0.0033	0.0437

surface with a higher speed. Therefore, the present controller guarantees effectively the robustness of the closed loop system.

6. CONCLUDING REMARKS

In this paper, a discrete time sliding mode control design has been developed for nonlinear uncertain systems. A disturbance compensator scheme allows to estimates both parameter uncertainties and external disturbances. The proposed controller guarantees asymptotic stability of the system dynamics as well as chattering reduction and ensures robustness of the closed loop system against disturbances and system uncertainties. Simulation results have been conducted to show the effectiveness of the present controller to stabilise a cart-inverted pendulum system. It should also be noted that the performances of the controller obtained through simulations aims to be verified experimentally in future work on the actual cart-inverted pendulum.

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