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# Control for Underactuated Systems Using Sliding Mode Observer

**Djamila Zehar*, Khier Benmahammed, and Khalissa Behih**

**Abstract:** In this work, first we estimate all the system's state vector, with guarantied precision, for a category of second order underactuated mechanical systems (UMS), exploiting the triangular observer (TO) model that suits to the structure of these systems. Then we propose a sliding mode controller (SMC). The latter uses the estimated states given by the observer. The underactuated system is decomposed into two subsystems, where the sliding surface is constructed in two levels for each subsystem. The proposed controller guaranties the tracking performances, with minimization of chattering phenomenon, due to the constructed observer, even for system with uncertainties. Simulation results show the effectiveness of this strategy of control.

**Keywords:** Observer, sliding mode control, stability anlysis, uncertainty, underactuated system.

## 1. INTRODUCTION

Control tools which are used in laboratories, in order to prove the effectiveness of the developed strategies, become more and more complicated, which means that this control technics for delicate problems are difficult to enforce. Among the most used systems to validate the new approaches of control, we mention the UMS, these latter are non-linear and complex systems; admit less number of actuators than their degrees of freedom. Those systems use less energy, have a lighter weight structure, and therefore they can be constructed with a low cost. We can find this type of systems in a lot of applications such that: robotics [1], underwater vehicle and ships [2–4], flexible systems [5–7] and aerospace systems [8].The lack of actuators and nonlinearities complicate the control task for this category of systems.

For years, scientific researchers on control have tried to approach the complex nonlinear systems behaviour, as linear models. However this linear approximation is valid only in limited operating range, and even can lead to the lose of their physical meaning. Analysis and control of UMS is not a simple task and the elaborated technics are not generalized for all systems in this category, they are limited to particular forms. For this reason, UMS are classified in several classes according to their dynamics and structures, and each class are studied separately as in [9, 10]. In fact, some already established methods for the control of these systems, such that the passivity control (PC) proposed in [11] and the control based on energy of system (EBC) found in [12], which generally switch to linear controllers, have disadvantages because these methods are slow and don't have a good precision, what explains their limited applications, and force the researchers to consider other directions of control of these systems.

In order to improve the performances of UMS in terms of stability and robustness, many technics are proposed. Among them, we can mention the SMC [13] which has known a very large success because of its robustness and simplicity of implantation for controlling uncertain nonlinear systems. The main idea behind it is to define an attractive sliding surface in terms of the state variables of the system.

Second order UMS are widely studied by researchers because of their nonlinear nature, their simplicity to implement in laboratories as benchmark systems, which allows developing control for high order systems and those with more complex dynamics. An important number of control scheme such that SMC [14–16], backstepping control (BC) [17], adaptive control (AC) [18, 19], fuzzy logic control (FLC) [20, 21] and neural network control (NNC) [22], were been applied to UMS. In order to improve performances of those systems in terms of stability, convergence rapidity, precision and robustness against structured and unstructured uncertainties, a combination of SMC and the other different technics cited above, has been known a prominent development, among these technics we can mention: backstepping sliding mode control

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Djamila Zehar is with the Electromechanical Department at the University of Bordj Bou Arreridj 34000, the Department of Electrical Engineering and member in LSI laboratory at University of Sétif, 19000, Algeria (e-mail: z.dj2005@hotmail.com). Khier Benmahammed is with the Electronic Department and with the LSI Laboratory at University of Setif 19000, Algeria (e-mail: khierben@gmail.com). Khalissa Behih is with the Department of Electrical Engineering and member in LSI Laboratory at University of Sétif, 19000, Algeria (e-mail: behihkhalissa@yahoo.fr)
* Corresponding author.

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(BSMC) [23], which exploit the advantage of the backstepping, where nonlinearities are used to compensate those of system, and operate robustness of the SMC. There is also the fuzzy adaptive sliding mode control (FASMC) [24, 25], this combination allows to do fuzzy approximation of the unknown dynamic of system in direct or indirect way, and to obtain a robust control against uncertainties and minimizing the chattering effect. In the same context, we find the works based on neural network sliding mode control (NNSMC) [26].

Among the requirements involved in the control of systems, is the knowledge of states which is practically difficult to be satisfied, because these state variables do not always have a physical meaning, and sometimes their measurement is tricky, even impossible technically to achieve. Furthermore it's often desirable to use a minimum of sensors, in order to reduce cost and maintenance. Therefore, if control requests the use of the not measured state variables, it will be essential to construct fully or partially state vector, using a dynamic system which is called state observer (SO), its role is to provide a real time estimate of the state vector.

The main contribution of this work, consist of introducing an approach for a category of second order UMS, which is based on the combination of the two technics: the SMC and the nonlinear observer (NO), the constructed observer is decomposed on two sub systems, where the estimation of the variables of each sub system is done in parallel and the constructed sliding surface is calculated simultaneously with the estimated variables, which produces a reduced observation error dynamic, and consequently overcomes the influence of uncertainties and reduce the chattering phenomenon in the control signal and the state variables.

This paper is organised as follows: Section 2 gives the state space model of a category of second order UMS, Section 3 presents the conception of the TO, Section 4 describe the synthesis of the SMC, by defining an appropriate sliding surface using the estimated state variables and we give the stability analysis based on Lyapunov function (LF) of this control system. We discuss simulation results in Section 5. Finally, concluding remarks is given in Section 6.

## 2. MATHEMATIC MODEL OF A SECOND ORDER UNDERACTUATED SYSTEMS CATEGORY

A category of class of second order UMS (as the cart-pole system, the pendubot, the rotating pendulum, the crane system) are systems with two degree of freedom, and have the following Lagrangian [10]:

$$L(q,\dot{q}) = E_c - E_p, \tag{1}$$

$$L = \frac{1}{2}\dot{q}^T \begin{pmatrix} m_{11}(q_2) & m_{12}(q_2) \\ m_{21}(q_2) & m_{22}(q_2) \end{pmatrix} \dot{q} - E_p(q), \tag{2}$$

where $E_c$ is the kinetic energy and $E_p$ is the potential energy. $q = (q_1, q_2)^T$ is the configuration vector. $M = \begin{pmatrix} m_{11}(q_2) & m_{12}(q_2) \\ m_{21}(q_2) & m_{22}(q_2) \end{pmatrix}$ is symmetric invertible inertia matrix.

