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Reliable Gain Scheduling Output Tracking Control for Spacecraft Rendezvous

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Abstract: This paper has proposed a discrete gain scheduling output tracking control method for the homing phase of the spacecraft rendezvous based on the parametric Lyapunov equation. Considering the actuator saturation, output tracking and the partial loss of thruster effectiveness, we establish a relative dynamic model based on C-W equation and transform the orbital transfer control problem into a stabilization problem. The proposed gain scheduling approach is to improve the state convergence rate by increasing the introduced parameters gradually and remove the affect of the partial loss of thruster effectiveness. To obtain the designed controller, we only need to solve a nonlinear equation. Numerical simulations illustrate the usefulness and effectiveness of the proposed method.

Keywords: Actuator saturation, gain scheduling, output tracking, parametric Lyapunov equation, thruster failure.

1. INTRODUCTION

Spacecraft rendezvous is a useful operation and a prerequisite for many astronautic missions [1]. The relative motion of the two spacecrafts can be described by the autonomous nonlinear differential equations. When the distance of the two spacecrafts is much smaller than the orbit radius, the relative motion can be described by C-W equation [2]. In the past decade, many efforts have been made to solve the control problem of the spacecraft rendezvous and lots of results have been obtained. For example, a parametric Lyapunov differential equation approach to the elliptical rendezvous with constrained control was proposed in [3], a decentralized adaptive control was studied for spacecraft rendezvous in [4], the hybrid multi-objective optimisation for the two docked spacecrafts can be found in [5] and a model predictive control approach was developed for the spacecraft rendezvous in [6].

In recent years, many results have been obtained on designing the optimal terminal rendezvous orbit [7]. Most of the studies divide the terminal phase into the final orbital transfer and the docking process. This approach can simplify the terminal rendezvous orbit design, but the required docking direction is ignored. In order to solve this problem, it is better to further divide the terminal process into the homing phase and the docking phase. In the homing phase, the chaser spacecraft enters into the target orbit and keeps a distance from the target spacecraft. Then, the docking phase starts by considering the spacecraft rendezvous requirements. In this paper, we will propose a new method for the homing phase spacecraft rendezvous by regarding the holding point as a reference output signal and transfer the control problem in this phase to an output tracking control problem.

Saturation nonlinearity is unavoidable in practice and makes the whole system essentially nonlinear [8]. Among all the constraints of the spacecraft orbital control input, the thrust constraint from the physical device is very important. In recent years, there are some results on the spacecraft rendezvous system with limited-thrust. For instance, a gain scheduling method based on parametric Lyapunov equation [9] was proposed in [10] to solve the stabilization problem for spacecraft rendezvous system with actuator saturation and [11] studied the robust output feedback control for a class of spacecraft rendezvous systems with input constraints. In addition to the actuator saturation, the partial loss of thruster effectiveness during the process of the spacecraft rendezvous is also necessary to consider. The existence of the thruster failure will affect the safety and accuracy of the spacecraft rendezvous.

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Therefore, the study on reliable control against the possible thruster failure is important and challengeable [12].

The gain-scheduling approach has been widely applied in many fields, such as, aerospace, process control and so on [13, 14]. Many results have been obtained on this topic [10, 15–19].

In this paper, we designed a reliable discrete static gain scheduling output tracking controller for the homing phase of the spacecraft rendezvous based on the parametric Lyapunov equation and low gain feedback (LGF). Based on the C-W equations, a dynamic model is established by considering the actuator saturation, thruster failure and the requirement of output tracking. With the obtained controller, the homing phase of the rendezvous can be completed. A numerical example illustrates the effectiveness of the proposed method.

Notation: Throughout the paper, the notation used is fairly standard. We use T to denote the transpose and $\mathrm{I}[m,n]$ to denote the integers sets $[m, m+1, ..., n]$. Biag$\{...\}$ refers to a block-diagonal matrix. Let I denote the identity matrix and $\|\cdot\|$ denote the 2− norm. The definition of the function sign is $\mathrm{sign}(z) = 1$ when $z \geq 0$ and $\mathrm{sign}(z) = -1$ when $z < 0$. The function $\mathrm{sat}: \mathrm{R}^m \to \mathrm{R}^m$ is a standard saturation function,

$$\mathrm{sat}(u) = \begin{bmatrix} \mathrm{sat}(u_1) & \mathrm{sat}(u_2) & \cdots & \mathrm{sat}(u_m) \end{bmatrix}^{\mathrm{T}},$$

and $\mathrm{sat}(u_i) = \mathrm{sign}(u_i)\min\{1, |u_i|\}, i = 1, 2, ..., m$.

2. DYNAMIC MODEL AND PROBLEM FORMULATION

2.1. Description of relative motion

The circular orbit coordinate $O-XYZ$ is given in Fig. 1, where X axis shows the circular orbit radial direction, Y axis denotes the target spacecraft flight direction, and Z axis is out of the orbit plane. We assume that the two spacecrafts (the target and chaser) are adjacent,the radius of the target circular orbit is R and the vector r is from the target spacecraft to the chaser spacecraft. The gravitational parameter is denoted as $\mu = GM$ where M denotes the center planet mass and G denotes the gravitational constant. Then the target orbit rate is $\omega = \mu^{1/2}/R^{3/2}$. The homing phase of the spacecraft rendezvous studied in this paper is depicted in Fig. 2.

