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Robust H_∞ Control of Uncertain Stochastic Systems with Time-varying Interval Delays

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Abstract: A robust delay-dependent stability criterion for discrete-time uncertain stochastic systems is proposed to achieve asymptotical stability and H_∞ performance in this paper. Based on modeling approaches, Linear Parameter Varying (LPV) system with multiplicative noise term is built to represent uncertain stochastic systems. Moreover, state and input delays are considered as two individual time-varying interval cases for general effect. Employing a novel Lyapunov-Krasovskii function, Jensen inequality and transform technology, some relaxed sufficient conditions are derived into Linear Matrix Inequality (LMI) forms to apply convex optimization algorithm. Through solving the derived conditions, Gain-Scheduled (GS) controller can be designed such that robust asymptotical stability and H_∞ performance of closed-loop system are achieved in the mean square. At last, two numerical examples are provided to demonstrate applicability and effectiveness of this paper.

Keywords: Discrete Jensen ineqaulity, H_∞ control scheme, LPV systems, Lyapunov-Krasovskii function, Stochastic systems.

1. INTRODUCTION

Practically, stochastic behavior often appears around operated environment and it causes poor performance in control engineering. Thus, control problem of stochastic systems is an important issue and has been widely discussed by [1–5]. In [1, 2], an external disturbance was employed to describe stochastic behavior of systems. Referring to [3, 4], some sliding-mode control schemes have been proposed to stabilize stochastic systems described by Markovian jumping parameters. Different to [1–4], the stochastic behavior was formulated as multiplicative noise term by using stochastic modeling approach [5]. Based on the modeling approach, many stability criteria have been developed for linear stochastic systems [6–8] and nonlinear stochastic systems [9]. Moreover, H_∞ controller design methods have been proposed by [10–16] to guarantee stability and attenuation performance of stochastic systems. Robust stability criteria [12–14, 17, 18] have been developed for stochastic systems with admissible uncertainties. Some delay-dependent control problems of stochastic systems have been discussed by [19]. Studying the above literature, it should be noted that stability criteria of stochastic systems can be easily developed by extending fundamental results of deterministic systems. Therefore, the stochastic modeling approach is applied to describe stochastic systems.

Generally, uncertainty of stochastic systems is considered as a bounded description [12–14, 17, 18]. However, the bounded description limits characterization of uncertain systems. Besides, LPV system [20] provides a general description of linear system whose elements are depending on a set of time-varying parameters. Based on structure of the LPV system, uncertain systems can be completely described via combining several linear systems and weighting functions. Therefore, many robust stability criteria [21–27] have been proposed via LPV systems. Furthermore, GS design method [25–27] has been applied for controller synthesis of the LPV systems. Applying the GS design method, a H_∞ GS controller design method [25, 26] and the delay-dependent stability criteria [27] have been developed for the LPV systems, respectively. Unfortunately, control issue of uncertain stochastic systems has been discussed in few literature [28]. To extend the results of [28], the delay-dependent control issue of uncertain stochastic systems is disucssed in this paper.

It is well known that time delay in great number of dynamic processes involving propagation/transportation of material, information or energy [29–36] via common and wicked phenomenon in industries and engineering systems. For the reason, it is an important issue for discussing control problems of delayed dynamic systems. In the ex-

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isting researches, delay-independent criteria [29–31] and delay-dependent criteria [32–37] are proposed to deal with the effect of time-delay on dynamic systems. According to length of delays, the delay-dependent criteria [31–33] are often developed for analyzing stability of delayed systems, small delays especially. In addition, interval delay [34–36] is a particular case that varies in a region whose lower bound is not restricted to zero. From [27], a delay-dependent stability criterion was proposed for LPV systems with state and input delays. However, the state and input delays were considered as the same case in [27]. To consider general delay effect, state and input delays are concerned as individual time-varying interval delay in this paper. To the best of our knowledge, the robust H_∞ delay-dependent stability criterion of LPV stochastic systems with state and input delays is still an open problem.

Motivated the above illustration, the H_∞ delay-dependent robust criterion is proposed to deal with stability and stabilization problem of uncertain stochastic systems with time delays. To propose the delay-dependent criterion for LPV stochatic systems with time-delays, some sufficient conditions are derived and converted into extended LMI form to apply convex optimization algorithm [37]. Through solving those LMI conditions, one can find some feasible solutions to build GS controller such that the uncertain stochastic system with time delays is asymptotically stable with H_∞ performance index in the mean square. The advantages of this paper are furtherly concluded as follows: 1) A more general robust stability criterion than the related works [20–25, 27] is developed according to the consideration of stochastic behaviors. 2) State and input delays are considered as two individual time-varying interval cases for general delay effect on the system. 3) The disturbance attenuation performance is dealt with H_∞ control shceme. 4) A novel parameter dependent Lyapunov-Krasovskii function is proposed to derive some sufficient conditions to propose a relaxed GS controller design method. Finally, two examples are proposed to demonstrate the advantages of this paper.

Notation: The following notations are applied in this paper. The $\mathbf{I}$ denotes identity matrix. The $diag\{\bullet,\bullet\}$ denotes a block-diagonal matrix with element $\bullet$. The $\mathbf{\Lambda}_{2\times 2}^{diag}$ denotes the two blocks in diagonal matrix with element $\mathbf{\Lambda}$, such as $\mathbf{\Lambda}_{2\times 2}^{diag}=\begin{bmatrix}\mathbf{\Lambda} & 0\\ 0 & \mathbf{\Lambda}\end{bmatrix}$. The $\mathbf{\Lambda}_{i\times j}$ denotes a matrix with dimension $i\times j$ and its elements are $\mathbf{\Lambda}$, for example: $\mathbf{\Lambda}_{2\times 2}=\begin{bmatrix}\mathbf{\Lambda} & \mathbf{\Lambda}\\ \mathbf{\Lambda} & \mathbf{\Lambda}\end{bmatrix}$. The $E\{Q(\bullet)\}$ denotes the expected value of $Q(\bullet)$.

2. SYSTEMS DESCRIPTIONS AND PROBLEM FORMULATIONS

In this section, the following discrete-time LPV stochastic systems with time-varying interval delays are considered.

$$\begin{aligned} &x(k+1)\\ &=\mathbf{A}(\alpha(k))x(k)+\mathbf{A}_t(\alpha(k))x(k-h(k))\\ &\quad+\mathbf{B}(\alpha(k))u(k)+\mathbf{B}_t(\alpha(k))u(k-g(k))\\ &\quad+\mathbf{E}(\alpha(k))w(k)\\ &\quad+\sum_{e=1}^{q}\Big(\big(\bar{\mathbf{A}}_e(\alpha(k))x(k)+\bar{\mathbf{A}}_{t_e}(\alpha(k))x(k-h(k))\\ &\quad+\bar{\mathbf{B}}_e(\alpha(k))u(k)+\bar{\mathbf{B}}_{t_e}(\alpha(k))u(k-g(k))\\ &\quad+\bar{\mathbf{E}}_e(\alpha(k))w(k)\big)\beta_e(k)\Big), \qquad (1)\\ &x(k)=\xi(k),\ \ k\in[-\tau,-\tau+1,\ldots,0],\\ &\qquad\qquad \tau=max(h_{\max},g_{\max}), \end{aligned}$$

where $x(k)\in\mathbb{R}^n$ is the state vector, $u(k)\in\mathbb{R}^m$ is the control input vector, $x(k-h(k))\in\mathbb{R}^n$ and $u(k-g(k))\in\mathbb{R}^m$ are respectively the state delay vector and the control input delay vector which satisfy, $0\le h_{\min}\le h(k)\le h_{\max}$ and $0\le g_{\min}\le g(k)\le g_{\max}$, $w(k)\in\mathbb{R}^p$ is the exogenous input, $\beta_e(k)$ are discrete type Brownian motion satisfying independent increment property [5], i.e., $E\{x(k)\beta_e(k)\}=0$, $E\{\beta_i(k)\beta_j(k)\}=0$ for $i\neq j$ and $E\{\beta_e^2(k)\}=\rho_e^2$, and $\xi(k)$ is the initial condition of system (1). $\mathbf{A}(\alpha(k))\in\mathbb{R}^{n\times n}$, $\mathbf{A}_t(\alpha(k))\in\mathbb{R}^{n\times n}$, $\mathbf{B}(\alpha(k))\in\mathbb{R}^{n\times m}$, $\mathbf{B}_t(\alpha(k))\in\mathbb{R}^{n\times m}$, $\mathbf{E}(\alpha(k))\in\mathbb{R}^{n\times p}$, $\bar{\mathbf{A}}_e(\alpha(k))\in\mathbb{R}^{n\times n}$, $\bar{\mathbf{A}}_{t_e}(\alpha(k))\in\mathbb{R}^{n\times n}$, $\bar{\mathbf{B}}_e(\alpha(k))\in\mathbb{R}^{n\times m}$, $\bar{\mathbf{B}}_{t_e}(\alpha(k))\in\mathbb{R}^{n\times m}$ and $\bar{\mathbf{E}}_e(\alpha(k))\in\mathbb{R}^{n\times p}$ are the matrices depending on time-varying parameters vector $\alpha(k)=\begin{bmatrix}\alpha_1(k) & \alpha_2(k) & \cdots & \alpha_r(k)\end{bmatrix}$. Moreover, these matrices can be furtherly defined by the following equation:

$$\begin{aligned} &\begin{bmatrix}\mathbf{A}(\alpha(k)) & \mathbf{A}_t(\alpha(k)) & \mathbf{B}(\alpha(k)) & \mathbf{B}_t(\alpha(k))\\ \bar{\mathbf{A}}_e(\alpha(k)) & \bar{\mathbf{A}}_{t_e}(\alpha(k)) & \bar{\mathbf{B}}_e(\alpha(k)) & \bar{\mathbf{B}}_{t_e}(\alpha(k))\end{bmatrix.\\ &\qquad\left.\begin{matrix}\mathbf{E}(\alpha(k))\\ \bar{\mathbf{E}}_e(\alpha(k))\end{matrix}\right]\\ &=\sum_{i=1}^{\mathbf{N}}\vartheta_i(k)\begin{bmatrix}\mathbf{A}_i & \mathbf{A}_{t_i} & \mathbf{B}_i & \mathbf{B}_{t_i} & \mathbf{E}_i\\ \bar{\mathbf{A}}_{ie} & \bar{\mathbf{A}}_{t_{ie}} & \bar{\mathbf{B}}_{ie} & \bar{\mathbf{B}}_{t_{ie}} & \bar{\mathbf{E}}_{ie}\end{bmatrix}, \qquad (2) \end{aligned}$$

where $N=2^r$, and $\vartheta_i(k)$ is measurable at each time instant and satisfies $\sum_{i=1}^{\mathbf{N}}\vartheta_i(k)=1$ and $0\le\vartheta_i(k)\le 1$. The $\mathbf{A}_i$, $\mathbf{A}_{t_i}$, $\mathbf{B}_i$, $\mathbf{B}_{t_i}$, $\mathbf{E}_i$, $\bar{\mathbf{A}}_{ie}$, $\bar{\mathbf{A}}_{t_{ie}}$, $\bar{\mathbf{B}}_{ie}$, $\bar{\mathbf{B}}_{t_{ie}}$ and $\bar{\mathbf{E}}_{ie}$ are the constant matrices with appropriate dimensions. For simplifying the following context, $\alpha(k)\triangleq\alpha$ and $\vartheta_i(k)\triangleq\vartheta_i$ are defined. Based on (2), system (1) can be furtherly rewritten as follows:

$$\begin{aligned} x(k+1)=&\sum_{i=1}^{\mathbf{N}}\vartheta_i\big(\mathbf{A}_i x(k)+\mathbf{A}_{t_i}x(k-h(k))\\ &+\mathbf{B}_i u(k)+\mathbf{B}_{t_i}u(k-g(k))+\mathbf{E}_i w(k)\\ &+\sum_{e=1}^{q}\big(\big(\bar{\mathbf{A}}_{ie}x(k)+\bar{\mathbf{A}}_{t_{ie}}x(k-h(k))\\ &+\bar{\mathbf{B}}_{ie}u(k)+\bar{\mathbf{B}}_{t_{ie}}u(k-g(k)) \end{aligned}$$

$$+\bar{\mathbf{E}}_{ie}w(k)\big)\beta_e(k)\big)\big). \quad (3)$$

In this paper, stabilization problem of system (1) is dealt with the following state feedback GS controller.