The Euler-Lagrange (EL) equation is given as:

$$\begin{aligned} &\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_1} - \frac{\partial L}{\partial q_1} = U, \\ &\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_2} - \frac{\partial L}{\partial q_2} = 0, \end{aligned} \tag{3}$$

where $U$ is the control force.

From (2) we have:

$$\begin{aligned} L = &\frac{1}{2}m_{11}(q_2)\dot{q}_1^2 + \frac{1}{2}m_{21}(q_2)\dot{q}_1\dot{q}_2 \\ &+ \frac{1}{2}m_{12}(q_2)\dot{q}_1\dot{q}_2 + \frac{1}{2}m_{22}(q_2)\dot{q}_2^2 - E_p(q), \end{aligned} \tag{4}$$

$$\frac{\partial L}{\partial \dot{q}_1} = m_{11}(q_2)\dot{q}_1 + \frac{1}{2}m_{21}(q_2)\dot{q}_2 + \frac{1}{2}m_{12}(q_2)\dot{q}_2. \tag{5}$$

Because the inertia matrix is symmetric we have $m_{12} = m_{21}$, so we get:

$$\begin{aligned} \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_1} = &m_{11}(q_2)\ddot{q}_1 + \frac{dm_{11}(q_2)}{dq_2}\dot{q}_2\dot{q}_1 + m_{12}\ddot{q}_2 \\ &+ \frac{dm_{12}(q_2)}{dq_2}\dot{q}_2^2, \end{aligned} \tag{6}$$

$$\frac{\partial L}{\partial q_1} = \frac{\partial E_p(q)}{\partial q_1} = g_1(q), \tag{7}$$

$$\frac{\partial L}{\partial \dot{q}_2} = m_{22}(q_2)\dot{q}_2 + \frac{1}{2}m_{21}(q_2)\dot{q}_1 + \frac{1}{2}m_{12}(q_2)\dot{q}_1, \tag{8}$$

$$\begin{aligned} \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_2} = &m_{22}(q_2)\ddot{q}_2 + \frac{dm_{22}(q_2)}{dq_2}\dot{q}_2^2 \\ &+ m_{21}(q_2)\ddot{q}_1 + \frac{dm_{21}(q_2)}{dq_2}\dot{q}_1\dot{q}_2, \end{aligned} \tag{9}$$

$$\frac{\partial L}{\partial q_2} = \frac{\partial E_p(q)}{\partial q_2} = g_2(q). \tag{10}$$

From (3), we get:

$$\begin{aligned} &m_{11}(q_2)\ddot{q}_1 + m_{12}(q_2)\ddot{q}_2 + \frac{dm_{11}(q_2)}{dq_2}\dot{q}_2\dot{q}_1 \\ &\quad + \frac{dm_{12}(q_2)}{dq_2}\dot{q}_2^2 + g_1(q) = U, \end{aligned} \tag{11}$$

$$\begin{aligned} &m_{22}(q_2)\ddot{q}_2 + m_{21}(q_2)\ddot{q}_1 + \frac{dm_{21}(q_2)}{dq_2}\dot{q}_1\dot{q}_2 \\ &\quad + \frac{dm_{22}(q_2)}{dq_2}\dot{q}_2^2 + g_2(q) = 0. \end{aligned} \tag{12}$$

According to the mathematical development mentioned above , we can obtain the following matrix representation (as an exemple you can see the development of the cart-pole system given in Appendix):

$$\begin{pmatrix} m_{11}(q_2) & m_{12}(q_2) \\ m_{21}(q_2) & m_{22}(q_2) \end{pmatrix} \begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{pmatrix} + \begin{pmatrix} H_1(q,\dot{q}) \\ H_2(q,\dot{q}) \end{pmatrix} = \begin{pmatrix} U \\ 0 \end{pmatrix}. \quad (13)$$

$H_i$ ($i = 1, 2$) is the vector which represents centrifugal, corioli and gravity terms, where

$$H_1(q,\dot{q}) = \frac{dm_{11}(q_2)}{dq_2}\dot{q}_2\dot{q}_1 + \frac{dm_{12}(q_2)}{dq_2}\dot{q}_2^2 + g_1(q), \quad (14)$$

$$H_2(q,\dot{q}) = \frac{dm_{21}(q_2)}{dq_2}\dot{q}_1\dot{q}_2 + \frac{dm_{22}(q_2)}{dq_2}\dot{q}_2^2 + g_2(q). \quad (15)$$

With some manipulations of these second order equations, we can get the following state space representation:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} = \begin{pmatrix} x_2 \\ f_1(x) \\ x_4 \\ f_2(x) \end{pmatrix} + \begin{pmatrix} 0 \\ g_1(x) \\ 0 \\ g_2(x) \end{pmatrix} U + \begin{pmatrix} 0 \\ d_1(x) \\ 0 \\ d_2(x) \end{pmatrix}, \quad (16)$$

$$y(t) = (x_1, x_3)^T, \quad (17)$$

where $x = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \end{pmatrix}^T$ is the state space vector, such that $x_1 = q_1$, $x_2 = \dot{q}_1$, $x_3 = q_2$ and $x_4 = \dot{q}_2$. $f_1$, $f_2$, $g_1$ and $g_2$ are nominal bounded nonlinear functions.

$$f_1(x) = \frac{1}{m_{11}(x_3)m_{22}(x_3) - m_{12}(x_3)m_{21}(x_3)} \times (m_{12}(x_3)H_2 - m_{12}(x_3)H_1), \quad (18)$$

$$f_2(x) = \frac{1}{m_{22}(m_{11}(x_3)m_{22}(x_3) - m_{12}(x_3)m_{21}(x_3))} \times (-m_{21}m_{12} + m_{22}m_{21} - (m_{11}(x_3)m_{22}(x_3) - m_{12}(x_3)m_{21}(x_3))H_2), \quad (19)$$

$$g_1(x) = \frac{m_{22}}{m_{11}(x_3)m_{22}(x_3) - m_{12}(x_3)m_{21}(x_3)}, \quad (20)$$

$$g_2(x) = \frac{-m_{21}}{m_{11}(x_3)m_{22}(x_3) - m_{12}(x_3)m_{21}(x_3)}, \quad (21)$$

$y(t)$ is the outputs vector. $d_1(x)$ and $d_2(x)$ present the uncertainties and disturbances.