By considering the thrust constraint and Newton's equations, the rendezvous dynamic model is [20]

$$\begin{cases} \ddot{x} = 2\omega\dot{y} + \omega^2(R+x) - \sigma\mu(R+x) + \mathrm{sat}_{\alpha_X}(a_x), \\ \ddot{y} = -2\omega\dot{x} + \omega^2 y - \sigma\mu y + \mathrm{sat}_{\alpha_Y}(a_y), \\ \ddot{z} = -\sigma\mu z + \mathrm{sat}_{\alpha_Z}(a_z), \end{cases} \tag{1}$$

where $\sigma = ((R+x)^2 + y^2 + z^2)^{-\frac{3}{2}}$ and

$$a = \begin{bmatrix} a_x & a_y & a_z \end{bmatrix}^{\mathrm{T}}, \tag{2}$$

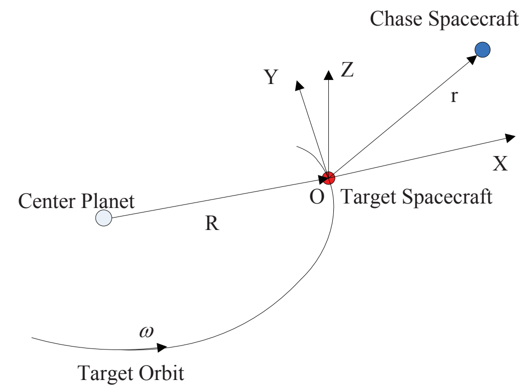

Fig. 1. Circular orbit coordinate system.

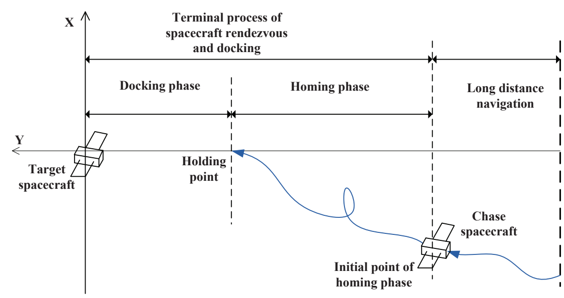

Fig. 2. Spacecraft rendezvous and docking process.

here a is the acceleration vector generated by the chaser's thrust forces, α_X, α_Y and α_Z denote the maximal accelerations that the thruster can generate. The linearized equation of (1) is

$$\begin{cases} \ddot{x} = 2\omega\dot{y} + 3\omega^2 x + \mathrm{sat}_{\alpha_X}(a_x), \\ \ddot{y} = -2\omega\dot{x} + \mathrm{sat}_{\alpha_Y}(a_y), \\ \ddot{z} = -\omega^2 z + \mathrm{sat}_{\alpha_Z}(a_z), \end{cases} \tag{3}$$

which is Clohessy-Wiltshire equation [2]. By denoting $D = \mathrm{Biag}\{\alpha_X, \alpha_Y, \alpha_Z\}$, we have

$$\begin{aligned} u' &\triangleq \begin{bmatrix} \mathrm{sat}_{\alpha_X}(a_x) & \mathrm{sat}_{\alpha_Y}(a_y) & \mathrm{sat}_{\alpha_Z}(a_z) \end{bmatrix}^{\mathrm{T}} \\ &= D \begin{bmatrix} \mathrm{sat}(\frac{a_x}{\alpha_X}) & \mathrm{sat}(\frac{a_y}{\alpha_Y}) & \mathrm{sat}(\frac{a_z}{\alpha_Z}) \end{bmatrix}^{\mathrm{T}} \\ &= D\mathrm{sat}(D^{-1}a). \end{aligned} \tag{4}$$

By the state vector

$$x_p = \begin{bmatrix} x & y & z & \dot{x} & \dot{y} & \dot{z} \end{bmatrix}^{\mathrm{T}}, \tag{5}$$

and output vector

$$y_p = \begin{bmatrix} x & y & z \end{bmatrix}^{\mathrm{T}}, \tag{6}$$

system (3) can be written as

$$\begin{cases} \dot{x}_p = A_p x_p + B_p \mathrm{sat}(u), \\ y_p = C_p x_p, \end{cases} \tag{7}$$

where $u = D^{-1}a$ and

$$A_p = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 3\omega^2 & 0 & 0 & 0 & 2\omega & 0 \\ 0 & 0 & 0 & -2\omega & 0 & 0 \\ 0 & 0 & -\omega^2 & 0 & 0 & 0 \end{bmatrix}, \quad (8)$$

$$B_p = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} D, \; C_p = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}}. \quad (9)$$

In view of the thruster failures, system (7) can be written as

$$\begin{cases} \dot{x}_p = A_p x_p + B_p \mathrm{sat}(\mathscr{U}), \\ y_p \; = C_p x_p, \end{cases} \quad (10)$$

where $\mathscr{U} = Hu$, $H = \mathrm{Biag}\{1-h_X, 1-h_Y, 1-h_Z\}$ with $0 \le h_i \le 0.5, i = X,Y,Z$. We use h_i to denote the possible failure of the actuator along X axis, Y axis and Z axis, respectively. That is, $h_i = 0$ denotes that there is no fault in the $i-$th thruster. Furthermore, $0 < h_i \le 0.5$ denotes that there is a partial failure which is not greater than half invalidation in the corresponding thruster.