$$u(k) = -\mathbf{F}(\alpha)x(k), \quad (4a)$$

or

$$u(k) = -\left(\sum_{j=1}^{\mathbf{N}} \vartheta_j \mathbf{F}_j\right) x(k). \quad (4b)$$

Since the existence of input delay, the follwing delayed GS controller is naturally and necessarily assumed.

$$u(k-g(k)) = -\mathbf{F}(\alpha)x(k-g(k)), \quad (5a)$$

or

$$u(k-g(k)) = -\left(\sum_{j=1}^{\mathbf{N}} \vartheta_j \mathbf{F}_j\right) x(k-g(k)). \quad (5b)$$

It should be pointed out that $g(k)$ is not strictly required equal to $h(k)$. Substituting (4) and (5) into system (1), the following closed-loop system can be inferred.

$$\begin{aligned} x(k+1) =& \mathbf{X}(\alpha)x(k) + \mathbf{A}_t(\alpha)x(k-h(k)) \\ & -\mathbf{Y}_t(\alpha)x(k-g(k)) + \mathbf{E}(\alpha)w(k) \\ & +\sum_{e=1}^{q}\big(\big(\bar{\mathbf{X}}_e(\alpha)x(k) + \bar{\mathbf{A}}_{t_e}(\alpha)x(k-h(k)) \\ & -\bar{\mathbf{Y}}_{t_e}(\alpha)x(k-g(k)) \\ & +\bar{\mathbf{E}}_e(\alpha)w(k)\big)\beta_e(k)\big) \\ =& \sum_{i=1}^{\mathbf{N}}\sum_{j=1}^{\mathbf{N}} \vartheta_i\vartheta_j\big(\mathbf{X}_{ij}x(k) + \mathbf{A}_{t_i}x(k-h(k)) \\ & -\mathbf{Y}_{t_{ij}}x(k-g(k)) + \mathbf{E}_i w(k) \\ & +\sum_{e=1}^{q}\big(\big(\bar{\mathbf{X}}_{ije}x(k) + \bar{\mathbf{A}}_{t_{ie}}x(k-h(k)) \\ & -\bar{\mathbf{Y}}_{t_{ije}}x(k-g(k)) + \bar{\mathbf{E}}_{ie}w(k)\big)\beta_e(k)\big)\big). \end{aligned} \quad (6)$$

where $\mathbf{X}(\alpha) = \mathbf{A}(\alpha) - \mathbf{Y}(\alpha)$, $\bar{\mathbf{X}}_e(\alpha) = \bar{\mathbf{A}}_e(\alpha) - \bar{\mathbf{Y}}_e(\alpha)$, $\mathbf{Y}(\alpha) = \mathbf{B}(\alpha)\mathbf{F}(\alpha)$, $\bar{\mathbf{Y}}_e(\alpha) = \bar{\mathbf{B}}_e(\alpha)\mathbf{F}(\alpha)$, $\mathbf{Y}_t(\alpha) = \mathbf{B}_t(\alpha)\mathbf{F}(\alpha)$, $\bar{\mathbf{Y}}_{t_e}(\alpha) = \mathbf{B}_{t_e}(\alpha)F(\alpha)$, $\mathbf{X}_{ij} = \mathbf{A}_i - \mathbf{Y}_{ij}$, $\bar{\mathbf{X}}_{ije} = \bar{\mathbf{A}}_{ie} - \bar{\mathbf{Y}}_{ije}$, $\mathbf{Y}_{ij} = \mathbf{B}_i\mathbf{F}_j$, $\bar{\mathbf{Y}}_{ij} = \bar{\mathbf{B}}_i\mathbf{F}_j$, $\mathbf{Y}_{t_{ij}} = \mathbf{B}_{t_i}\mathbf{F}_j$ and $\bar{\mathbf{Y}}_{t_{ije}} = \bar{\mathbf{B}}_{t_{ie}}\mathbf{F}_j$.

Remark 1: It should be noted that Takagi-Sugeno (T-S) fuzzy system and LPV system are expressed as the similar polytopic-type description. Referring to [17–19], the combination of states is important index to build a membership function of T-S fuzzy system. Thus, the final output of fuzzy system is determined by the stated function of time. For LPV system [20–22, 28], the combination of time-varying parameters is applied to design the weighting functions to determine output of LPV system. Therefore, the structural component of T-S fuzzy system is not the same one of LPV system. However, according to the similar polytopic-type description, most analysis methods used to T-S fuzzy systems can also be applied to LPV systems.

In order to discuss control issue of the closed-loop system (6), the following lemmas and definitions are proposed for the derivative of this paper. Firstly, the following lemmas are introduced to deal with delay terms.

Lemma 1 [33]**:** For any compatible constant matrices $\mathbf{R} = \mathbf{R}^T > 0$, scalars $h_{\min}$ and $h_{\max}$ satisfying $0 \le h_{\min} \le h(k) \le h_{\max}$ and vector function ϖ : $[h_{\min},\ h_{\min}+1,\ \ldots,\ h_{\max}] \to \mathbb{R}^n$ such that the following sums are well-defined, it holds that

$$\begin{aligned} &-(\Delta h+1)\sum_{k-h_{\max}}^{k-h_{\min}} \varpi^T(k)\mathbf{R}\varpi(k) \\ &< -\left(\sum_{k-h_{\max}}^{k-h_{\min}} \varpi(k)\right)^T \mathbf{R}\left(\sum_{k-h_{\max}}^{k-h_{\min}} \varpi(k)\right), \end{aligned} \quad (7)$$

where $\Delta h = h_{\max} - h_{\min}$. □

Lemma 2 [36]**:** For symmetric positive definite matrix $\mathbf{R}$ and any matrix $\mathbf{M}$ satisfying $\begin{bmatrix} \mathbf{R} & \mathbf{M} \\ \mathbf{M}^T & \mathbf{R} \end{bmatrix} \ge 0$, scalars $h_{\min} > 0$ and $h_{\max} > 0$ satisfying $h_{\min} < h_{\max}$, and vector function $\tilde{\varpi} : \begin{bmatrix} h_{\min}, h_{\min}+1, \ldots, & h_{\max} \end{bmatrix} \to \mathbb{R}^n$ such that the following sums are well-defined, it holds that

$$\begin{aligned} &-\Delta h\sum_{k-h_{\max}}^{k-h_{\min}} \varpi^T(k)\mathbf{R}\varpi(k) \\ &\le -\begin{bmatrix} \sum\limits_{s=k-h_{\max}}^{k-h(k)-1} \varpi(s) \\ \sum\limits_{s=k-h(k)}^{k-h_{\min}-1} \varpi(s) \end{bmatrix}^T \begin{bmatrix} \mathbf{R} & \mathbf{M} \\ \mathbf{M}^T & \mathbf{R} \end{bmatrix} \begin{bmatrix} \sum\limits_{s=k-h_{\max}}^{k-h(k)-1} \varpi(s) \\ \sum\limits_{s=k-h(k)}^{k-h_{\min}-1} \varpi(s) \end{bmatrix}, \end{aligned} \quad (8)$$

where $\Delta h = h_{\max} - h_{\min}$. □

Besides, the following definitions are applied to ensure H_∞ performance index and stability concept of the closed-loop system (6).

Definition 1 [32]**:** Given a positive real number η, the H_∞ performance constraint for the closed-loop system (6) is introduced in the following form:

$$\sum_{0}^{\infty} x^T(k)\mathbf{S}x(k) < \eta^2\sum_{0}^{\infty} w^T(k)w(k),\ \forall w(k) \ne 0, \quad (9)$$

where η is a prescribed value which denotes the worst case effect of $w(k)$ on $x(k)$, and S is a positive definite weighting matrix. □

Definition 2 [5]**:** If the following condition holds, the closed-loop system (6) is asymptotically stable in the mean square.

$$E\left\{\|x(k)\|^2\right\} < E\left\{\|x(0)\|^2\right\}. \quad (10)$$

□

Based on the above lemmas and definitions, some sufficient conditions are derived in the following section for guaranteeing the robust asymptotical stability and H_∞ performance of the closed-loop system (6) in the mean square.

3. ROBUST H_∞ DELAY-DEPENDENT STABILITY CRITERION

In this section, a relaxed robust H_∞ delay-dependent stability criterion is developed via using a novel Lyapunov-Krasovskii function and the above lemmas. Based on the proposed criterion, the asymptotical stability and H_∞ performance of the closed-loop system (6) are verified via achieving the above definitions.

Theorem 1: Given a value η and time delay constants $h_{\max}$, $h_{\min}$, $g_{\max}$ and $g_{\min}$, if there exists the feedback gains $\mathbf{F}_i$, positive definite matrices $\mathbf{P}_i$, $\mathbf{S}$ and $\mathbf{Q}_b$ for $b = 1, 2, \ldots, 10$, and any matrices $\mathbf{M}$ and $\mathbf{N}$ satisfying the following conditions, then the closed-loop system (6) is asymptotically stable with disturbance attenuation η in the mean square.

$$\boldsymbol{\Theta} + \mathbf{Z} + \boldsymbol{\Gamma} < 0 \ \text{ for } i, j, l = 1, 2, \ldots, N, \tag{11a}$$

$$\begin{bmatrix} \mathbf{Q}_8 & \mathbf{M} \\ \mathbf{M}^T & \mathbf{Q}_8 \end{bmatrix} \geq 0, \tag{11b}$$

and

$$\begin{bmatrix} \mathbf{Q}_{10} & \mathbf{N} \\ \mathbf{N}^T & \mathbf{Q}_{10} \end{bmatrix} \geq 0, \tag{11c}$$

where

$$\varphi = \begin{bmatrix} \varphi_X & \varphi_E & 0 & 0 & \varphi_{A_t} & 0 & 0 & \varphi_{Y_t} \end{bmatrix},$$
$$\tilde{\mathbf{X}}_{ij} = \mathbf{X}_{ij} - \mathbf{I},$$
$$\varphi_{\mathbf{X}}^T = \begin{bmatrix} \mathbf{X}_{ij}^T & \tilde{\mathbf{X}}_{ij}^T \times \boldsymbol{\Lambda}_{1\times 4} & \bar{\mathbf{X}}_{ije}^T \times \boldsymbol{\Lambda}_{1\times 5} \end{bmatrix},$$
$$\varphi_{\mathbf{E}}^T = \begin{bmatrix} \mathbf{E}_{ij}^T \times \boldsymbol{\Lambda}_{1\times 5} & \bar{\mathbf{E}}_{ije}^T \times \boldsymbol{\Lambda}_{1\times 5} \end{bmatrix},$$
$$\boldsymbol{\Gamma} = diag\left\{\mathbf{S},\ -\eta^2\mathbf{I},\ 0_{6\times 6}\right\},$$
$$\varphi_{\mathbf{A}_t}^T = \left[\ \mathbf{A}_{t_{ij}}^T \times \boldsymbol{\Lambda}_{1\times 5} \ \middle|\ \bar{\mathbf{A}}_{t_{ije}}^T \times \boldsymbol{\Lambda}_{1\times 5}\ \right],$$
$$\varphi_{\mathbf{Y}_t}^T = \left[\ -\mathbf{Y}_{t_{ij}}^T \times \boldsymbol{\Lambda}_{1\times 5} \ \middle|\ -\bar{\mathbf{Y}}_{t_{ije}}^T \times \boldsymbol{\Lambda}_{1\times 5}\ \right],$$
$$\boldsymbol{\Theta} = \sum_{e=1}^{q} \rho_e^2 \varphi^T \aleph_{2\times 2}^{diag} \varphi,$$
$$\aleph = diag\left\{\mathbf{P}_l,\ h_{\min}^2\mathbf{Q}_7,\ \Delta h^2\mathbf{Q}_8,\ g_{\min}^2\mathbf{Q}_9,\ \Delta g^2\mathbf{Q}_{10}\right\},$$