**Assumption 1:** The system (16) is observable.

**Assumption 2:** The system in (16) is bounded input bounded output and stable for $t \in [0, T]$.

**Assumption 3:** The uncertain terms are bounded by: $|d_1(x)| \le \rho_1$ and $|d_2(x)| \le \rho_2$, where $\rho_1$ and $\rho_2$ are known positive constants.

## 3. DESIGN OF THE TRIANGULAR OBSERVER

The implementation of the control laws based on the system nonlinear model requires the knowledge of the system complete state vector at every instant of time. usually, only part of the state vector can be accessed through sensors. To have the whole system state, one uses a soft sensor, called observer. Generally the dynamics of non linear systems are approximated using fuzzy systems (FS), nevertheless the latter need an exact model of the system and fuzzy rules estabilished by an expert. Furthermore, in most cases, stability analysis for FS is difficult, since they lack mathematical descriptions. In the literature, many strategies with FLC have been introduced, such as the fuzzy PID controller (FPIDC) [27], the fuzzy adaptive controller (FAC) [28] the fuzzy neural network controller (FNNC) [29], and the work [30] where the system is described by a discrete time Takagi-Sugeno (T-S) fuzzy affine model using a Markov chain to describe the actuator fault behavior. However these combinations may make the analysis and the control procedure tedious and long, which require a long computation time and a powerful calculator. The presented observer simplifies the computation of the estimated states and the study of the closed loop system stability, since it is obtained separately from the system controller.

The most of synthesized observers, for nonlinear systems have the following structure [31]:

$$\dot{\hat{x}} = f(\hat{x}, U) + \varphi(y, \hat{x}), \quad y = h(x). \quad (22)$$

It's a copy of the state space model plus a corrector term $\varphi(y, \hat{x})$ which establishes the convergence of the estimated state $\hat{x}$ to its real state $x$ in finite time.

In the Extended Leunberger Observer (LO) [32], a linearized model is needed, or a change of coordinates is made, which requires a set of nonlinear partial differential equations, and that is so difficult to achieve. Another technique of estimation which is widely studied for the estimation for dynamical system is the Extended Kalman Filter (EKF) [33], unfortunately, stability and convergence proofs, established for linear systems couldn't be extended to nonlinear systems. One of the most known classes of robust observer is Sliding Mode Observer (SMO) as described in [34] and [31].

The principle of SMO is to constrained the "n" order system dynamics to converge to the sliding surface S of "n-p" dimension, using discontinuous functions (p is the sensor vector dimension) [35].

We propose the following TO dynamics, which is decomposed in two sub system:

$$\begin{pmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \\ \dot{\hat{x}}_3 \\ \dot{\hat{x}}_4 \end{pmatrix} = \begin{pmatrix} \hat{x}_2 \\ \hat{f}_1(\bar{x}_1,\bar{x}_2,\bar{x}_3,\bar{x}_4)+\hat{g}_1(\bar{x}_3)U \\ \hat{x}_4 \\ \hat{f}_2(\bar{x}_1,\bar{x}_2,\bar{x}_3,\bar{x}_4)+\hat{g}_2(\bar{x}_3)U \end{pmatrix.} \begin{matrix} +\mu_1 sign_{11}(x_1-\hat{x}_1) \\ +\mu_2 sign_{12}(\bar{x}_2-\hat{x}_2) \\ +\mu_3 sign_{23}(x_3-\hat{x}_3) \\ +\mu_4 sign_{24}(\bar{x}_4-\hat{x}_4) \end{matrix}\Bigg), \tag{23}$$

where $\hat{f}_1$, $\hat{f}_2$, $\hat{g}_1$ and $\hat{g}_2$ are respectively the functions $f_1$, $f_2$, $g_1$ and $g_2$ but with the new estimated state variables. Function $sign_{ij}$ is the classical function "sign" of the $i$th subsystem if $\bar{x}_j-\hat{x}_j=0$, where $i=\{1,2\}$ and $j=\{1,2\}$ for the first subsystem, $j=\{3,4\}$ for the second subsystem, else $sign_{ij}$ is set to zero.

$$\begin{cases} \bar{x}_1 = x_1, \\ \bar{x}_2 = \hat{x}_2+\mu_1 \operatorname{atan}\left(\frac{\pi}{2}(\bar{\mathrm{x}}_1-\hat{\mathrm{x}}_1)\right), \end{cases}$$

and

$$\begin{cases} \bar{x}_3 = x_3, \\ \bar{x}_4 = \hat{x}_4+\mu_3 \operatorname{atan}\left(\frac{\pi}{2}(\bar{\mathrm{x}}_3-\hat{\mathrm{x}}_3)\right). \end{cases}$$

atan is usual arc tangente function, which is known to be a continuous approximation to the sign function.

Estimation errors are: $e_{1e}=x_1-\hat{x}_1$, $e_{2e}=x_2-\hat{x}_2$, $e_{3e}=x_3-\hat{x}_3$, $e_{4e}=x_4-\hat{x}_4$ .

**Theorem 1:** Suppose the observer (23) is constructed for the system (16), which state variables are estimated by choosing appropriate parameters $\mu_i$ ($i=1, 2, ..., n$) for any initial conditions and Assumptions 1, 2 and 3 are verified, then the estimated state variables converge to the real system state variables in finite time.