In the homing phase, we use a reference output $y_r = [0; y_r; 0]$ to denote the chaser's terminal position. Therefore, we can design an output tracking controller to solve the control problem in homing phase, such that the output y_p of the closed-loop system can track the reference signal y_r, namely,

$$\lim_{t\to\infty} y_p(t) - y_r = 0. \quad (11)$$

For eliminating the tracking error, we carry out an integral operation. Let

$$q(t) = \int_0^t (y_p(\tau) - y_r)\,\mathrm{d}\tau,$$

then

$$\dot{q} = y_p(t) - y_r.$$

Hence, we obtain the augmented system

$$\begin{cases} \dot{x} = Ax + B\mathrm{sat}(\mathscr{U}) + Gy_r, \\ y_p \; = Cx, \end{cases} \quad (12)$$

where

$$A = \begin{bmatrix} A_p & 0 \\ C_p & 0 \end{bmatrix}, B = \begin{bmatrix} B_p \\ 0 \end{bmatrix}, G = \begin{bmatrix} 0 \\ -I \end{bmatrix},$$

$$C = \begin{bmatrix} C_p & 0 \end{bmatrix}, x = \begin{bmatrix} x_p \\ q \end{bmatrix}.$$

Then, we consider the state-feedback control law in the following form

$$u = Kx = K_x x_p + K_q q, \quad (13)$$

where $K = [K_x \; K_q]$. Thus, the augmented closed-loop system can be written as

$$\dot{x} = Ax + B\mathrm{sat}(HKx) + Gy_r. \quad (14)$$

2.2. Problem formulation

According to [21], the output tracking requirement in (11) can be satisfied if the closed-loop system $\dot{x} = Ax + BHKx + Gy_r$ which is (14) removing the saturation function is stable. Thus, if there exists a controller in the form of (13) which can stabilize the following system

$$\begin{cases} \dot{x} = Ax + B\mathscr{U} + Gy_r, \\ y_p \; = Cx, \end{cases}$$

which is system (12) without saturation, then (11) can be obtained, namely, the output y_p in (10) can track the reference signal y_r.

In this paper, we will study the homing phase control problem by considering the following issues:

- The actuator saturation;
- Partial loss of thruster effectiveness;
- Output tracking;
- An estimation of the domain of attraction as large as possible;
- Improving the dynamic performance of the closed-loop system.

According to the dynamic model prescribed previously and control objects given in the above, the control problem to be studied can be expressed as follows.

Consider the rendezvous dynamic model (12), design a discrete gain scheduling output tracking controller u to make the closed-loop system (14) asymptotically stable (meaning that the output tracking requirement in (11) is satisfied). The designed controller can remove the affect of the partial loss of thruster effectiveness, maximize the size of attraction region, improve the dynamic performance, meanwhile, ensure that the control input will not saturate.

3. MAIN RESULTS

In this section we design a discrete static gain scheduling controller to solve the output tracking control problem for homing phase of the spacecraft rendezvous based on the parametric Lyapunov equation method which is one of the methods to construct the LGF [22]. In view of the parametric Lyapunov equation method, the control law for system (14) is

$$K = -B^{\mathrm{T}} P(\varsigma), \quad (15)$$

where $P(\varsigma)$ is the unique positive definite solution to the following parametric ARE

$$PA + A^{\mathrm{T}}P - PBB^{\mathrm{T}}P = -\varsigma P \tag{16}$$

and $\varsigma > 0$ is the low gain parameter.

Lemma 1 gives the properties of the ARE (16).

Lemma 1 [22]**:** Noticing that (A,B) is controllable and A has all its eigenvalues on the imaginary axis. Then, for the low gain parameter $\varsigma > 0$

1) there exists a unique matrix $P(\varsigma) > 0$ which solves the ARE (16), $P(\varsigma) = W^{-1}(\varsigma)$, where $W(\varsigma)$ is the unique positive-definite solution to the following Lyapunov matrix equation

$$W(A + \frac{\varsigma}{2}I)^{\mathrm{T}} + (A + \frac{\varsigma}{2}I)W = BB^{\mathrm{T}}; \tag{17}$$

2) the matrix $A - BB^{\mathrm{T}}P(\varsigma)$ is a Hurwitz matrix, $\mathrm{Re}\left\{\lambda_i\left(A - BB^{\mathrm{T}}P(\varsigma)\right)\right\} = -\varsigma,\ i \in [1,9]$;

3) $\lim\limits_{\varsigma \to 0} P(\varsigma) = 0$;

4) $P(\varsigma)$ is continuously differentiable and strictly increasing with respect to ς, i.e. $\mathrm{d}P(\varsigma)/\mathrm{d}\varsigma > 0$;

5) $\mathrm{tr}\left(B^{\mathrm{T}}P(\varsigma)B\right) = 9\varsigma$.