$$\mathbf{Z} = \begin{bmatrix}
\mathbf{Z}_{11} & * & * & * & * & * & * & * \\
0 & 0 & * & * & * & * & * & * \\
0 & 0 & -(\mathbf{Q}_1+\mathbf{Q}_8) & * & * & * & * & * \\
\mathbf{Q}_7 & 0 & \mathbf{M} & \mathbf{Z}_{44} & * & * & * & * \\
0 & 0 & \mathbf{Q}_8-\mathbf{M} & \mathbf{Q}_8-\mathbf{M}^T & \mathbf{Z}_{55} & * & * & * \\
0 & 0 & 0 & 0 & 0 & -\mathbf{Q}_4-\mathbf{Q}_{10} & * & * \\
\mathbf{Q}_9 & 0 & 0 & 0 & 0 & \mathbf{N} & \mathbf{Z}_{77} & * \\
0 & 0 & 0 & 0 & 0 & \mathbf{Q}_{10}-\mathbf{N} & \mathbf{Q}_{10}-\mathbf{N}^T & \mathbf{Z}_{88}
\end{bmatrix},$$

$$\mathbf{Z}_{11} = \mathbf{Q}_1 + \mathbf{Q}_2 + (1+\Delta h)\mathbf{Q}_3 + \mathbf{Q}_4 + \mathbf{Q}_5 + (1+\Delta g)\mathbf{Q}_6 - (\mathbf{Q}_7 + \mathbf{Q}_9 + \mathbf{P}_j),$$
$$\mathbf{Z}_{44} = -(\mathbf{Q}_2 + \mathbf{Q}_7 + \mathbf{Q}_8),$$
$$\mathbf{Z}_{55} = \mathbf{M} + \mathbf{M}^T - (\mathbf{Q}_3 + 2\mathbf{Q}_8),$$
$$\mathbf{Z}_{77} = -(\mathbf{Q}_5 + \mathbf{Q}_9 + \mathbf{Q}_{10})$$

and

$$\mathbf{Z}_{88} = \mathbf{N} + \mathbf{N}^T - (\mathbf{Q}_6 + 2\mathbf{Q}_{10}).$$

Proof: Choose the following Lyapunov-Krasovskii function:

$$V(x(k)) = \sum_{j=1}^{9} V_j(x(k)), \tag{12}$$

where

$$V_1(x(k)) = x^T(k)\mathbf{P}(\alpha(k))x(k),$$
$$\eta(k) = x(k+1) - x(k),$$
$$V_2(x(k)) = \sum_{s=k-h_{\max}}^{k-1} x^T(s)\mathbf{Q}_1x(s) + \sum_{s=k-h_{\min}}^{k-1} x^T(s)\mathbf{Q}_2x(s) + \sum_{s=k-h(k)}^{k-1} x^T(s)\mathbf{Q}_3x(s),$$
$$V_3(x(k)) = \sum_{s=k-g_{\max}}^{k-1} x^T(s)\mathbf{Q}_4x(s) + \sum_{s=k-g_{\min}}^{k-1} x^T(s)\mathbf{Q}_5x(s) + \sum_{s=k-g(k)}^{k-1} x^T(s)\mathbf{Q}_6x(s),$$
$$V_4(x(k)) = \sum_{d=-h_{\max}+2}^{-h_{\min}+1} \sum_{s=k+d-1}^{k-1} x^T(s)\mathbf{Q}_3x(s),$$
$$V_5(x(k)) = \sum_{d=-g_{\max}+2}^{-g_{\min}+1} \sum_{s=k+d-1}^{k-1} x^T(s)\mathbf{Q}_6x(s),$$
$$V_6(x(k)) = \sum_{d=-h_{\min}}^{-1} \sum_{s=k+d}^{k-1} h_{\min}\eta^T(s)\mathbf{Q}_7\eta(s),$$
$$V_7(x(k)) = \sum_{d=-h_{\max}}^{-h_{\min}-1} \sum_{s=k+d}^{k-1} \Delta h\eta^T(s)\mathbf{Q}_8\eta(s),$$

$$V_8(x(k)) = \sum_{d=-g_{\min}}^{-1} \sum_{s=k+d}^{k-1} g_{\min} \eta^T(s) \mathbf{Q}_9 \eta(s) \quad \text{and}$$
$$V_9(x(k)) = \sum_{d=-g_{\max}}^{-g_{\min}-1} \sum_{s=k+d}^{k-1} \Delta g \eta^T(s) \mathbf{Q}_{10} \eta(s).$$

Calculating the difference of $V(x(k))$ along the trajectories of (6) and taking the mathematical expectation of it, one has

$$\begin{aligned}
E\{\Delta V_1(x(k))\} &= E\{x^T(k+1)\mathbf{P}(\alpha(k+1))x(k+1) \\
&\quad -x^T(k)\mathbf{P}(\alpha(k))x(k)\} \\
&= E\{(\mathbf{X}(\alpha)x(k)+\mathbf{A}_t(\alpha)x(k-h(k)) \\
&\quad -\mathbf{Y}_t(\alpha)x(k-g(k))+\mathbf{E}(\alpha)w(k) \\
&\quad +\sum_{e=1}^{q}((\bar{\mathbf{X}}_e(\alpha)x(k)+\bar{\mathbf{A}}_{t_e}(\alpha)x(k-h(k)) \\
&\quad -\bar{\mathbf{Y}}_{t_e}(\alpha)x(k-g(k)) \\
&\quad +\bar{\mathbf{E}}_e(\alpha)w(k))\beta_e(k)))^T\mathbf{P}(\varepsilon)(\mathbf{X}(\alpha)x(k) \\
&\quad +\mathbf{A}_t(\alpha)x(k-h(k))-\mathbf{Y}_t(\alpha)x(k-g(k)) \\
&\quad +\mathbf{E}(\alpha)w(k)+\sum_{e=1}^{q}((\bar{\mathbf{X}}_e(\alpha)x(k) \\
&\quad +\bar{\mathbf{A}}_{t_e}(\alpha)x(k-h(k))-\bar{\mathbf{Y}}_{t_e}(\alpha)x(k-g(k)) \\
&\quad +\bar{\mathbf{E}}_e(\alpha)w(k))\beta_e(k)))-x^T(k)\mathbf{P}(\alpha)x(k)\},
\end{aligned} \tag{13}$$

where $\mathbf{P}(\varepsilon)=\mathbf{P}(\alpha(k+1))$ and $\varepsilon \triangleq \varepsilon(k)$. Without loss of generality, $\mathbf{P}(\varepsilon)=\sum_{l=1}^{\mathbf{N}}\varepsilon_l\mathbf{P}_l$ is well defined with $\varepsilon_l \triangleq \varepsilon_l(k)$. And, ε_l is a time varying parameter satisfying $\sum_{l=1}^{\mathbf{N}}\varepsilon_l=1$ and $0\le\varepsilon_l\le 1$.

$$\begin{aligned}
&E\{\Delta V_2(x(k))\} \\
&= E\left\{\sum_{s=k-h_{\max}+1}^{k} x^T(s)\mathbf{Q}_1x(s) - \sum_{s=k-h_{\max}}^{k-1} x^T(s)\mathbf{Q}_1x(s)\right. \\
&\quad + \sum_{s=k-h_{\min}+1}^{k} x^T(s)\mathbf{Q}_2x(s) - \sum_{s=k-h_{\min}}^{k-1} x^T(s)\mathbf{Q}_2x(s) \\
&\quad \left.+ \sum_{s=k-h(k+1)+1}^{k} x^T(s)\mathbf{Q}_3x(s) - \sum_{s=k-h(k)}^{k-1} x^T(s)\mathbf{Q}_3x(s)\right\} \\
&= E\{x^T(k)(\mathbf{Q}_1+\mathbf{Q}_2+\mathbf{Q}_3)x(k) \\
&\quad -x^T(k-h_{\max})\mathbf{Q}_1x(k-h_{\max}) \\
&\quad -x^T(k-h_{\min})\mathbf{Q}_2x(k-h_{\min}) \\
&\quad -x^T(k-h(k))\mathbf{Q}_3x(k-h(k)) \\
&\quad \left.+ \sum_{s=k-h(k+1)+1}^{k-1} x^T(s)\mathbf{Q}_3x(s) - \sum_{s=k-h(k)+1}^{k-1} x^T(s)\mathbf{Q}_3x(s)\right\} \\
&= E\{x^T(k)(\mathbf{Q}_1+\mathbf{Q}_2+\mathbf{Q}_3)x(k) \\
&\quad -x^T(k-h_{\max})\mathbf{Q}_1x(k-h_{\max}) \\
&\quad -x^T(k-h_{\min})\mathbf{Q}_2x(k-h_{\min}) \\
&\quad -x^T(k-h(k))\mathbf{Q}_3x(k-h(k)) \\
&\quad \left.+ \sum_{s=k-h(k+1)+1}^{k-h(k)} x^T(s)\mathbf{Q}_3x(s)\right\} \\
&\le E\{x^T(k)(\mathbf{Q}_1+\mathbf{Q}_2+\mathbf{Q}_3)x(k) \\
&\quad -x^T(k-h_{\max})\mathbf{Q}_1x(k-h_{\max}) \\
&\quad -x^T(k-h_{\min})\mathbf{Q}_2x(k-h_{\min}) \\
&\quad -x^T(k-h(k))\mathbf{Q}_3x(k-h(k)) \\
&\quad \left.+ \sum_{s=k-g_{\max}+1}^{k-g_{\min}} x^T(s)\mathbf{Q}_6x(s)\right\},
\end{aligned} \tag{14}$$

$$\begin{aligned}
&E\{\Delta V_3(x(k))\} \\
&\le E\{x^T(k)(\mathbf{Q}_4+\mathbf{Q}_5+\mathbf{Q}_6)x(k) \\
&\quad -x^T(k-g_{\max})\mathbf{Q}_4x(k-g_{\max}) \\
&\quad -x^T(k-g_{\min})\mathbf{Q}_5x(k-g_{\min}) \\
&\quad -x^T(k-g(k))\mathbf{Q}_6x(k-g(k)) \\
&\quad \left.+ \sum_{s=k-g_{\max}+1}^{k-g_{\min}} x^T(s)\mathbf{Q}_6x(s)\right\},
\end{aligned} \tag{15}$$

$$\begin{aligned}
&E\{\Delta V_4(x(k))\} \\
&= E\left\{\sum_{d=-h_{\max}+2}^{-h_{\min}+1}\sum_{s=k+d}^{k} x^T(s)\mathbf{Q}_3x(s)\right. \\
&\quad \left.- \sum_{d=-h_{\max}+2}^{-h_{\min}+1}\sum_{s=k+d-1}^{k-1} x^T(s)\mathbf{Q}_3x(s)\right\} \\
&= E\left\{\Delta h x^T(k)\mathbf{Q}_3x(k) - \sum_{s=k-h_{\max}+1}^{k-h_{\min}} x^T(s)\mathbf{Q}_3x(s)\right\},
\end{aligned} \tag{16}$$

$$\begin{aligned}
&E\{\Delta V_5(x(k))\} \\
&= E\left\{\sum_{d=-g_{\max}+2}^{-g_{\min}+1}\sum_{s=k+d}^{k} x^T(s)\mathbf{Q}_6x(s)\right. \\
&\quad \left.- \sum_{d=-g_{\max}+2}^{-g_{\min}+1}\sum_{s=k+d-1}^{k-1} x^T(s)\mathbf{Q}_6x(s)\right\} \\
&= E\left\{\Delta g x^T(k)\mathbf{Q}_6x(k) - \sum_{s=k-g_{\max}+1}^{k-g_{\min}} x^T(s)\mathbf{Q}_6x(s)\right\},
\end{aligned} \tag{17}$$

$$\begin{aligned}
&E\{\Delta V_6(x(k))\} \\
&= h_{\min}E\left\{\sum_{d=-h_{\min}}^{-1}\sum_{s=k+d+1}^{k} \eta^T(s)\mathbf{Q}_7\eta(s)\right. \\
&\quad \left.- \sum_{d=-h_{\min}}^{-1}\sum_{s=k+d}^{k-1} \eta^T(s)\mathbf{Q}_7\eta(s)\right\} \\
&= E\{h_{\min}^2\eta^T(k)\mathbf{Q}_7\eta(k) \\
&\quad \left.-h_{\min}\sum_{s=k-h_{\min}}^{k-1} \eta^T(s)\mathbf{Q}_7\eta(s)\right\}.
\end{aligned} \tag{18}$$