**Proof:** The dynamic of the observer errors is:

$$\begin{pmatrix} \dot{e}_{1e} \\ \dot{e}_{2e} \\ \dot{e}_{3e} \\ \dot{e}_{4e} \end{pmatrix} = \Bigg( \begin{matrix} e_{2e} \\ f_1(x_1,x_2,x_3,x_4)-\hat{f}_1(\bar{x}_1,\bar{x}_2,\bar{x}_3,\bar{x}_4) \\ e_{4e} \\ f_2(x_1,x_2,x_3,x_4)-\hat{f}_2(\bar{x}_1,\bar{x}_2,\bar{x}_3,\bar{x}_4) \end{matrix} \begin{matrix} -\mu_1 sign_{11}(x_1-\hat{x}_1) \\ -\mu_2 sign_{12}(\bar{x}_2-\hat{x}_2)+d_1(x) \\ -\mu_3 sign_{23}(x_3-\hat{x}_3) \\ -\mu_4 sign_{24}(\bar{x}_4-\hat{x}_4)+d_2(x) \end{matrix}\Bigg). \tag{24}$$

Step 1:

For the first subsystem, we have:

$$\dot{e}_{1e} = \dot{x}_1-\dot{\hat{x}}_1 = x_2-\hat{x}_2-\mu_1 sign_{11}(x_1-\hat{x}_1), \tag{25}$$

$$\dot{e}_{1e} = e_{2e}-\mu_1 sign_{11}(x_1-\hat{x}_1). \tag{26}$$

The LF is given by:

$$V_{11} = \frac{1}{2}e_{1e}^2, \tag{27}$$

$$\dot{V}_{11} = e_{1e}\dot{e}_{1e} = e_{1e}(e_{2e}-\mu_1 sign_{11}(x_1-\hat{x}_1)). \tag{28}$$

We choose $\mu_1>\max|e_{2e}| \to e_{1e}$ tends toward zero in finite time, and consequently $\dot{e}_{1e}=0$ , which implies that: $e_{2e}=\mu_1 sign_{11}(x_1-\hat{x}_1)$.

In other hand, we have: $\bar{x}_2=\hat{x}_2+e_{2e}=x_2$.

For the second subsystem, we have the following LF:

$$V_{21} = \frac{1}{2}e_{3e}^2, \tag{29}$$

$$\dot{V}_{21} = e_{3e}\dot{e}_{3e} = e_{3e}(e_{4e}-\mu_3 sign_{23}(x_3-\hat{x}_3)). \tag{30}$$

We choose $\mu_3>\max|e_{4e}| \to e_{3e}$ tends toward zero in finite time, and consequently $\dot{e}_{3e}=0$, which implies that $e_{4e}=\mu_3 sign_{23}(x_3-\hat{x}_3)$.

In other hand, we have: $\bar{x}_4=\hat{x}_4+e_4=x_4$.

Step2:

We have:

$$\dot{e}_{1e} = 0, \tag{31}$$

$$\dot{e}_{2e} = -\mu_2 sign_{12}(\bar{x}_2-\hat{x}_2)+d_1(x), \tag{32}$$

and

$$\dot{e}_{3e} = 0, \tag{33}$$

$$\dot{e}_{4e} = -\mu_4 sign_{24}(\bar{x}_4-\hat{x}_4)+d_2(x). \tag{34}$$

The new LF for the first subsystem will be:

$$V_1 = \frac{1}{2}e_{1e}^2+\frac{1}{2}e_{2e}^2, \tag{35}$$

$$\dot{V}_1 = e_{1e}\dot{e}_{1e}+e_{2e}\dot{e}_{2e}. \tag{36}$$

We have

$$\begin{aligned} &\dot{e}_{1e}=0 \\ &\Rightarrow\ \dot{V}_1 = e_{2e}\dot{e}_{2e} = e_{2e}(-\mu_2 sign_{12}(\bar{x}_2-\hat{x}_2)+d_1(x)) \\ &\Rightarrow\ \dot{V}_1 \le -\mu_2|e_{2e}|+\rho_1|e_{2e}|. \end{aligned}$$

For the second subsystem, we have

$$V_2 = \frac{1}{2}e_{3e}^2+\frac{1}{2}e_{4e}^2, \tag{37}$$

$$\dot{V}_2 = e_{3e}\dot{e}_{3e}+e_{4e}\dot{e}_{4e}. \tag{38}$$

We have

$$\begin{aligned} &\dot{e}_{3e}=0 \\ &\Rightarrow \dot{V}_2 = e_{4e}\dot{e}_{4e} = e_{4e}(-\mu_4 sign_{24}(\bar{x}_4-\hat{x}_4)+d_2(x)) \\ &\Rightarrow\ \dot{V}_2 \le -\mu_4|e_{4e}|+\rho_2|e_{4e}|. \end{aligned}$$

So in order to obtain $\dot{V}_1<0$ and $\dot{V}_2<0$ we have just to choose $\mu_2>\rho_1$ and $\mu_4>\rho_2$. □

## 4. CONTROL DESIGN

The procedure of constructing observer is separated from the control. After estimating the system states, for the first level, one can construct the sliding surfaces as follows:

$$S_1 = c_1 e_1 + e_2, \tag{39}$$

$$S_2 = c_2 e_3 + e_4, \tag{40}$$

where $c_1$ and $c_2$are positive constants.

Dynamic errors are:

$$e_1 = \hat{x}_1 - x_{1d},\ e_2 = \hat{x}_2 - x_{2d},\ e_3 = \hat{x}_3 - x_{3d},$$
$$e_4 = \hat{x}_4 - x_{4d}.$$

The desired vector is:

$$X_d = (x_{1d}, x_{2d},\ x_{3d}, x_{4d}).$$

The equivalent control can be extracted from the annulation of the sliding surface derivative, as follows:

In the first level we differentiate $S_1$and $S_2$ with respect to time, we get:

$$\dot{S}_1 = c_1 \dot{e}_1 + \dot{e}_2, \tag{41}$$

$$\dot{S}_2 = c_2 \dot{e}_3 + \dot{e}_4. \tag{42}$$

The equivalent control laws are:

$$U_{eq1} = \frac{-1}{\hat{g}_1}\left(\hat{f}_1 + c_1 \dot{e}_1 - \dot{x}_{2d}\right), \tag{43}$$

$$U_{eq2} = \frac{-1}{\hat{g}_2}\left(\hat{f}_2 + c_2 \dot{e}_3 - \dot{x}_{4d}\right). \tag{44}$$

For the second level, we propose the following sliding surface:

$$S = \sigma_1 S_1 + \sigma_2 S_2. \tag{45}$$

Parameters $\sigma_1$and $\sigma_2$ are positive constants, are chosen such that:

$$\sigma_1 \sigma_2 S_1 S_2 \geq 0. \tag{46}$$

The control law is given as:

$$U = U_{eq1} + U_{eq2} + U_{sw}. \tag{47}$$

The switching control is given by:

$$U_{sw} = \frac{-1}{\sigma_1 \hat{g}_1 + \sigma_2 \hat{g}_2}\left(k.signe(S) + \beta.S\right), \tag{48}$$

such that $k$ and $\beta$ are positive constants.