Assume that the initial state of system (14) is from $\Omega \in \mathbf{R}^9$ which is bounded. Define ς_0 as

$$\varsigma_0 = \varsigma_0(\Omega) = \min_{x \in \Omega}\left\{\varsigma : 9\varsigma x^{\mathrm{T}}P(\varsigma)x = 1\right\}. \tag{18}$$

The existence of $\varsigma(0)$ can be guaranteed by $\lim\limits_{\varsigma \to 0^+} P(\varsigma) = 0$ and $\frac{\mathrm{d}P}{\mathrm{d}\varsigma} > 0$.

Consider a real number set

$$\Lambda_N = \{\varsigma_0, \varsigma_1, \cdots, \varsigma_N\},\ \varsigma_{i-1} < \varsigma_i,\ i \in \mathrm{I}[1,N], \tag{19}$$

where N is any given positive integer. For any $\varsigma_j \in \Lambda_N$, define the following ellipsoids

$$\mathscr{E}(P_{\varsigma_j}) = \left\{x \in \mathbf{R}^9 : 9\varsigma_j x^{\mathrm{T}}P(\varsigma_j)x \le 1\right\}, j \in \mathrm{I}[0,N], \tag{20}$$

where $P_{\varsigma_j} = 9\varsigma_j P(\varsigma_j)$.

Proposition 1: The ellipsoids $\mathscr{E}(P_{\varsigma_i})$ in (20) are nested, that is, $\mathscr{E}(P_{\varsigma_2}) \subset \mathscr{E}(P_{\varsigma_1})$, if $\varsigma_1 < \varsigma_2$.

Proof: According to property 4 of Lemma 1, we know that $\mathrm{d}P(\varsigma)/\mathrm{d}\varsigma > 0$. Hence

$$\frac{\mathrm{d}}{\mathrm{d}\varsigma}P_\varsigma = 9P(\varsigma) + 9\varsigma \mathrm{d}P(\varsigma)/\mathrm{d}\varsigma > 0. \tag{21}$$

So the ellipsoids $\mathscr{E}(P_\varsigma)$ are nested. □

Then the main result of this paper can be stated as follows.

Theorem 1: Let $P(\varsigma)$ be the unique positive definite solution to the ARE in (16). Then the designed discrete gain scheduling controller in the following

$$u = \begin{cases} u_{i-1} = -K_{i-1}x,\ x \in \mathscr{S}_{i-1}, \\ u_N = -K_N x,\ x \in \mathscr{E}\left(P_{\varsigma_N}\right), \end{cases} \tag{22}$$

where $K_{i-1} = B^{\mathrm{T}}P(\varsigma_{i-1}), i \in \mathrm{I}[1,N]$, $K_N = B^{\mathrm{T}}P(\varsigma_N)$ and

$$\mathscr{S}_{i-1} = \mathscr{E}\left(P_{\varsigma_{i-1}}\right) \backslash \mathscr{E}\left(P_{\varsigma_i}\right),\ i \in \mathrm{I}[1,N],$$

solves the reliable output tracking control problem for the homing phase of the spacecraft rendezvous. The ellipsoid set $\mathscr{E}(P_{\varsigma_0}) = \{x : 9\varsigma_0 x^{\mathrm{T}}P(\varsigma_0)x \le 1\}$ is the maximal domain of attraction of the closed-loop system with controller (22). Moreover, T_{i-1} denotes the working time of the controller $u = u_{i-1},\ i \in \mathrm{I}[1,N]$ which satisfy the following relation

$$T_{i-1} \le \frac{1}{\varsigma_{i-1}} \ln\left(\frac{\varsigma_i}{\varsigma_{i-1}} \lambda_{\max}\left\{P(\varsigma_i)P^{-1}(\varsigma_{i-1})\right\}\right), \tag{23}$$

and after

$$T(N) = \sum_{i=1}^{N} T_{i-1}, \tag{24}$$

the closed-loop system becomes a linear one under the controller $u = u_N$.

Proof: The proof is divided into three steps.

Step 1: The proof of the stability

Consider the set

$$\mathscr{L}_j \triangleq \left\{x : \left\|HB^{\mathrm{T}}P(\varsigma_j)x\right\| \le 1\right\}, j \in \mathrm{I}[0,N] \tag{25}$$

which is the area in the state space where the actuators with the control $u = -B^{\mathrm{T}}P(\varsigma_j)x$ will not saturate.

According to Lemma 1 and the fact that $\left\|H^{\mathrm{T}}H\right\| \le 1$, for any $x \in \mathscr{E}(P_{\varsigma_j})$, the following inequality can be obtained

$$\begin{aligned} \|Hu\|^2 &= \left\|HB^{\mathrm{T}}P(\varsigma_j)x\right\|^2 \\ &\le \|H\|^2 \left\|B^{\mathrm{T}}P(\varsigma_j)x\right\|^2 \\ &= \left\|H^{\mathrm{T}}H\right\| x^{\mathrm{T}}P(\varsigma_j)BB^{\mathrm{T}}P(\varsigma_j)x \\ &\le x^{\mathrm{T}}P^{\frac{1}{2}}(\varsigma_j)\mathrm{tr}\left(P^{\frac{1}{2}}(\varsigma_j)BB^{\mathrm{T}}P^{\frac{1}{2}}(\varsigma_j)\right)P^{\frac{1}{2}}(\varsigma_j)x \\ &= \mathrm{tr}\left(B^{\mathrm{T}}P(\varsigma_j)B\right)x^{\mathrm{T}}P(\varsigma_j)x \\ &= 9\varsigma_j x^{\mathrm{T}}P(\varsigma_j)x \le 1. \end{aligned} \tag{26}$$