Substituting (6) into the first term of the right-hand side in (18), i.e., $E\left\{h_{\min}^2\eta^T(k)\mathbf{Q}_7\eta(k)\right\}$, one has

$$\begin{aligned}
&\mathbf{E}\left\{h_{\min}^2\eta^T(k)\mathbf{Q}_7\eta(k)\right\}\\
&\le h_{\min}^2\mathbf{E}\Big\{\big(\tilde{\mathbf{X}}(\alpha)x(k)+\mathbf{A}_t(\alpha)x(k-h(k))\\
&\quad-\mathbf{Y}_t(\alpha)x(k-g(k))+\mathbf{E}(\alpha)w(k)\\
&\quad+\sum_{e=1}^{q}\big(\big(\bar{\mathbf{X}}_e(\alpha)x(k)+\bar{\mathbf{A}}_{t_e}(\alpha)x(k-h(k))\\
&\quad-\bar{\mathbf{Y}}_{t_e}(\alpha)x(k-g(k))+\bar{\mathbf{E}}_e(\alpha)w(k)\big)\\
&\quad\times\beta_e(k)\big)\big)^T\mathbf{Q}_7\big(\tilde{\mathbf{X}}(\alpha)x(k)\\
&\quad+\mathbf{A}_t(\alpha)x(k-h(k))-\mathbf{Y}_t(\alpha)x(k-g(k))\\
&\quad+\mathbf{E}(\alpha)w(k)+\sum_{e=1}^{q}\big(\big(\bar{\mathbf{X}}_e(\alpha)x(k)\\
&\quad+\bar{\mathbf{A}}_{t_e}(\alpha)x(k-h(k))-\bar{\mathbf{Y}}_{t_e}(\alpha)x(k-g(k))\\
&\quad+\bar{\mathbf{E}}_e(\alpha)w(k)\big)\beta_e(k)\big)\big)\Big\},
\end{aligned}\tag{19}$$

where $\tilde{\mathbf{X}}(\alpha)=\mathbf{X}(\alpha)-\mathbf{I}$.

Based on Lemma 1, the following inequality can be obtained from the second term of the right-hand side in (18), i.e., $-h_{\min}\sum_{s=k-h_{\min}}^{k-1}\eta(s)^T\mathbf{Q}_7\eta(s)$.

$$\begin{aligned}
&-h_{\min}\sum_{s=k-h_{\min}}^{k-1}\eta(s)^T\mathbf{Q}_7\eta(s)\\
&\le-\sum_{s=k-h_{\min}}^{k-1}\eta(s)^T\mathbf{Q}_7\sum_{s=k-h_{\min}}^{k-1}\eta(s)\\
&=-(x(k)-x(k-h_{\min}))^T\mathbf{Q}_7(x(k)-x(k-h_{\min})).
\end{aligned}\tag{20}$$

According to (19) and (20), $\Delta V_6(x(k))$can be furtherly inferred as follows:

$$\begin{aligned}
&\mathbf{E}\{\Delta V_6(x(k))\}\\
&\le h_{\min}^2\mathbf{E}\Big\{\big(\tilde{\mathbf{X}}(\alpha)x(k)+\mathbf{A}_t(\alpha)x(k-h(k))\\
&\quad-\mathbf{Y}_t(\alpha)x(k-g(k))\\
&\quad+\mathbf{E}(\alpha)w(k)\big)^T\mathbf{Q}_7\big(\tilde{\mathbf{X}}(\alpha)x(k)\\
&\quad+\mathbf{A}_t(\alpha)x(k-h(k))-\mathbf{Y}_t(\alpha)x(k-g(k))\\
&\quad+\mathbf{E}(\alpha)w(k)\big)+\sum_{e=1}^{q}\rho_e^2\big(\bar{\mathbf{X}}_e(\alpha)x(k)\\
&\quad+\bar{\mathbf{A}}_{t_e}(\alpha)x(k-h(k))-\bar{\mathbf{Y}}_{t_e}(\alpha)x(k-g(k))\\
&\quad+\bar{\mathbf{E}}_e(\alpha)w(k)\big)^T\mathbf{Q}_7\big(\bar{\mathbf{X}}_e(\alpha)x(k)\\
&\quad+\bar{\mathbf{A}}_{t_e}(\alpha)x(k-h(k))-\bar{\mathbf{Y}}_{t_e}(\alpha)x(k-g(k))\\
&\quad+\bar{\mathbf{E}}_e(\alpha)w(k)\big)\Big\}\\
&\quad-(x(k)-x(k-h_{\min}))^T\mathbf{Q}_7(x(k)-x(k-h_{\min})).
\end{aligned}\tag{21}$$

In addition, one has

$$\begin{aligned}
&E\{\Delta V_7(x(k))\}\\
&=\Delta h\left(\sum_{d=-h_{\max}}^{-h_{\min}-1}\sum_{s=k+d+1}^{k}\eta^T(s)\mathbf{Q}_8\eta(s)\right.\\
&\quad\left.-\sum_{d=-h_{\max}}^{-h_{\min}-1}\sum_{s=k+d}^{k-1}\eta^T(s)\mathbf{Q}_8\eta(s)\right)\\
&=\left(\Delta h^2\eta^T(k)\mathbf{Q}_8\eta(k)-\Delta h\sum_{s=-h_{\max}}^{-h_{\min}-1}\eta^T(s)\mathbf{Q}_8\eta(s)\right).
\end{aligned}\tag{22}$$

Substituting (6) into the first term of the right-hand side in (22), i.e., $E\left\{\Delta h^2\eta^T(k)\mathbf{Q}_8\eta(k)\right\}$, the following inequality can be inferred.

$$\begin{aligned}
&E\left\{\Delta h^2\eta^T(k)\mathbf{Q}_8\eta(k)\right\}\\
&\le\Delta h^2E\Big\{\big(\tilde{\mathbf{X}}(\alpha)x(k)+\mathbf{A}_t(\alpha)x(k-h(k))\\
&\quad-\mathbf{Y}_t(\alpha)x(k-g(k))+\mathbf{E}(\alpha)w(k)\\
&\quad+\sum_{e=1}^{q}\big(\big(\bar{\mathbf{X}}(\alpha)x(k)+\bar{\mathbf{A}}_t(\alpha)x(k-h(k))\\
&\quad-\bar{\mathbf{Y}}_{t_e}x(k-g(k))\ +\bar{\mathbf{E}}_e(\alpha)w(k)\big)\beta_e(k)\big)\big)^T\\
&\quad\times\mathbf{Q}_8\big(\tilde{\mathbf{X}}(\alpha)\,x(k)+\mathbf{A}_t(\alpha)x(k-h(k))\\
&\quad-\mathbf{Y}_t(\alpha)x(k-g(k))+\mathbf{E}(\alpha)w(k)\\
&\quad+\sum_{e=1}^{q}\big(\big(\bar{\mathbf{X}}_e(\alpha)x(k)+\bar{\mathbf{A}}_{t_e}(\alpha)x(k-h(k))\\
&\quad-\bar{\mathbf{Y}}_{t_e}(\alpha)x(k-g(k))+\bar{\mathbf{E}}_e(\alpha)w(k)\big)\beta_e(k)\big)\big)\Big\}.
\end{aligned}\tag{23}$$

Based on Lemma 2, the following inequality can be obtained from the second term of the right-hand side in (22), i.e., $-\Delta h\sum_{s=-h_{\max}}^{k-h_{\min}-1}\eta^T(s)\mathbf{Q}_8\eta(s)$.

$$\begin{aligned}
&-\Delta h\sum_{s=-h_{\max}}^{-h_{\min}-1}\eta^T(s)\mathbf{Q}_8\eta(s)\\
&\le-\begin{bmatrix}\sum_{s=k-h_{\max}}^{k-h(k)-1}\eta(s)\\ \sum_{s=k-h(k)}^{k-h_{\min}-1}\eta(s)\end{bmatrix}^T\begin{bmatrix}\mathbf{R}&\mathbf{M}\\ \mathbf{M}^T&\mathbf{R}\end{bmatrix}\begin{bmatrix}\sum_{s=k-h_{\max}}^{k-h(k)-1}\eta(s)\\ \sum_{s=k-h(k)}^{k-h_{\min}-1}\eta(s)\end{bmatrix}\\
&=-\xi_h^T(k)\mathbf{Z}_h\xi_h(k),
\end{aligned}\tag{24}$$

where $\xi_h(k)=\left[\begin{array}{ccc}x(k-h_{\max}) & x(k-h_{\min}) & x(k-h(k))\end{array}\right]^T$ and $\mathbf{Z}_h=\begin{bmatrix}-\mathbf{Q}_8&*&*\\ \mathbf{M}&-\mathbf{Q}_8&*\\ \mathbf{Q}_8-\mathbf{M}&\mathbf{Q}_8-\mathbf{M}^T&-2\mathbf{Q}_8\end{bmatrix}$.

Based on (23) and (24), $\Delta V_7(x(k))$ can be inferred.

$$\begin{aligned}
&E\{\Delta V_7(x(k))\}\\
&\le\Delta h^2E\Big\{\big(\tilde{\mathbf{X}}(\alpha)x(k)\ +\mathbf{A}_t(\alpha)x(k-h(k))\\
&\quad-\mathbf{Y}_t(\alpha)x(k-g(k))+\mathbf{E}(\alpha)w(k)\big)^T\\
&\quad\times\mathbf{Q}_8\big(\tilde{\mathbf{X}}(\alpha)x(k)+\mathbf{A}_t(\alpha)x(k-h(k))\\
&\quad-\mathbf{Y}_t(\alpha)x(k-g(k))+\mathbf{E}(\alpha)w(k)\big)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{e=1}^{q}\rho_e^2\left(\bar{\mathbf{X}}_e(\alpha)x(k)+\bar{\mathbf{A}}_{t_e}(\alpha)x(k-h(k))\right.\\
&-\bar{\mathbf{Y}}_{t_e}(\alpha)x(k-g(k))+\bar{\mathbf{E}}_e(\alpha)\\
&\times w(k))^T\mathbf{Q}_8\left(\bar{\mathbf{X}}_e(\alpha)x(k)+\bar{\mathbf{A}}_{t_e}(\alpha)x(k-h(k))\right.\\
&\left.-\bar{\mathbf{Y}}_{t_e}(\alpha)x(k-g(k))+\bar{\mathbf{E}}_e(\alpha)w(k)\right)\}\\
&-\xi_h^T(k)\mathbf{Z}_h\xi_h(k).
\end{aligned} \tag{25}
$$

Furthermore, the following relations can be obtained via the similar deriving process of $\Delta V_6(x(k))$ and $\Delta V_7(x(k))$, respectively.