The main objective of this work can be resumed by the diagram presented in Fig. 1:

**Theorem 2:** for the UMS given by equation (16), the sliding surfaces are given as equations (39), (40) and (45) with the control law defined by equation (47), then the output vector $y(t)$ could track the desired trajectories and the whole system is globally asymptotically stable.

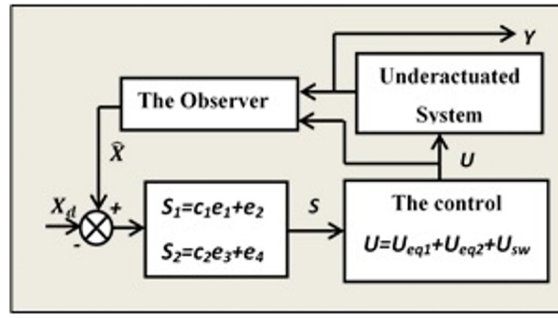


Fig. 1. The controller structure.

**Proof:** The derivative of LF is:

$$\dot{V} = S\dot{S}, \tag{49}$$

$$\dot{V} = S\left(\sigma_1 \hat{f}_1 + \sigma_1 \hat{g}_1 U + \sigma_1 c_1 \dot{e}_1 + \sigma_1 d_1 + \sigma_2 \hat{f}_2 + \sigma_2 \hat{g}_2 U + \sigma_2 c_2 \dot{e}_3 + \sigma_2 d_2 - \sigma_1 \dot{x}_{2d} - \sigma_2 \dot{x}_{4d}\right). \tag{50}$$

Applying the control of equation (47), we have:

$$\dot{V} = S\left(\sigma_1 d_1 + \sigma_2 d_2 - ksign(S) - \beta.S\right),$$
$$\dot{V} = (\sigma_1 d_1 + \sigma_2 d_2)|S| - k|S| - \beta.S^2, \tag{51}$$

where $\rho = \sup(\sigma_1 \rho_1 + \sigma_2 \rho_2)$ and choosing $k > \rho$, then

$$\dot{V} = -(k-\rho)|S| - \beta.S^2 \leq 0. \tag{52}$$

Integrating both sides of (52), we get:

$$\int_0^t \dot{V} d\tau = \int_0^t \left(-(k-\rho)|S| - \beta.S^2\right) d\tau, \tag{53}$$

$$V(t) - V(0) = -\int_0^t \left((k-\rho)|S| + \beta.S^2\right) d\tau < \infty,$$
$$\forall t \geq 0. \tag{54}$$

We have $V(t)$ is positive definite, which means:

$$0 \leq \int_0^t \left((k-\rho)|S| + \beta.S^2\right) d\tau \leq V(0) < \infty. \tag{55}$$

This implies that

$$0 \leq (k-\rho)\int_0^\infty (|S|)\, d\tau < \infty. \tag{56}$$

So

$S \in L_1$ ($L_1$: space of the function 1_Norm),

and

$$0 \leq \int_0^\infty \left(\beta.S^2\right) d\tau < \infty. \tag{57}$$

So,

$S \in L_2$ ($L_2$: space of the function 2_Norm).

From (57), we have

$$\int_0^{\infty} S^2 d\tau = \int_0^{\infty} \left(\sigma_1^2 S_1^2 + \sigma_2^2 S_2^2 + 2\sigma_1\sigma_2 S_1 S_2\right) d\tau. \tag{58}$$

Thus,

$$\int_0^{\infty} (2\sigma_1\sigma_2 S_1 S_2)\, d\tau \le \int_0^{\infty} \left(\sigma_1^2 S_1^2 + \sigma_2^2 S_2^2\right) d\tau. \tag{59}$$

From (58) we have

$$0 < \int_0^{\infty} (4\sigma_1\sigma_2 S_1 S_2)\, d\tau \le \int_0^{\infty} S^2 d\tau. \tag{60}$$

And

$$\sigma_1 \int_0^{\infty} (S_1^2) + \sigma_2 \int_0^{\infty} S_2^2) d\tau \le \int_0^{\infty} S^2 d\tau < \infty. \tag{61}$$

We have $S_1 \in L_2$ and $S_2 \in L_2$, which means that $\int_0^{\infty} S_1^2 d\tau < \infty$ and $\int_0^{\infty} S_2^2 d\tau < \infty$.

From (56), we have

$$\int_0^{\infty} (|S|)\, d\tau = \int_0^{\infty} |S_1|\, d\tau + \int_0^{\infty} |S_2|\, d\tau < \infty, \tag{62}$$

$$\int_0^{\infty} |S_1|\, d\tau < \infty, \text{ i.e., } S_1 \in L_1, \tag{63}$$

$$\int_0^{\infty} |S_2|\, d\tau < \infty, \text{ i.e., } S_2 \in L_1. \tag{64}$$

Since the desired trajectory vector $y_d(t)$ and its derivatives are bounded, the control input is bounded too, and $y(t) \in L_\infty$, and from equations (41) and (42) we have $(\dot{e}_1, \dot{e}_2, \dot{e}_3, \dot{e}_4) \in L_\infty$, hence $\dot{S}_1 \in L_\infty$, $\dot{S}_2 \in L_\infty$ using Barbarat's lemma we get $\lim_{t\to\infty} S_1 = 0$ and $\lim_{t\to\infty} S_2 = 0$ , we can conclude that sliding surfaces and tracking errors converge asymptotically to zero, which means that the dynamic estimated states converge to their references.