Then it follows from (20), (25) and (26) that

$$\mathscr{E}(P_{\varsigma_j}) \subseteq \mathscr{L}_j. \tag{27}$$

Thus the actuators will not saturate for any $x \in \mathscr{E}(P_{\varsigma_j})$, that is

$$x \in \mathscr{E}(P_{\varsigma_j}) \Longrightarrow \mathrm{sat}(Hu) = Hu. \tag{28}$$

Therefore, for any $x \in \mathscr{E}(P_{\varsigma_j})$, the system (14) is shown as

$$\dot{x} = (A + BHK)x + Gy_r. \tag{29}$$

By using the designed controller (22), we have the closed-loop system

$$\dot{x} = Ax + B\text{sat}(Hu) + Gy_r. \tag{30}$$

In view of (28) and (29), the closed-loop system (30) can be written as

$$\dot{x} = \left(A - BHB^{\mathrm{T}}P(\varsigma_{i-1})\right)x + Gy_r, \forall x \in \mathscr{S}_{i-1}. \tag{31}$$

According to [21], we can prove the stability of

$$\dot{x} = (A - BHB^{\mathrm{T}}P(\varsigma_{i-1}))x \tag{32}$$

instead of the system (31).

Assume that at $t = t_{i-1}, i \in \mathrm{I}[1,N]$, the state $x(t_{i-1})$ is on the ellipsoid boundary $\partial\mathscr{E}(P_{\varsigma_{i-1}})$. Obviously, we have

$$\begin{aligned} V_{i-1}(x(t_{i-1})) &= x^{\mathrm{T}}(t_{i-1})P(\varsigma_{i-1})x(t_{i-1}) \\ &= \frac{1}{9\varsigma_{i-1}}, \ i \in \mathrm{I}[1,N]. \end{aligned} \tag{33}$$

Now, the following Lyapunov function is selected

$$V_{i-1}(x) = x^{\mathrm{T}}P(\varsigma_{i-1})x. \tag{34}$$

Then, for any $x \in \mathscr{S}_{i-1}$, in view of Lemma 1 and the structure of the matrix H, the time-derivative of $V(x)$ is

$$\begin{aligned} \dot{V}_{i-1}(x) =& 2\dot{x}^{\mathrm{T}}P(\varsigma_{i-1})x \\ =& 2\left(Ax - BHB^{\mathrm{T}}P(\varsigma_{i-1})x\right)^{\mathrm{T}}P(\varsigma_{i-1})x \\ =& x^{\mathrm{T}}\left(P(\varsigma_{i-1})A + A^{\mathrm{T}}P(\varsigma_{i-1})\right)x \\ & - 2x^{\mathrm{T}}P(\varsigma_{i-1})BH^{\mathrm{T}}B^{\mathrm{T}}P(\varsigma_{i-1})x \\ =& x^{\mathrm{T}}\left(P(\varsigma_{i-1})BB^{\mathrm{T}}P(\varsigma_{i-1}) - \varsigma_{i-1}P(\varsigma_{i-1})\right)x \\ & - 2x^{\mathrm{T}}P(\varsigma_{i-1})BH^{\mathrm{T}}B^{\mathrm{T}}P(\varsigma_{i-1})x \\ =& -\varsigma_{i-1}x^{\mathrm{T}}P(\varsigma_{i-1})x + x^{\mathrm{T}}P(\varsigma_{i-1}) \\ & \times B\left(I - 2H^{\mathrm{T}}\right)B^{\mathrm{T}}P(\varsigma_{i-1})x \end{aligned} \tag{35}$$

In view of $0 \leq h_i \leq 0.5, i = X, Y, Z$, we have

$$I - 2H^{\mathrm{T}} \leq 0.$$

Then, (35) can be continued as

$$\begin{aligned} \dot{V}_{i-1}(x) \leq & -\varsigma_{i-1}x^{\mathrm{T}}P(\varsigma_{i-1})x \\ =& -\varsigma_{i-1}V_{i-1}(x) \\ <& 0, \ \forall x \in \mathscr{S}_{i-1}\backslash\{0\}. \end{aligned} \tag{36}$$

Hence, the state will convergent to the ellipsoid $\mathscr{E}\left(P_{\varsigma_i}\right)$ at limited time and finally move to the ellipsoid $\mathscr{E}(P_{\varsigma_N})$ and hold in the ellipsoid $\mathscr{E}(P_{\varsigma_N})$ thereafter.

When $x \in \mathscr{E}(P_{\varsigma_N})$, we have $u = -B^{\mathrm{T}}P(\varsigma_N)x$ and the closed-loop system becomes

$$\dot{x} = (A - BHB^{\mathrm{T}}P(\varsigma_N))x. \tag{37}$$

Similarly to (35) and (36), the time-derivative of the Lyapunov function $V_N(x) = x^{\mathrm{T}}P(\varsigma_N)x$ is

$$\dot{V}_N(x) < 0, \ \forall x \in \mathscr{E}(P_{\varsigma_N})\backslash\{0\}, \tag{38}$$

which indicates that the state x will converge to zero as $t \to \infty$, i.e., the closed-loop system is asymptotically stable.