$$
\begin{aligned}
&E\{\Delta V_8(x(k))\}\\
&\le g_{\min}^2E\left\{\left(\tilde{\mathbf{X}}(\alpha)x(k)+\mathbf{A}_t(\alpha)x(k-h(k))\right.\right.\\
&-\mathbf{Y}_t(\alpha)x(k-g(k))+\mathbf{E}(\alpha)w(k))^T\mathbf{Q}_9\left(\tilde{X}(\alpha)x(k)\right.\\
&+\mathbf{A}_t(\alpha)x(k-h(k))-\mathbf{Y}_t(\alpha)x(k-g(k))\\
&+\mathbf{E}(\alpha)w(k))+\sum_{e=1}^{q}\rho_e^2\left(\bar{\mathbf{X}}_e(\alpha)x(k)+\bar{\mathbf{A}}_{t_e}(\alpha)\right.\\
&\times x(k-h(k))-\bar{\mathbf{Y}}_{t_e}(\alpha)x(k-g(k))+\bar{\mathbf{E}}_e(\alpha)w(k))^T\\
&\times\mathbf{Q}_9\left(\bar{\mathbf{X}}_e(\alpha)x(k)+\bar{\mathbf{A}}_{t_e}(\alpha)x(k-h(k))\right.\\
&\left.-\bar{\mathbf{Y}}_{t_e}(\alpha)x(k-g(k))+\bar{\mathbf{E}}_e(\alpha)w(k)\right)\}\\
&-(x(k)-x(k-g_{\min}))^T\mathbf{Q}_9(x(k)-x(k-g_{\min})),
\end{aligned} \tag{26}
$$

and

$$
\begin{aligned}
&E\{\Delta V_9(x(k))\}\\
&\le\Delta g^2E\left\{\left(\tilde{\mathbf{X}}(\alpha)x(k)+\mathbf{A}_t(\alpha)x(k-h(k))\right.\right.\\
&-\mathbf{Y}_t(\alpha)x(k-g(k))\\
&+\mathbf{E}(\alpha)w(k))^T\mathbf{Q}_{10}\left(\tilde{X}(\alpha)x(k)\right.\\
&+\mathbf{A}_t(\alpha)x(k-h(k))-\mathbf{Y}_t(\alpha)x(k-g(k))\\
&+\sum_{e=1}^{q}\rho_e^2\left(\bar{\mathbf{X}}_e(\alpha)x(k)+\bar{\mathbf{A}}_{t_e}(\alpha)x(k-h(k))\right.\\
&-\bar{\mathbf{Y}}_{t_e}(\alpha)x(k-g(k))+\bar{\mathbf{E}}_e(\alpha)w(k))^T\mathbf{Q}_{10}\\
&\times\left(\bar{\mathbf{X}}_e(\alpha)x(k)+\bar{\mathbf{A}}_{t_e}(\alpha)x(k-h(k))-\bar{\mathbf{Y}}_{t_e}(\alpha)\right.\\
&\left.\times x(k-g(k))+\bar{\mathbf{E}}_e(\alpha)w(k)\right)\}-\xi_g^T(k)\mathbf{Z}_g\xi_g(k),
\end{aligned} \tag{27}
$$

where $\xi_g(k)=\left[\begin{array}{ccc} x(k-g_{\max}) & x(k-g_{\min}) & x(k-g(k))\end{array}\right]^T$ and $\mathbf{Z}_g=\left[\begin{array}{ccc}-\mathbf{Q}_{10} & * & *\\ \mathbf{N} & -\mathbf{Q}_{10} & *\\ \mathbf{Q}_{10}-\mathbf{N} & \mathbf{Q}_{10}-\mathbf{N}^T & -2\mathbf{Q}_{10}\end{array}\right]$. Via combing $\Delta V_i(x(k))$ for $i=1,2,\ldots,9$, one can find the following inequality:

$$
E\{\Delta V(x(k))\}\le\sum_{i=1}^{\mathbf{N}}\sum_{j=1}^{\mathbf{N}}\sum_{l=1}^{\mathbf{N}}\vartheta_i\vartheta_j\varepsilon_l(\mathbf{\Theta}+\mathbf{Z}). \tag{28}
$$

Let us define the following performance function with zero initial condition for all nonzero $w(k)$.

$$
J_D=\sum_{0}^{t_f}\left(x^T(k)\mathbf{S}x(k)-\eta^2w^T(k)w(k)\right). \tag{29}
$$

Then, for any nonzero $w(k)$, one have

$$
\begin{aligned}
J_D=&E\left\{\sum_{0}^{t_f}\left(x^T(k)\mathbf{S}x(k)-\eta^2w^T(k)w(k)\right)\right.\\
&\left.+\sum_{0}^{t_f}\Delta V(x(k))-V(x(t_f))\right\}\\
\le&E\left\{\sum_{0}^{t_f}\left(x^T(k)\mathbf{S}x(k)-\eta^2w^T(k)w(k)\right.\right.\\
&\left.\left.+\Delta V(x(k))\right)\right\}\\
=&E\left\{\xi^T(k)\mathbf{\Psi}(\alpha)\xi(k)\right\},
\end{aligned} \tag{30}
$$

where $\mathbf{\Psi}(\alpha)=\sum_{i=1}^{\mathbf{N}}\sum_{j=1}^{\mathbf{N}}\sum_{l=1}^{\mathbf{N}}\vartheta_i\vartheta_j\varepsilon_l(\mathbf{\Theta}+\mathbf{Z}+\mathbf{\Gamma})$ and $\xi(k)=\left[\begin{array}{cccc} x(k) & w(k) & \xi_h(k) & \xi_g(k)\end{array}\right]^T$.

According to $0\le\vartheta_i\le 1$, $0\le\varepsilon_l\le 1$, $\sum_{i=1}^{\mathbf{N}}\vartheta_i=1$ and $\sum_{l=1}^{\mathbf{N}}\varepsilon_l=1$, the following inequality can be obtained from (11a).

$$
\sum_{i=1}^{\mathbf{N}}\sum_{j=1}^{\mathbf{N}}\sum_{l=1}^{\mathbf{N}}\vartheta_i\vartheta_j\varepsilon_l(\mathbf{\Theta}+\mathbf{Z}+\mathbf{\Gamma})<0. \tag{31}
$$

Obviously, because (11) holds, then $\mathbf{\Psi}(\alpha)<0$ can be found from (30). Due to $\mathbf{\Psi}(\alpha)<0$, the following inequalities can also be inferred.

$$
J_D<0, \tag{32}
$$

or

$$
E\left\{\sum_{0}^{t_f}x^T(k)\mathbf{S}x(k)\right\}<E\left\{\eta^2\sum_{0}^{t_f}w^T(k)w(k)\right\}. \tag{33}
$$

According to (32) and (33), it is obvious to show that the closed-loop system (6) driven by (4) and (5) satisfies H_∞ performance in Definition 1 for all nonzero external disturbance. Next, the asymptotical stability of (6) is proven. If the sufficient conditions of this theorem are satisfied, then $\mathbf{\Psi}(\alpha)<0$ is held. By assuming $w(k)=0$, the following inequalities can be found from (30).

$$
E\{\Delta V(x(k))\}<E\left\{-x^T(k)\mathbf{S}x(k)\right\}. \tag{34}
$$

According to $\mathbf{S}>0$, one can find $E\{\Delta V(x(k))\}<0$ from (34). Referring to Definition 2, the closed-loop system (6) is asymptotically stable in the mean square due to $E\{\Delta V(x(k))\}<0$. The proof of this theorem is complete. □

In Theorem 1, some sufficient conditions are derived to analyze the asymptotical stability and H_∞ performance of the closed-loop system (6). However, those sufficient conditions are not standard LMI problems that cannot be directly solved by the convex optimization algorithm. In the following theorem, the sufficient conditions are converted into extended LMI form.

Theorem 2: Given a value η and time delay constants $h_{\max}$, $h_{\min}$, $g_{\max}$ and $g_{\min}$, if there exists the feedback gains $\mathbf{K}_i$, positive definite matrices $\bar{\mathbf{P}}_i$, $\bar{\mathbf{S}}$ and $\bar{\mathbf{Q}}_b$ for $b = 1, 2, \cdots, 10$, any matrices $\bar{\mathbf{M}}$ and $\bar{\mathbf{N}}$, and a non-singular matrix $\mathbf{G}$ to satisfy the following conditions, then closed-loop system (6) is asymptotically stable with disturbance attenuation η in the mean square.

$$\begin{bmatrix} \bar{\mathbf{Z}} & * & * & * & * & * \\ \mathbf{\Phi} & \prod_{\mathbf{P}} & * & * & * & * \\ h_{\min}\times\tilde{\mathbf{\Phi}} & 0 & \prod_{\bar{\mathbf{Q}}7} & * & * & * \\ \Delta h\times\tilde{\mathbf{\Phi}} & 0 & 0 & \prod_{\bar{\mathbf{Q}}8} & * & * \\ g_{\min}\times\tilde{\mathbf{\Phi}} & 0 & 0 & 0 & \prod_{\bar{\mathbf{Q}}9} & * \\ \Delta g\times\tilde{\mathbf{\Phi}} & 0 & 0 & 0 & 0 & \prod_{\bar{\mathbf{Q}}10} \end{bmatrix} < 0$$

for $i = j = l = 1, 2, \ldots, N$, (35a)

$$\begin{bmatrix} \bar{\mathbf{Q}}_8 & \bar{\mathbf{M}} \\ \bar{\mathbf{M}}^T & \bar{\mathbf{Q}}_8 \end{bmatrix} \geq 0, \tag{35b}$$

and

$$\begin{bmatrix} \bar{\mathbf{Q}}_{10} & \bar{\mathbf{N}} \\ \bar{\mathbf{N}}^T & \bar{\mathbf{Q}}_{10} \end{bmatrix} \geq 0, \tag{35c}$$

where

$$\bar{\mathbf{Z}} = \begin{bmatrix} \bar{\mathbf{Z}}_{11} & * & * & * & * & * & * & * \\ 0 & -\eta^2 & * & * & * & * & * & * \\ 0 & 0 & -\left(\bar{\mathbf{Q}}_1+\bar{\mathbf{Q}}_8\right) & * & * & * & * & * \\ \bar{\mathbf{Q}}_7 & 0 & \bar{\mathbf{M}} & \bar{\mathbf{Z}}_{44} & * & * & * & * \\ 0 & 0 & \bar{\mathbf{Q}}_8-\bar{\mathbf{M}} & \bar{\mathbf{Q}}_8-\bar{\mathbf{M}}^T & \bar{\mathbf{Z}}_{55} & * & * & * \\ 0 & 0 & 0 & 0 & 0 & -\left(\bar{\mathbf{Q}}_4+\bar{\mathbf{Q}}_{10}\right) & * & * \\ \bar{\mathbf{Q}}_9 & 0 & 0 & 0 & 0 & \bar{\mathbf{N}} & \bar{\mathbf{Z}}_{77} & * \\ 0 & 0 & 0 & 0 & * & \bar{\mathbf{Q}}_{10}-\bar{\mathbf{N}} & \bar{\mathbf{Q}}_{10}-\bar{\mathbf{N}}^T & \bar{\mathbf{Z}}_{88} \end{bmatrix},$$

$$\bar{\mathbf{Z}}_{11} = \bar{\mathbf{Q}}_1+\bar{\mathbf{Q}}_2+(1+\Delta h)\bar{\mathbf{Q}}_3+\bar{\mathbf{Q}}_4+\bar{\mathbf{Q}}_5 + (1+\Delta g)\bar{\mathbf{Q}}_6-\bar{\mathbf{Q}}_7-\bar{\mathbf{Q}}_9-\bar{\mathbf{P}}_j+\bar{\mathbf{S}},$$

$$\mathbf{Z}_{55} = \bar{\mathbf{M}}+\bar{\mathbf{M}}^T-\left(\bar{\mathbf{Q}}_3+2\bar{\mathbf{Q}}_8\right),$$

$$\bar{\mathbf{Z}}_{77} = -\left(\bar{\mathbf{Q}}_5+\bar{\mathbf{Q}}_9+\bar{\mathbf{Q}}_{10}\right),$$

$$\bar{\mathbf{Z}}_{88} = \bar{\mathbf{N}}+\bar{\mathbf{N}}^T-\left(\bar{\mathbf{Q}}_6+2\bar{\mathbf{Q}}_{10}\right),$$

$$\bar{\mathbf{Z}}_{44} = -\left(\bar{\mathbf{Q}}_2+\bar{\mathbf{Q}}_7+\bar{\mathbf{Q}}_8\right),$$

$$\bar{\mathbf{P}}_l = \mathbf{G}^T\mathbf{P}_l\mathbf{G},\ \bar{\mathbf{Q}}_b = \mathbf{G}^T\mathbf{Q}_b\mathbf{G},\ \bar{\mathbf{M}} = \mathbf{G}^T\mathbf{M}\mathbf{G},$$

$$\bar{\mathbf{N}} = \mathbf{G}^T\mathbf{N}\mathbf{G}, \bar{\mathbf{S}} = \mathbf{G}^T\mathbf{S}\mathbf{G},$$