## 5. SIMULATION RESULTS

The proposed control strategy is applied to a cart-pole system, the dynamic equations are given as (16) (for the detail see Appendix), where

$x_1 = x$ is the position of the cart.

$x_2 = \dot{x}$ is the velocity of the cart.

$x_3 = \theta$ is the angle of the pole from the vertical axis.

$x_4 = \dot{\theta}$ is the velocity of the pole.

$M$ and $m$ are respectively the masses of the cart and the pole, $l$ is the length of the pole and $g$ is acceleration of gravity.

In these simulations, the parameters are chosen as $M = 2$ kg, $m = 0.1$ kg, $l = 0.25$ m, and $g = 9.81$.

The simulations are done using Matlab environnement, where we have used the ODE45 for the computation and resolution of simple first order differential equations (the mathematical model of the observer and the real system) with a step $\Delta t = 0.001$.

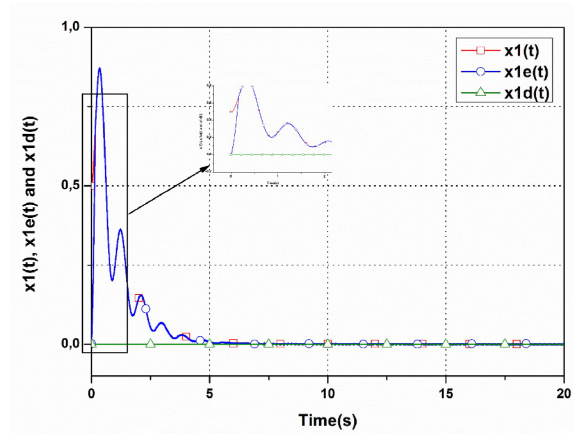


Fig. 2. The position of the cart, its estimatation and the reference.

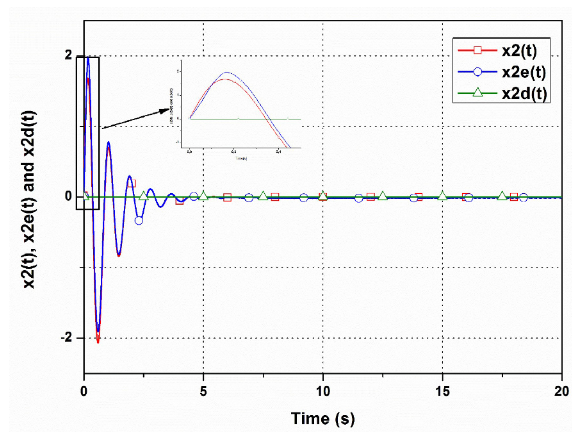


Fig. 3. The velocity of the cart, its estimatation and the reference.

### 5.1. Case I: Without parameter uncertainties:

The initial conditions are $x = \left(0.5, 0, \frac{\pi}{12}, 0\right)^T$, $x_e = (0,0,0,0)^T$, and the desired output vector is $y(t) = (0,0)^T$, we choose $c_1 = 0.9$, $c_2 = 2.5$,$k = 0.05$ and $\beta = 20$, $\sigma_1 = \sigma_2 = 1.5$.

Figs. 2-7 represent respectively the trajectories of $x(t)$, $\dot{x}(t)$, $\theta(t)$, $\dot{\theta}(t)$ their estimators, and their references, the sliding surfaces $S(t)$, $S_1(t)$ and $S_2(t)$ and the control signal $U(t)$.

### 5.2. Case II: With parameter uncertainties:

The initial conditions are:$x = \left(0.2, 0, \frac{-\pi}{12}, 0\right)^T$, $x_e = (0.8, 0, 0.5, 0)^T$ and the desired output vector is $y(t) = \left(0, \frac{\pi}{30}\sin(t)\right)^T$, we have the uncertainty on the two masses: $M = 2 + 0.0 randn(1,tf)$, $m = 0.1 + 0.05 randn(1,tf)$, we choose $c_1 = 0.9$, $c_2 = 2.5$, and $k = 0.05$, $\beta = 20$, $\sigma_1 = \sigma_2 = 1.5$.

Figs. 8-13 represent respectively the trajectories of $x(t)$, $\dot{x}(t)$, $\theta(t)$, $\dot{\theta}(t)$ their estimators, and their references, the

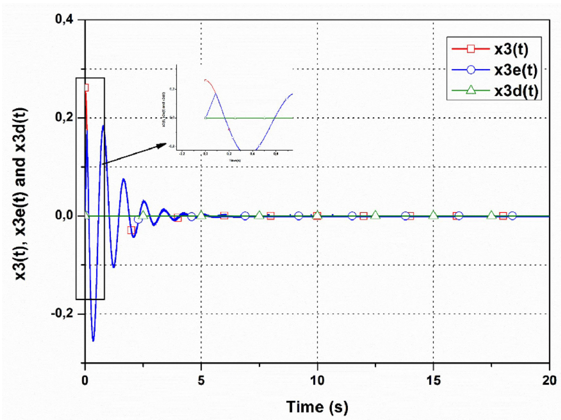


Fig. 4. The angle of the pole, its estimatation and the reference.

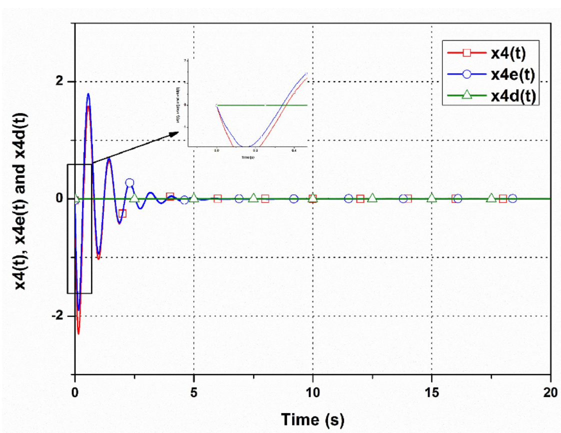


Fig. 5. The velocity of the pole, its estimatation and the reference.