When $t_i = t_{i-1} + T_{i-1}$, the states from the ellipsoid boundary $\partial\mathscr{E}(P_{\varsigma_{i-1}})$ move to the ellipsoid boundary $\partial\mathscr{E}(P_{\varsigma_i})$. So

$$V_i(x(t_{i-1} + T_{i-1})) = \frac{1}{9\varsigma_i}. \tag{39}$$

Let

$$\begin{aligned} \varpi_{i-1} &\triangleq \frac{1}{\varsigma_{i-1}}\lambda_{\max}\left\{P^{-\frac{1}{2}}(\varsigma_{i-1})P(\varsigma_i)P^{-\frac{1}{2}}(\varsigma_{i-1})\right\} \\ &= \frac{1}{\varsigma_{i-1}}\lambda_{\max}\left\{P(\varsigma_i)P^{-1}(\varsigma_{i-1})\right\}, \forall i \in \mathbf{I}[1,N]. \end{aligned} \tag{40}$$

According to $P^{-\frac{1}{2}}(\varsigma_{i-1})P(\varsigma_i)P^{-\frac{1}{2}}(\varsigma_{i-1}) > 0$, we have

$$P(\varsigma_{i-1}) \geq \frac{1}{\varpi_{i-1}\varsigma_{i-1}}P(\varsigma_i). \tag{41}$$

It is easy to see from (36) that, for any $t \in [t_{i-1}, t_i)$, the following inequality holds

$$V_{i-1}(x(t)) \leq V_{i-1}(x(t_{i-1}))\mathrm{e}^{-\varsigma_{i-1}(t-t_{i-1})}. \tag{42}$$

Then according to (33), (39) and (42), we obtain

$$\begin{aligned} \mathrm{e}^{-\varsigma_{i-1}(t_i-t_{i-1})} &= 9\varsigma_{i-1}\mathrm{e}^{-\varsigma_{i-1}(t_i-t_{i-1})}V_{i-1}(x(t_{i-1})) \\ &\geq 9\varsigma_{i-1}V_{i-1}(x(t_{i-1}+T_{i-1})) \\ &= 9\varsigma_{i-1}x^{\mathrm{T}}(t_{i-1}+T_{i-1})P(\varsigma_{i-1})x \\ &\quad \times (t_{i-1}+T_{i-1}) \\ &\geq \frac{9}{\varpi_{i-1}}x^{\mathrm{T}}(t_{i-1}+T_{i-1})P(\varsigma_i)x \\ &\quad \times (t_{i-1}+T_{i-1}) \\ &= \frac{9}{\varpi_{i-1}}V_i(X(t_{i-1}+T_{i-1})) \\ &= \frac{1}{\varpi_{i-1}\varsigma_i}. \end{aligned} \tag{43}$$

Then, we can get

$$\begin{aligned} T_{i-1} &\leq \frac{1}{\varsigma_{i-1}}\ln(\varpi_{i-1}\varsigma_i) \\ &= \frac{1}{\varsigma_{i-1}}\ln\left(\frac{\varsigma_i}{\varsigma_{i-1}}\lambda_{\max}\left\{P(\varsigma_i)P^{-1}(\varsigma_{i-1})\right\}\right). \end{aligned} \tag{44}$$

Step 2: The proof of $\lim_{t\to\infty} y_p(t) - y_r = 0$

Take derivative on both sides of system (37) with respect to t, we can acquire

$$\ddot{x} = (A - BHB^{\mathrm{T}}P(\varsigma_N))\dot{x} = A_c\dot{x}, \forall x \in \mathscr{E}\left(P_{\varsigma_N}\right)\setminus\{0\}. \quad (45)$$

It follows from the above proof that A_c is stable. Obviously, we can get

$$\dot{x} = \begin{bmatrix} \dot{x}_p \\ \dot{q} \end{bmatrix} \to 0, t \to \infty.$$

Hence, we have

$$\lim_{t\to\infty} y_p(t) - y_r = 0.$$

Step 3: An estimation of the maximal domain of attraction

When $\varsigma \to 0$, the ARE (16) can be written as

$$P(0)A + A^{\mathrm{T}}P(0) - P(0)BB^{\mathrm{T}}P(0) = 0, \quad (46)$$

which has an unique positive definite solution $P(0)$ since (A,B) is controllable [23]. Therefore, the maximal domain of attraction for system (12) is the ellipsoid $\mathscr{E}(P(0)) = 9\varsigma_0 x^{\mathrm{T}}P(0)x$ based on the controller (22). □

Remark 1: From Theorem 1, the controller switching order is $u_0 \to u_1 \to \cdots \to u_{N-1} \to u_N$. The introducing parameter ς indicates the state convergence speed. Therefore, the control capability of the controller will become stronger and stronger with the increase of the parameter ς. Thus, the dynamic performance can be improved.

Remark 2: In this paper, we only consider the case that the thruster invalidation is not greater than 50% ($h_i \in (0, 0.5], i = X, Y, Z$). Actually, the case that the thruster invalidation is greater than half, namely, $h_i \in (0.5, 1], i = X, Y, Z$ is very important and deserves further study in the future.