$$\prod_{\bar{\mathbf{P}}} = \left(\bar{\mathbf{P}}_l-\mathbf{G}^T-\mathbf{G}\right)\times\mathbf{I}_{2\times2}^{diag},$$

$$\prod_{\bar{\mathbf{Q}}7} = \left(\bar{\mathbf{Q}}_7-\mathbf{G}^T-\mathbf{G}\right)\times\mathbf{I}_{2\times2}^{diag},$$

$$\prod_{\bar{\mathbf{Q}}8} = \left(\bar{\mathbf{Q}}_8-\mathbf{G}^T-\mathbf{G}\right)\times\mathbf{I}_{2\times2}^{diag},$$

$$\prod_{\bar{\mathbf{Q}}9} = \left(\bar{\mathbf{Q}}_9-\mathbf{G}^T-\mathbf{G}\right)\times\mathbf{I}_{2\times2}^{diag},$$

$$\prod_{\bar{\mathbf{Q}}10} = \left(\bar{\mathbf{Q}}_{10}-\mathbf{G}^T-\mathbf{G}\right)\times\mathbf{I}_{2\times2}^{diag},$$

$$\mathbf{\Phi} = \sum_{e=1}^{q}\rho_e^2\times\begin{bmatrix} \mathbf{A}_i\mathbf{G}-\mathbf{B}_i\mathbf{K}_j & \mathbf{E}_i & \mathbf{A}_{t_i}\mathbf{G} & -\mathbf{B}_i\mathbf{K}_j & 0 & 0 & 0 & 0 \\ \bar{\mathbf{A}}_{ie}\mathbf{G}-\bar{\mathbf{B}}_{ie}\mathbf{K}_j & \bar{\mathbf{E}}_{ie} & \bar{\mathbf{A}}_{t_{ie}}\mathbf{G} & -\bar{\mathbf{B}}_{ie}\mathbf{K}_j & 0 & 0 & 0 & 0 \end{bmatrix},$$

and

$$\tilde{\mathbf{\Phi}} = \sum_{e=1}^{q}\rho_e^2\times\begin{bmatrix} \mathbf{A}_i\mathbf{G}-\mathbf{B}_i\mathbf{K}_j-\mathbf{G} & \mathbf{E}_i & \mathbf{A}_{t_i}\mathbf{G} & -\mathbf{B}_i\mathbf{K}_j & 0 & 0 & 0 & 0 \\ \bar{\mathbf{A}}_{ie}\mathbf{G}-\bar{\mathbf{B}}_{ie}\mathbf{K}_j & \bar{\mathbf{E}}_{ie} & \bar{\mathbf{A}}_{t_{ie}}\mathbf{G} & -\bar{\mathbf{B}}_{ie}\mathbf{K}_j & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Proof: According to $\mathbf{P}_l^{-1} > 0$ and $\mathbf{Q}_b^{-1} > 0$, the following inequalities can be held.

$$\mathbf{G}^T\mathbf{P}_l\mathbf{G}-\mathbf{G}^T-\mathbf{G} \geq -\mathbf{P}_l^{-1}, \tag{36}$$

and

$$\mathbf{G}^T\mathbf{Q}_b\mathbf{G}-\mathbf{G}^T-\mathbf{G} \geq -\mathbf{Q}_b^{-1}. \tag{37}$$

Based on (36) and (37), if (35a) is held, the following inequality can be held.

$$\begin{bmatrix} \bar{\mathbf{Z}} & * & * & * & * & * \\ \mathbf{\Phi} & \coprod_{\mathbf{P}} & * & * & * & * \\ h_{\min}\times\tilde{\mathbf{\Phi}} & 0 & \coprod_{Q7} & * & * & * \\ \Delta h\times\tilde{\mathbf{\Phi}} & 0 & 0 & \coprod_{Q8} & * & * \\ g_{\min}\times\tilde{\mathbf{\Phi}} & 0 & 0 & 0 & \coprod_{Q9} & * \\ \Delta g\times\tilde{\mathbf{\Phi}} & 0 & 0 & 0 & 0 & \coprod_{Q10} \end{bmatrix} < 0, \tag{38}$$

where $\coprod_{\mathbf{P}} = -\mathbf{P}_l^{-1}\times\mathbf{I}_{2\times2}$, $\coprod_{\mathbf{Q7}} = -\mathbf{Q}_7^{-1}\times\mathbf{I}_{2\times2}$, $\coprod_{\mathbf{Q8}} = -\mathbf{Q}_8^{-1}\times\mathbf{I}_{2\times2}$, $\coprod_{\mathbf{Q9}} = -\mathbf{Q}_9^{-1}\times\mathbf{I}_{2\times2}$, $\coprod_{\mathbf{Q10}} = -\mathbf{Q}_{10}^{-1}\times\mathbf{I}_{2\times2}$.

Pre- and pose- multiplying (38) by ϕ^T and ϕ with $\phi = diag\left\{\left(\mathbf{G}^{-1}\right)^T\times\mathbf{I}_{8\times8}^{diag}, \mathbf{I}_{10\times10}^{diag}\right\}$, the following inequality can be obtained as follows:

$$\begin{bmatrix} \mathbf{Z}+\mathbf{\Gamma} & * & * & * & * & * \\ \boldsymbol{\chi} & \coprod_{\mathbf{P}} & * & * & * & * \\ h_{\min}\times\tilde{\boldsymbol{\chi}} & 0 & \coprod_{\mathbf{Q7}} & * & * & * \\ \Delta h\times\tilde{\boldsymbol{\chi}} & 0 & 0 & \coprod_{\mathbf{Q8}} & * & * \\ g_{\min}\times\tilde{\boldsymbol{\chi}} & 0 & 0 & 0 & \coprod_{\mathbf{Q9}} & * \\ \Delta g\times\tilde{\boldsymbol{\chi}} & 0 & 0 & 0 & 0 & \coprod_{\mathbf{Q10}} \end{bmatrix} < 0, \tag{39}$$

where

$$\tilde{\boldsymbol{\chi}} = \sum_{e=1}^{q} \rho_e^2 \times \begin{bmatrix} \mathbf{A}_i - \mathbf{B}_i\mathbf{F}_j & \mathbf{E}_i & \mathbf{A}_{t_i} & -\mathbf{B}_i\mathbf{F}_j & 0 & 0 & 0 & 0 \\ \bar{\mathbf{A}}_{ie} - \bar{\mathbf{B}}_{ie}\mathbf{F}_j & \bar{\mathbf{E}}_{ie} & \bar{\mathbf{A}}_{t_{ie}} & -\bar{\mathbf{B}}_{ie}\mathbf{F}_j & 0 & 0 & 0 & 0 \end{bmatrix} \text{ and}$$

$$\tilde{\boldsymbol{\chi}} = \sum_{e=1}^{q} \rho_e^2 \times \begin{bmatrix} \mathbf{A}_i - \mathbf{B}_i\mathbf{F}_j - I & \mathbf{E}_i & \mathbf{A}_{t_i} & -\mathbf{B}_i\mathbf{F}_j & 0 & 0 & 0 & 0 \\ \bar{\mathbf{A}}_{ie} - \bar{\mathbf{B}}_{ie}\mathbf{F}_j & \bar{\mathbf{E}}_{ie} & \bar{\mathbf{A}}_{t_{ie}} & -\bar{\mathbf{B}}_{ie}\mathbf{F}_j & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Applying Shur compliment [37] to (39), one has

$$\boldsymbol{\Theta} + \mathbf{Z} + \boldsymbol{\Gamma} < 0. \tag{40}$$

Obviously, (40) is equal to (11a). Then, (11b) and (11c) can be directly found by pre- and post-multiplying (35b) and (35c) with $\left(\mathbf{G}^{-1}\right)^T$ and $\mathbf{G}^{-1}$. Therefore, the proofs of (35b) and (35c) are omitted here. Based on this proof, it is obviously known that if one can find feasible solutions to satisfy conditions (35) then the solutions can also satisfy the conditions (11). The proof of this theorem is completed. □

From Theorem 2, the convex optimization algorithm can be directly used to find the feasible solutions of sufficient conditions (35). With the obtained feasible solutions, the feedback gains can be obtained by $\mathbf{F}_j = \mathbf{K}_j\mathbf{G}^{-1}$ to establish GS controllers (4) and (5) such that asymptotical stability and H_∞ performance of the closed-loop system (6) are achieved in the mean square. In next section, some simulated resluts are proposed to demonstrate the applicability and effectiveness of the proposed design method.

Remark 2: In order to propose the relaxed GS controller design method, a novel parameter-dependent Lyapunov-Krasovskii function (12), Lemma 1 and Lemma 2 are applied to derive some sufficient conditions (11). Besides, these conditions are converted into extended LMI form (35) to use convex optimization algorithm. Although the conservatism of the sufficient conditions is reduced, computational complexity and demand are increased since many slack variables and huge dimension. It is a worth issue to be discussed in our future works.

4. SIMULATION RESULTS

In this section, two examples are empolyed to demonstrate the applicability and effectiveness of the proposed design method. The first example is to propose a comparison between the proposed design method and method of [27] to discuss their conservatism. In the second example, a state-feedback stabilization problem of truck-trailer system [38] with the added perturbation and multiplicative noise terms is discussed.

Table 1. Compared results with fixing $h_{\max} = 1$.

η	0.03	0.04	0.05
[27]	Infeasible	feasible	Feasible
This Paper	Feasible	Feasible	Feasible
η	0.03	0.04	0.05
[27]	Infeasible	Infeasible	Infeasible
This Paper	Infeasible	Feasible	Feasible

Example 1: In this example, two cases are proposed to discuss the conservatism of proposed design method by comparing with the method of [27]. Referring to [27], the delay-dependent criterion was developed for LPV deterministic system with state and input delays which are the same case. Let us consider the following LPV system.

$$\begin{aligned} x(k+1) = \sum_{i=1}^{2} \vartheta_i(k)\,(&\mathbf{A}_i x(k) + \mathbf{A}_{t_i} x(k-h(k)) + \mathbf{B}_i u(k) \\ &+ \mathbf{B}_i u(k) + \mathbf{B}_{t_i} u(k-h(k)) + \mathbf{E}_i w(k)), \end{aligned} \tag{41}$$

where $\mathbf{A}_1 = \begin{bmatrix} 0.013 & -0.013 \\ 0 & -0.117 \end{bmatrix}$, $\mathbf{A}_2 = \begin{bmatrix} 0.013 & -0.013 \\ 0 & -0.013 \end{bmatrix}$, $\mathbf{A}_{t_1} = \begin{bmatrix} 0 & 0.01 \\ 0 & 0.01 \end{bmatrix}$, $\mathbf{A}_{t_2} = \begin{bmatrix} 0 & 0.01 \\ 0 & -0.01 \end{bmatrix}$, $\mathbf{B}_1 = \begin{bmatrix} 0.013 \\ -0.052 \end{bmatrix}$, $\mathbf{B}_2 = \begin{bmatrix} 0.013 \\ -0.026 \end{bmatrix}$, $\mathbf{B}_{t_1} = \begin{bmatrix} 0 \\ 0.01 \end{bmatrix}$, $\mathbf{B}_{t_2} = \begin{bmatrix} 0 \\ -0.01 \end{bmatrix}$, $\mathbf{E}_1 = \begin{bmatrix} 0 \\ 0.01 \end{bmatrix}$, $\mathbf{E}_2 = \begin{bmatrix} 0 \\ 0.01 \end{bmatrix}$, $1 \le h(k) \le h_{\max}$, $\alpha_1(k) = |\sin(k)|$ and $\alpha_2(k) = 1 - |\sin(k)|$.

Case 1: Under fixing $h_{\max} = 1$, the proposed design method and the method of [27] are respectively applied to find their corresponding minimum attenuating value η. And, the results are stated in Table 1. From Table 1, the minimum value $\eta = 0.04$ can be found by the method of [27] when $h_{\max} = 1$. Besides, $\eta = 0.03$ can be found by using the proposed design method for $h_{\max} = 1$.