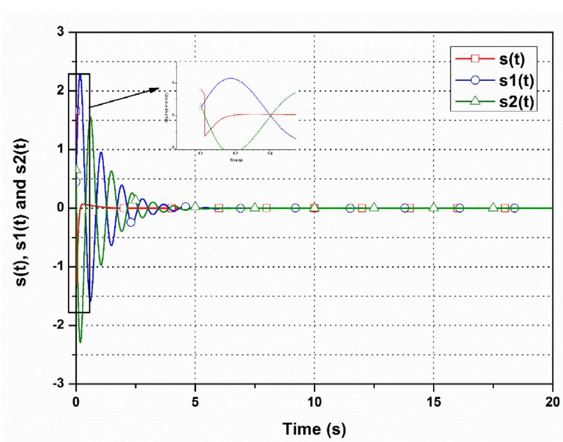


Fig. 6. The sliding surfaces.

sliding surfaces $S(t)$, $S_1(t)$ and $S_2(t)$ and the control signal $U(t)$.

From these simulation results, we can see clearly that the closed loop system with the control and the observer

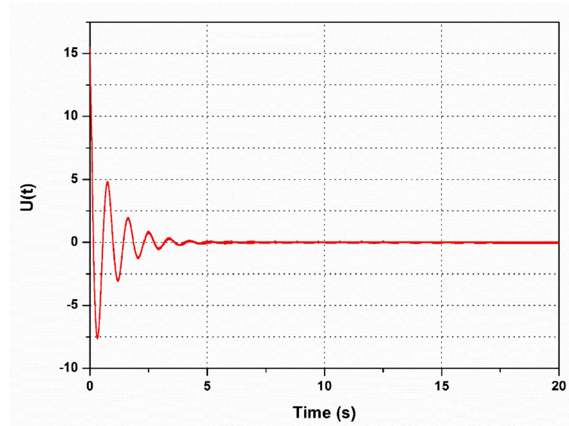


Fig. 7. The control signal.

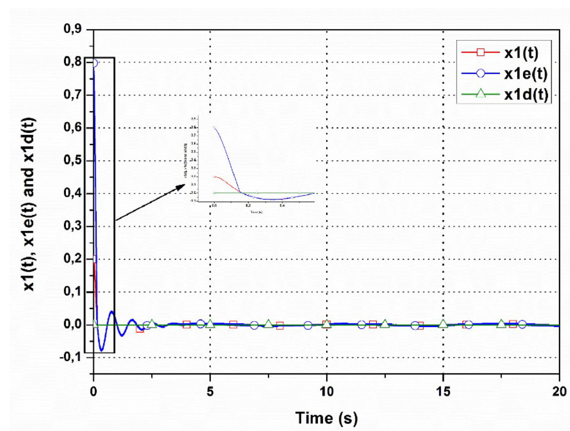


Fig. 8. The position of the cart, its estimatation and the reference.

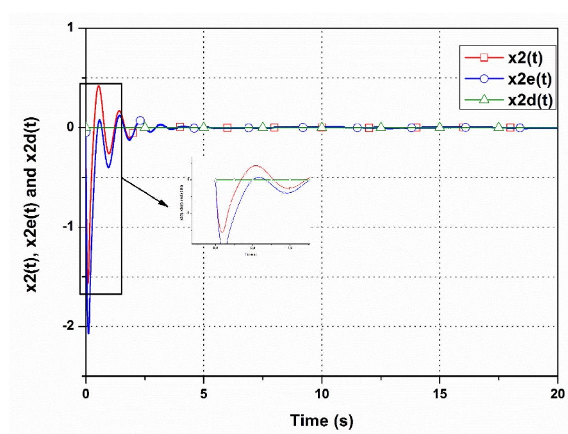


Fig. 9. The velocity of the cart, its estimatation and the reference.

assure a good estimation of the all states, where the output vector can track the desired output vector. A minimization of chattering has been detected in the control signal.

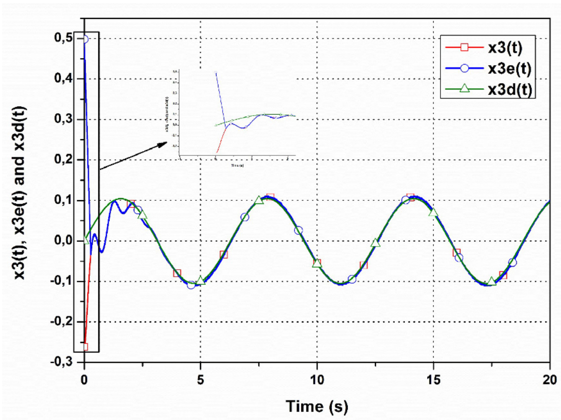


Fig. 10. The angle of the pole, its estimatation and the reference

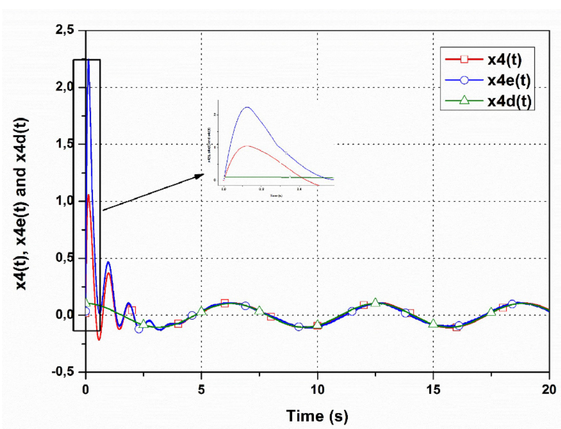


Fig. 11. The velocity of the pole, its estimatation and the reference.