Remark 3: (The implementation steps)

- **Step 1:** computing ς_0 by solving the following nonlinear equation

$$9\varsigma_0 x_0^{\mathrm{T}}P(\varsigma_0)x_0 = 1. \quad (47)$$

- **Step 2:** setting Λ_N in (19) by the exponential growth method

$$\varsigma_i = \varsigma_0 \Delta\varsigma^i,\ i \in \mathrm{I}[1,N], \quad (48)$$

where $\Delta\varsigma$ ($\Delta\varsigma > 1$) is a given constant. Actually, the other methods can be used to design Γ_N in (19), such as, a linear growth method:

$$\varsigma_i = \varsigma_0 + \frac{i}{N}(\varsigma_N - \varsigma_0), i \in \mathrm{I}[1,N] \quad (49)$$

where, ς_N is a known constant.

- **Step 3:** computing $P(\varsigma_i)$ by solving the Riccati equation (16), continuously, computing u in (22).
- **Step 4:** setting the initial value of the current variable i as $i = 0$ and the controller is $u = u_0$. If $i \le N-1$, for each $x(t)$, calculate

$$\Psi(x) = 1 - 9\varsigma_i x^{\mathrm{T}}P(\varsigma_i)x. \quad (50)$$

If $\Psi(x) \ge 0$, let $u = u_{i+1}$ and $i = i+1$; otherwise setting $u = u_i$.

4. NUMERICAL SIMULATIONS

In this section, an example is used to verify the effectiveness of the designed controller. Through considering a pair of adjacent spacecrafts, and the chaser transfers towards the target along the homing phase orbit. It is assumed that the target spacecraft is on a geosynchronous orbit whose radius is $R = 42241$km and orbital period is 24 hours. Then the computed orbit rate is $\omega = 7.2722 \times 10^{-5}$rad/s. Assume that the accelerations in the three directions satisfy, respectively, $|\alpha_X| \le 0.5$, $|\alpha_Y| \le 0.5$, and $|\alpha_Z| \le 0.5$. Suppose that the initial state is

$$x_p(0) = \begin{bmatrix} 10{,}000 & 10{,}000 & 8{,}000 & 8 & 6 & -5 \end{bmatrix}^{\mathrm{T}},$$

and the holding point is $y_{r=}\begin{bmatrix} 0 & -100 & 0 \end{bmatrix}^{\mathrm{T}}$.

Considering the given initial state, we get $\varsigma_0 = 0.002$. The exponential growth method is adopted to design Λ_N, where $\Delta\varsigma = 1.01$ and $N = 100$. And assume $h_i = 0.5, i = X, Y, Z$. With these parameters, the unique positive definite solution to the parametric ARE (16) can be computed. For the comparison, the closed-loop system will also be simulated for the proposed method with $N = 50$ and the LGF with the corresponding gain $K = -B^{\mathrm{T}}P(\varsigma_0)$ [22].

From Fig. 3 and Fig. 4, we can see that the closed-loop system is stable and the output tracking requirement is satisfied by using the proposed gain scheduling controller (22). Moreover, the convergent time of the homing phase mission of spacecraft rendezvous with $N = 100$ and $h_i = 0.5, i = X, Y, Z$ is shorter than the case of $N = 50$ and LGF. This result illustrates that the system dynamic performance becomes better and better as the increase of the switching number N in a certain range of switching number and shows the effectiveness of the proposed method. The control accelerations for the closed-loop system are recorded in Fig. 5 which shows that the control capability of the designed controller will become stronger and stronger with the increase of the parameter ς and the actuator does not saturate. The curve of the control gains can be found in Fig. 6.

Now we consider the different thruster invalidation cases by choosing $h_i = 0.5$, $h_i = 0.2$ and $h_i = 0$, respectively, where $i = X, Y, Z$. From Fig. 7 to Fig. 9, we can see that the homing phase mission of the spacecraft rendezvous can be finished in the three different cases. The

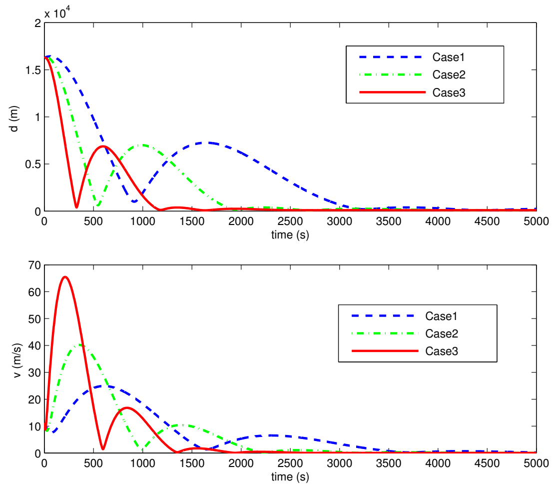

Fig. 3. Relative distances and velocities, where $d = \sqrt{x^2+y^2+z^2}$ and $v = \sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}$, Case 1 denotes LGF, Case 2 denotes the proposed method with $N = 50$, Case 3 denotes the proposed method with $N = 100$.