In addition, by setting $h_{\max} = 2$, an attenuating value η satisfying the sufficient conditions of [27] cannot be found. However, one can apply the proposed design method to establish controllers (4) and (5) with the following feedback gains under $\eta = 0.04$ and $h_{\max} = 2$.

$$\begin{aligned} \mathbf{F}_1 &= \begin{bmatrix} 7.0876 & 11.4640 \end{bmatrix}, \text{ and} \\ \mathbf{F}_2 &= \begin{bmatrix} 7.9402 & 15.6462 \end{bmatrix}. \end{aligned} \tag{42}$$

From this case, it is easy to find that the attenuating value η found by this paper is smaller than one found by [27] under the same $h_{\max}$. Besides, a case as finding $h_{\max}$ under the same η is also an interested issue of discussing the conservatism of the proposed design method. Therefore, a comparison with [27] in searching $h_{\max}$ is proposed in the following case.

Case 2: By fixing a attenuating value as $\eta = 0.1$, the proposed design method and the method of [27] are re-

Table 2. Compared results with fixing $\eta = 0.1$.

$h_{\max}$	$0.1\ldots1$	$1.1\ldots2.9$	3
[27]	Feasible	Infeasible	Infeasible
This Paper	Feasible	Feasible	Infeasible

spectively applied to find their corresponding allowable bound $h_{\max}$. Furthermore, the simulated results are stated in Table 2. From Table 2, it is easy to find that the maximum allowed value $h_{\max} = 1.9$ can be obtained by the method of [27]. Besides, the maximum upper bound as $h_{\max} = 2.9$ can be found by the proposed design method. In case as $\eta = 0.1$ and $h_{\max} = 2.9$, the following feedback gains are determined by this paper.

$$\mathbf{F}_1 = \begin{bmatrix} 3.8900 & 16.5327 \end{bmatrix}, \text{ and}$$
$$\mathbf{F}_2 = \begin{bmatrix} 4.5833 & 21.1669 \end{bmatrix}. \tag{43}$$

From the results of this case, the maximum value $h_{\max}$ found by the proposed design method is bigger than one found by [27].

In those cases, the relaxation of the proposed design method can be demonstrated. And, this paper provides the less conservative results than [27] for stabilizing (41). Besides, the following example is provided to show the importance of considering stochastic behavior and general time-delay case.

Example 2: In this section, a truck-trailer system is considered for discussing the practical applicability of the proposed design method. Moreover, a comparison with the method of [27] is proposed to emphasize the importance of considering stochastic behavior and general delay. Referring to [38], the following linearized differential equation of truck-trailer is obtained. Furthermore, the backing up speed v is assumed as time-varying parameter $v(k)$ for a possible uncertainty.

$$x_1(k+1) = \left(1 - \frac{v(k)\Delta t}{L_2}\right)x_1(k) + \left(\frac{v(k)\Delta t}{L_1}\right)u(k), \tag{44a}$$

$$x_2(k+1) = \left(\frac{v(k)\Delta t}{L_2}\right)x_1(k) + x_2(k), \tag{44b}$$

$$x_3(k+1) = \left(\frac{(v(k)\Delta t)^2}{2L_2}\right)x_1(k) + (v(k)\Delta t)x_2(k) + x_3(k). \tag{44c}$$

where $x_1(k)$ denotes angle difference between truck and trailer; $x_2(k)$ denotes angle of trailer; $x_3(k)$ denotes vertical position of rear end of trailer; $u(k)$ denotes steering angle; L_1 denotes the length of truck; L_2 is the length of trailer; Δt is the sampling time; and $v(k)$ is the speed of backing up. Assuming that $v(k)$ is time varying in an interval as $v(k) \in [-1.2, -0.8]$. Besides, $L_1 = 2.58$ m, $L_2 = 5.5$ m and $\Delta t = 2$ s are given from [38]. According to the time-varying parameter of (45) is one, $N = 2$ can be determined with $N = 2^r$ and $r = 1$. Besides, the multiplicative noise terms and disturbed terms are added to represent the stochastic behavior and external disturbance of the system, respectively. Moreover, the state and input delays are added to simulate general delay effects on the system. Then, the LPV model for (44) is proposed as follows:

$$\begin{aligned} x(k+1) = \sum_{i=1}^{2} \vartheta_i \Big(&\mathbf{A}_i x(k) + \mathbf{A}_{t_i} x(k-h(k)) + \mathbf{B}_i u(k) \\ &+ \mathbf{B}_{t_i} u(k-g(k)) + \mathbf{E}_i w(k) + \sum_{e=1}^{2} \Big(\big(\bar{\mathbf{A}}_{ie} x(k) \\ &+ \bar{\mathbf{A}}_{t_{ie}} x(k-h(k)) + \bar{\mathbf{B}}_{ie} u(k) \\ &+ \bar{\mathbf{B}}_{t_{ie}} u(k-g(k)) + \bar{\mathbf{E}}_{ie} w(k)\big)\beta_e(k)\Big)\Big), \end{aligned} \tag{45}$$

where $\mathbf{A}_1 = \begin{bmatrix} 1.43 & 0 & 0 \\ -0.43 & 1 & 0 \\ 0.5 & -2 & 1 \end{bmatrix}$, $\mathbf{A}_2 = \begin{bmatrix} 1.3 & 0 & 0 \\ -0.3 & 1 & 0 \\ 0.23 & -1.8 & 1 \end{bmatrix}$, $\mathbf{B}_1 = \begin{bmatrix} -0.86 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{B}_2 = \begin{bmatrix} -0.57 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{E}_1 = \begin{bmatrix} 0 \\ 0 \\ 0.01 \end{bmatrix}$, $\mathbf{E}_2 = \begin{bmatrix} 0 \\ 0 \\ 0.02 \end{bmatrix}$, $\mathbf{A}_{t_1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0.01 & 0.1 & 0 \end{bmatrix}$, $\mathbf{A}_{t_2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0.02 & 0.1 & 0 \end{bmatrix}$, $\mathbf{B}_{t_1} = \begin{bmatrix} -0.05 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{B}_{t_2} = \begin{bmatrix} -0.07 \\ 0 \\ 0 \end{bmatrix}$, $\bar{\mathbf{A}}_{11} = \bar{\mathbf{A}}_{21} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0.05 & 0.01 & 0 \end{bmatrix}$, $\bar{\mathbf{B}}_{11} = \bar{\mathbf{B}}_{21} = \begin{bmatrix} -0.006 \\ 0 \\ 0 \end{bmatrix}$, $\bar{\mathbf{A}}_{12} = \bar{\mathbf{A}}_{22} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0.1 & 0.02 & 0 \end{bmatrix}$, $\bar{\mathbf{B}}_{12} = \bar{\mathbf{B}}_{22} = \begin{bmatrix} -0.01 \\ 0 \\ 0 \end{bmatrix}$, $\bar{\mathbf{A}}_{t_{11}} = \bar{\mathbf{A}}_{t_{21}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0.005 & 0 \end{bmatrix}$, $\bar{\mathbf{B}}_{t_{11}} = \bar{\mathbf{B}}_{t_{21}} = \begin{bmatrix} -0.005 \\ 0 \\ 0 \end{bmatrix}$, $\bar{\mathbf{A}}_{t_{12}} = \bar{\mathbf{A}}_{t_{22}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0.01 & 0 \end{bmatrix}$, $\bar{\mathbf{B}}_{t_{12}} = \bar{\mathbf{B}}_{t_{22}} = \begin{bmatrix} -0.006 \\ 0 \\ 0 \end{bmatrix}$, $\bar{\mathbf{E}}_{11} = \bar{\mathbf{E}}_{21} = \begin{bmatrix} 0 \\ 0 \\ 0.001 \end{bmatrix}$, $\bar{\mathbf{E}}_{12} = \bar{\mathbf{E}}_{22} = \begin{bmatrix} 0 \\ 0 \\ 0.002 \end{bmatrix}$, $\vartheta_1 = |\sin(k)|$ and $\vartheta_2 = 1 - |\sin(k)|$, $h(k) = |2sin(k)| + 1$ and $g(k) = |3cos(k)| + 2$, and disturbance input $w(k)$ is chosen as zero-mean white noise with unit variance.

According to (45), $h_{\max} = 3$, $h_{\min} = 1$, $g_{\max} = 5$ and $g_{\min} = 2$ can be determined. Applying the proposed design method, the following GS controllers can be designed via

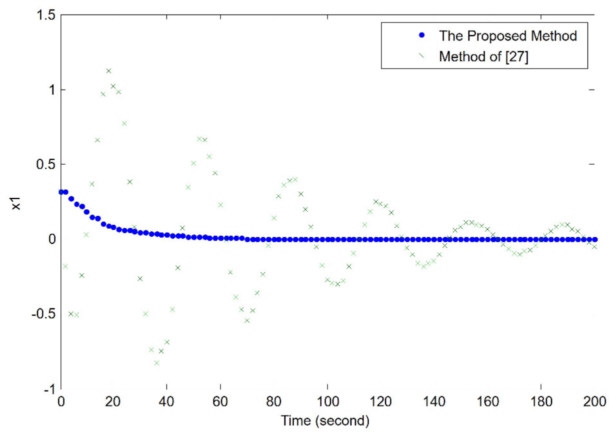

Fig. 1. Responses for $x_1(k)$ of Example 2.

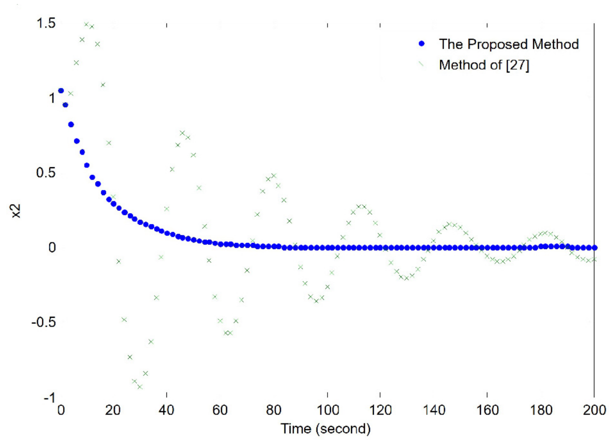

Fig. 2. Responses for $x_2(k)$ of Example 2.

setting $\rho_1 = 0.87$, $\rho_2 = 0.5$ and $\eta = 2$.

$$u(k) = -\left(\sum_{i=1}^{2} \vartheta_i \mathbf{F}_i\right) x(k),$$

and

$$u(k-g(k)) = -\left(\sum_{i=1}^{2} \vartheta_i \mathbf{F}_i\right) x(k-g(k)), \tag{46}$$

where $\mathbf{F}_1 = \begin{bmatrix} -0.7993 & 0.0521 & -0.0007 \end{bmatrix}$ and $\mathbf{F}_2 = \begin{bmatrix} -0.7338 & 0.0576 & -0.0006 \end{bmatrix}$.

Applying (46), the responses of (45) are stated in Figs. 1-3 with initial condition $x(0) = \begin{bmatrix} \frac{\pi}{6} & \frac{\pi}{3} & -3 \end{bmatrix}^T$. And, the following ratio function is used to check the satisfaction of Definition 1 by using the simulation results.

$$E\left\{\frac{\sum_0^\infty x^T(k)\mathbf{S}x(k)}{\sum_0^\infty w^T(k)w(k)}\right\} = 0.2867. \tag{47}$$

From Figs. 1-3, the responses of system (45) driven by (46) are respectively stabilized near zero. The asymptotical stability of the system (45) can be thus guaranteed in

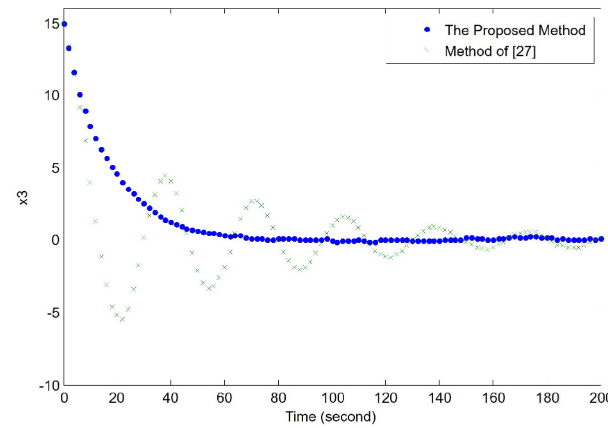

Fig. 3. Responses for $x_3(k)$ of Example 2.

the mean square via the designed controllers (46). In addition, from (47), it is easily found that the ratio value in (47) is smaller than given $\eta^2 = 4$ with $\eta = 2$. Thus, the system (45) driven by (46) achieves H_∞ performance.