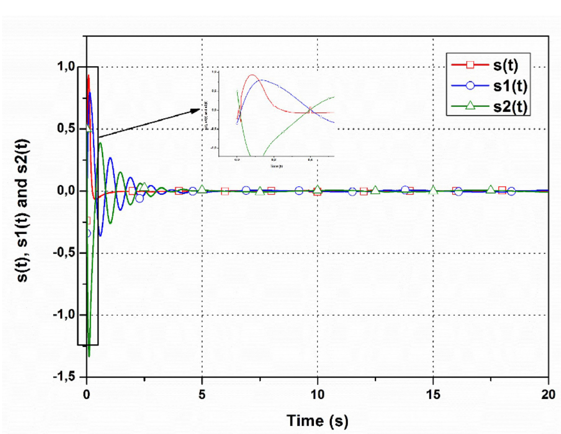


Fig. 12. The sliding surfaces.

## 6. CONCLUSION

In this paper we have presented a nonlinear SMO for a category of second order UMS, the observer is used in order to estimate all the states of the system. This controller gives a good tracking performance, where the errors converge to zero asymptotically, and minimises the chattering phenomenon in the control signal. Simulation results show the effectiveness and the robustness of this controller.

The proposed approach, can be exploited for the control of continuous and discrete time Markovian jump systems with state delays [36–39]. Extending this study to other class and higher order of UMS, deserve further investigations.

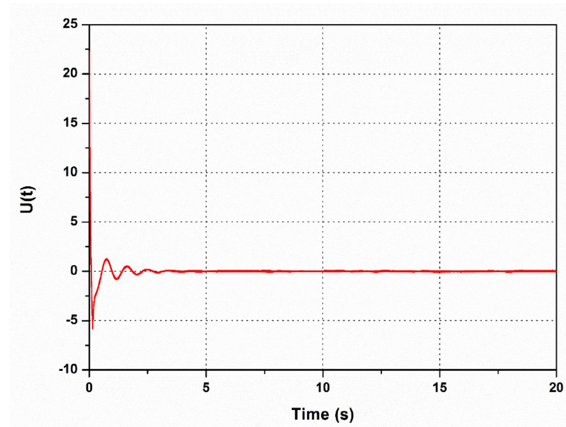


Fig. 13. The control signal.

## APPENDIX A

The simplified model of the cart-pote system is given in this section.

The global mechanical energy is given by

$$L = \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}ml^2\dot{\theta}^2 + ml\cos\theta\dot{\theta}\dot{x} - mgl(\cos\theta - 1). \qquad (A.1)$$

Applying (3) of Euler-lagrange, we get the following dynamic equations

$$(M+m)\ddot{x} - ml\sin\theta\dot{\theta}^2 + ml\cos\theta\ddot{\theta} = U, \qquad (A.2)$$

$$ml^2\ddot{\theta} + ml\cos\theta\ddot{x} - mgl\sin\theta = 0, \qquad (A.3)$$

such that

$$M(q) = \begin{pmatrix} M+m & ml\cos\theta \\ ml\cos\theta & ml^2 \end{pmatrix}, \qquad (A.4)$$

$$H(q) = \begin{pmatrix} -ml\sin\theta\dot{\theta}^2 \\ -mgl\sin\theta \end{pmatrix}. \qquad (A.5)$$

From (66) and (67), we have

$$\ddot{x} = \frac{l}{l(M+m\sin^2\theta)}U - \frac{\sin\theta\left(mgl\cos\theta - ml^2\dot{\theta}^2\right)}{l(M+m\sin^2\theta)}, \qquad (A.6)$$

$$\ddot{\theta} = \frac{-\cos\theta}{l\left(M + m\sin^2\theta\right)}U + \frac{(M+m)\,g - ml\cos\theta\,\dot{\theta}^2}{l\left(M + m\sin^2\theta\right)}, \tag{A.7}$$

$$f_1 = -\frac{\sin\theta\left(mgl\cos\theta - ml^2\dot{\theta}^2\right)}{l\left(M + m\sin^2\theta\right)}, \tag{A.8}$$

$$g_1 = \frac{l}{l\left(M + m\sin^2\theta\right)}, \tag{A.9}$$

$$f_2 = \frac{(M+m)\,g - ml\cos\theta\,\dot{\theta}^2}{l\left(M + m\sin^2\theta\right)}, \tag{A.10}$$

$$g_2 = \frac{-\cos\theta}{l\left(M + m\sin^2\theta\right)}. \tag{A.11}$$

Let $x_1 = x$, $x_2 = \dot{x}$, $x_3 = \theta$, $x_4 = \dot{\theta}$.

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**Djamila Zehar** has received her MSc in electrical engineering, (speciality: Automatic) at the University of Ferhat Abbes Sétif, Algeria, in 2006. She is pursuing her PhD degree in electrical engineering (speciality: Automatic) at the University of Ferhat Abbes Sétif, Algeria, where she is also a member of the intelligent systems laboratory. She is currently an assistant professor at the University of Mohamed Elbachir Elibrahimi Bordj Bou Arreridj, Algeria. Her research interests are in nonlinear complex systems, robust control, intelligent control and optimization.

**Khier Benmahammed** is an IEEE senior member, he has received the M.S. in electrical and computer engineering from the University of Colorado at Boulder, the M.A. in mathematics and the Ph.D. in electrical and computer engineering from the Louisiana State University at Baton Rouge, in 1980, 1984, and 1986, respectively. He has held teaching and research positions at Louisiana State University, Baton Rouge, the University of Minnesota, Duluth, Alfred University, New York, and University of Umm Al Qura, Mekka, Saudi Arabia, and visiting positions at the University of Reims, Champagne-Ardenne, University of Havre, France. He is currently a professor of electrical engineering and the director of the intelligent systems laboratory at the University Ferhat Abbas Setif 1, Algeria. His current research interests are in soft computing, signal/image processing, multidimensional systems, nonlinear systems and robotics.

**Khalissa Behih** has received her MSc in electrical engineering, (speciality: Automatic) at the University of Ferhat Abbes Sétif, Algeria, in 2005. She is pursuing her PhD degree in electrical engineering (speciality: Automatic) at the University of Ferhat Abbes Sétif, Algeria. She has held teaching at the University of Mohamed Elbachir Elibrahimi Bordj bou Arreridj, Algeria, from 2007 to 2016; she is currently an assistant professor and a member of the intelligent systems laboratory at the University of Ferhat Abbes Sétif, Algeria. Her research interests are in nonlinear complex systems, robotics, robust control and intelligent control.