In [27], the state and input delays were considered as the same case $h(k)$. Moreover, the stochastic behaviors did not considered in [27]. By applying the method of [27], the following GS controllers can be designed by setting $h_{min} = 1$, $h_{max} = 3$, $\eta = 2$ and $\rho_1 = \rho_2 = 0$.

$$u(k) = \left(\sum_{i=1}^{2} \vartheta_i \mathbf{K}_i\right) x(k),$$

and

$$u(k-h(k)) = \left(\sum_{i=1}^{2} \vartheta_i \mathbf{K}_i\right) x(k-h(k)), \tag{48}$$

where $\mathbf{K}_1 = \begin{bmatrix} 1.1741 & -0.5252 & 0.0775 \end{bmatrix}$ and $\mathbf{K}_2 = \begin{bmatrix} 1.295 & -0.5431 & 0.0742 \end{bmatrix}$.

Applying (48), the responses of (45) are also stated in Figs. 1-3 with the same initial condition. Form Figs. 1-3, one can find that the settling time and maximum overshoot of (45) driven by (48) are bigger than that driven by (46) according to the considerations of stochastic behaviors and general delay. Obviously, the proposed design method provides some improvements to the method of [27] in controlling LPV stochastic systems with time-delays.

Based on the simulated results in this section, this paper provides not only less conservative delay-dependent stability criterion than [27] but also some improvements to the method of [27] in stabilizing LPV stochastic systems with time-delays.

5. CONCLUSION

In this paper, a robust H_∞ control problem for discrete-time uncertain stochastic system with time delays is discussed via LPV modeling approach and stochastic difference equation. For proposing a general delay-dependent stability criterion, state and input delays were considered as different time-varying interval case. And, a novel Lyapunov-Krasovskii function and Jensen inequality were applied to derive the relaxed LMI sufficient conditions. By solving those conditions, the GS controller can be designed such that the asymptotical stability and H_∞ performance of closed-loop system are achieved in the mean square. According to the numerical simulations, the effectiveness and applicability of the proposed GS controller design method have been demonstrated.

REFERENCES

[1] A. Chammas and C. Leondes, "Optimal control of stochastic linear systems by discrete output feedback," *IEEE Trans. on Automatic Control*, vol. 23, no. 5, pp. 921-926, October 1978. [click]

[2] V. A. Ugrinovskii, "Observability of linear stochastic uncertain systems," *IEEE Trans. on Automatic Control*, vol. 48, no. 12, pp. 2264-2269, December 2003. [click]

[3] H. Li, P. Shi, D. Yao, and L. Wu, "Observer-based adaptive sliding mode control for nonlinear Markovian jump systems," *Automatica*, vol. 64, pp.133-142, February 2016. [click]

[4] H. Li, P. Shi, and D. Yao, "Adaptive sliding-mode control of Markov jump nonlinear systems with actuator faults," *IEEE Trans. on Automatic Control*, vol. 62, no. 4, pp. 1933-1939, April 2017. [click]

[5] G. Eli, S. Uri, and Y. Isaac, *H_∞ Control and Estimation of State-Multiplicative Linear Systems*, Springer, London, 2005.

[6] S. Xu and T. Chen, "Robust H_∞ control for uncertain stochastic systems with state delay," *IEEE Trans. on Automatic Control*, vol. 47, no. 12, pp. 2089-2094, December 2002. [click]

[7] T. Hou, W. Zhang, and H. Ma, "A game-based control design for discrete-time Markov jump systems with multiplicative noise," *IET Control Theory and Appl.*, vol. 7, no. 5, pp. 773-783, March 2013. [click]

[8] J. Song, Y. Niu, and Y. Zou, "Robust finite-time bounded control for discrete-time stochastic systems with communication constraint," *IET Control Theory and Appl.*, vol. 9, no. 13, pp. 2015-2021, August 2015. [click]

[9] W. J. Chang, M. W. Chen, and C. C. Ku, "Passive fuzzy controller design for discrete ship steering systems via Takagi-Sugeno fuzzy model with multiplicative noises," *Journal of Marine Science and Technology*, vol. 21, no. 2, pp. 159-165, April 2013.

[10] M. Guo, L. Sheng, and W. Zhang, "Finite horizon H_2/H_∞ control of time-varying stochastic systems with Markov jumps and (x,u,v)-dependent noise," *IET Control Theory and Appl.*, vol. 8, no. 14, pp. 1354-1363, September 2014. [click]

[11] L. Li, Q. Zhang, J. Li, and G. Wang "Robust finite-time H_∞ control for uncertain singular stochastic Markovian jump systems via proportional differential control law," *IET Control Theory and Appl.*, vol. 8, no. 16, pp. 1625-1638, November 2014. [click]

[12] L. Xie, "Robust H_∞ control for uncertain discrete-time delayed stochastic systems with saturating nonlinear actuators," *Proc. of the IEEE Conf. on Networking, Sensing and Control*, pp. 844-849, 2005.

[13] Q. Gao, G. Feng, Z. Xi, Y. Wang, and J. Qiu, "A new design of robust H_∞ sliding mode control for uncertain stochastic T-S fuzzy time-delay systems," *IEEE Trans. on Cybernetics*, vol. 44, no. 9, pp. 1556-1566, September 2014. [click]

[14] A. A. Pantelous and L. Yang, "Robust LMI stability, stabilization and H_∞ control for premium pricing models with uncertainties into a stochastic discrete-time framework," *Insurance: Mathematics and Economics*, vol. 59, no. 6, pp. 133-143, November 2014. [click]

[15] W. H. Chen, W. X. Zheng, and Y. Shen, "Delay-dependent stochastic stability and H_∞-control of uncertain neutral stochastic systems with time delay," *IEEE Trans. on Automatic Control*, vol. 54, no. 7, pp. 1660-1667, July 2009. [click]

[16] W. H. Chen, S. Luo, and W. X. Zheng, "Sampled-data distributed H_∞ control of a class of 1-D parabolic systems under spatially point measurements," *Journal of The Franklin Institute*, vol. 354, no. 1, pp. 197-214, January 2017.

[17] W. J. Chang, S. S. Jheng, and C. C. Ku, "Fuzzy control with robust and passive properties for discrete-time Takagi-Sugeno fuzzy systems with multiplicative noises," *Proc. of the Institution of Mechanical Engineers, Part I: Journal Systems and Control Engineering*, vol. 226, no. 4, pp. 476-485, October 2011.

[18] W. J. Chang, S. S. Jheng, and C. C. Ku, "Passive estimated state feedback fuzzy controller design for discrete perturbed fuzzy systems with multiplicative noises," *Journal of Chinese Institute of Engineers*, vol. 36, no. 6, pp. 684-695, November 2012. [click]

[19] Y. Tang, J. Fang, M. Xia, and X. Gu, "Synchronization of Takagi-Sugeno fuzzy stochastic discrete-time complex networks with mixed time-varying delays," *Applied Mathematical Modelling*, vol. 34, no. 4, pp. 843-855, April 2010.

[20] G. Franze, D. Famularo, E. Garone, and A. Casavola, "Dilated model predictive control strategy for linear parameter-varying systems with a time-varying terminal set," *IET Control Theory and Appl.*, vol. 3, no. 1, pp. 110-120 , January 2009. [click]

[21] A. Ilka and V. Vesely, "Robust gain-scheduled controller design for uncertain LPV systems: affine quadratic stability approach," *Journal of Electrical Systems and Information Technology*, vol. 1, no. 1, pp. 45-57, May 2014. [click]

[22] B. Kulcsár and M. Verhaegen, "Robust inversion based fault estimation for discrete-time LPV systems," *IEEE Trans. on Automatic Control*, vol. 57, no. 6, pp. 1581-1586, June 2012. [click]

[23] M. Fiacchini and G. Millerioux, "Dead-beat functional observers for discrete-time LPV systems with unknown inputs," *IEEE Trans. on Automatic Control*, vol. 58, no. 12, pp. 3230-3235, December 2013. [click]

[24] E. Garone and A. Casavola, "Receding horizon control strategies for constrained LPV systems based on a class of nonlinearly parameterized Lyapunov functions," *IEEE Trans. on Automatic Control*, vol. 57, no. 9, pp. 2354-2360, September 2012. [click]

[25] E. Prempain, I. Postlethwaite, and A. Benchaib, "A linear parameter variant H_∞ control design for an induction motor," *Control Engineering Practice*, vol. 10, no. 6, pp. 633-644, June 2002. [click]

[26] J. D. Caigny, J. F. Camino, R. C. L. F. Oliveira, P. L. D. Peres, and J. Swevers, "Gain-scheduled H_2 and H_∞ control of discrete-time polytopic time-varying systems," *IET Control Theory and Appl.*, vol. 4, no. 3, pp. 362-380, March 2010. [click]

[27] S. Zhou and W. X. Zheng, "A parameter-dependent Lyapunov function based approach to H_∞ control of LPV discrete-time systems with delays," *Proc. of the 47th IEEE Conf. on Decision and Control*, pp. 4669-4674, 2008. [click]

[28] C. C. Ku and G. W. Chen, "Gain-scheduled controller design for discrete-time linear parameter varying systems with multiplicative noises," *International Journal of Control, Automation and Systems*, vol. 13, no. 6, pp. 1382-1390, December 2015. [click]

[29] W. H. Chen, R. Jiang, X. Lu, and W. X. Zheng, "H_∞control of linear singular time-delay systems subject to impulsive perturbations," *IET Control Theory and Appl.*, vol. 11, no. 3, pp. 420-428, February 2017.

[30] Y. Niu, B. Chen, and X. Wang, "Sliding mode control for a class of nonlinear Itô stochastic systems with state and input delays," *International Journal of Control, Automation, and Systems*, vol. 7, no. 3, pp. 365-370, June 2009. [click]

[31] W. J. Chang and W. Chang, "Synthesis of nonlinear discrete control systems via time-delay affine Takagi-Sugeno fuzzy models," *ISA Trans.*, vol. 44, no. 2, pp. 243-257, April 2005.

[32] S. Wang, Y. Jiang, Y. Li, and D. Liu, "Reliable observer-based H_∞ control for discrete-time fuzzy systems with time-varying delays and stochastic actuator faults via scaled small gain theorem," *Neurocomputing*, vol. 147, no. 1, pp. 251-259, January 2015.

[33] J. An and G. Wen, "Improved stability criteria for time-varying delayed T-S fuzzy systems via delay partitioning approach," *Fuzzy Sets and Systems*, vol. 185, no. 1, pp. 83-94, December 2011. [click]

[34] C. Peng and M. R. Fei, "An improved result on the stability of uncertain T-S fuzzy systems with interval time-varying delay," *Fuzzy Sets and Systems*, vol. 212, no. 1, pp. 97-109, February 2013. [click]

[35] X. Su, P. Shi, L. Wu, and M. V. Basin, "Reliable filtering with strict dissipativity for T-S fuzzy time-delay systems," *IEEE Trans. on Cybernetics*, vol. 44, no. 12, pp. 2470-2483, December 2014. [click]

[36] J. H. Kim, "Delay-dependent robust H_∞ control for discrete-time uncertain sigular systems with interval time-varying delays in state and control input," *Journal of The Franklin Institute*, vol. 347, no. 9, pp. 1704-1722, November 2010. [click]

[37] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, SIAM, Philadelphia, PA, 1994.

[38] K. Tanaka and M. Sano, "A robust stabilization problem of fuzzy control systems and its application to backing up control of a truck-trailer," *IEEE Trans. on Fuzzy Systems*, vol. 2, no. 2, pp. 119-134, May 1994. [click]

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