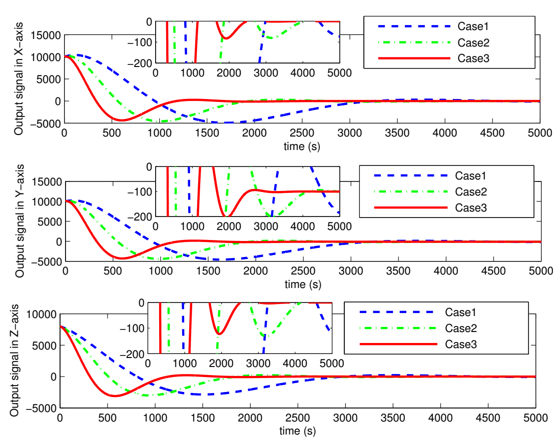

Fig. 4. The output signal $y_p(t)$, where Case 1 denotes LGF, Case 2 denotes the proposed method with $N = 50$, Case 3 denotes the proposed method with $N = 100$.

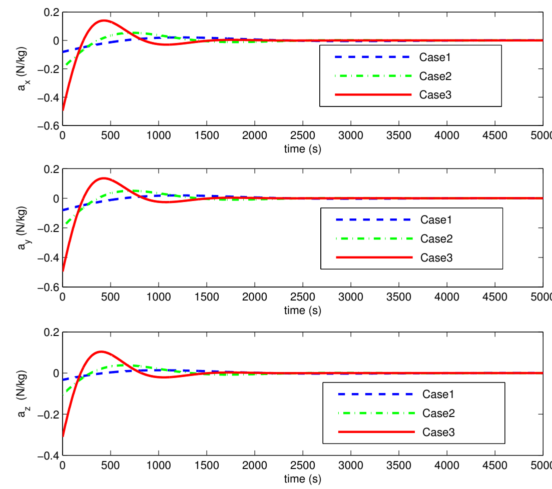

Fig. 5. Control accelerations in the X-axis, Y-axis and Z-axis, where Case 1 denotes LGF, Case 2 denotes the proposed method with $N = 50$, Case 3 denotes the proposed method with $N = 100$.

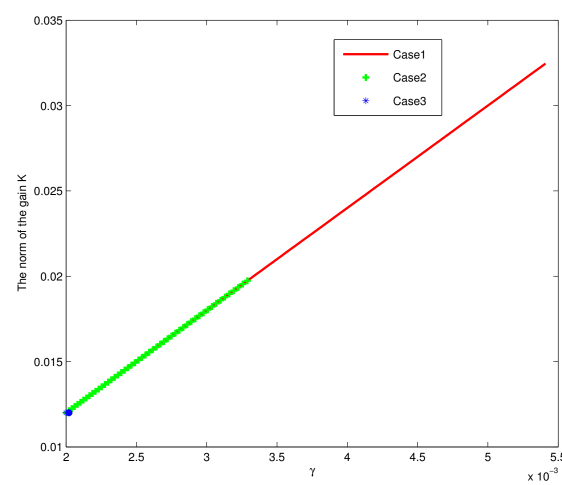

Fig. 6. The norm of the gain K, Case 1 denotes the proposed method with $N = 100$, Case 2 denotes the proposed method with $N = 50$, Case 3 denotes LGF.

state convergent time and the transfer orbit of the chaser would be much longer under the condition of the serious thruster invalidation.We compare the exponential growth method for Λ_i with the linear one with the same parameters $N = 100$, $\varsigma_0 = 0.002$, $\Delta\varsigma = 1.01$, $\varsigma_N = \varsigma_0 \Delta\varsigma^{100}$. Fig. 10 shows that the two different design methods for Λ_i do not influence the dynamic performance of the closed-loop system.

5. CONCLUSION

This paper has designed a discrete static gain scheduling output tracking controller for the homing phase of spacecraft rendezvous system with actuator saturation and thruster invalidation. The contributions of this paper mainly reflected in two aspects, that is, increasing the state convergence speed and removing the thruster invalidation. Simulation results have shown that the homing phase mission is finished successfully by using the proposed method.

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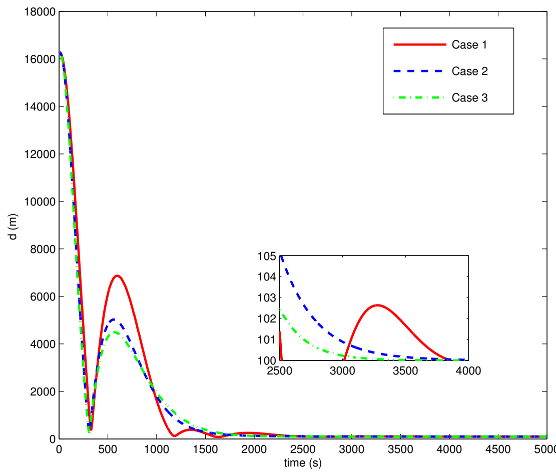

Fig. 7. Relative distances with $N = 100$, where $d = \sqrt{x^2+y^2+z^2}$, Case 1 denotes $h_i = 0.5$, Case 2 denotes $h_i = 0.2$ and Case 3 denotes $h_i = 0$, $i = X, Y, Z$

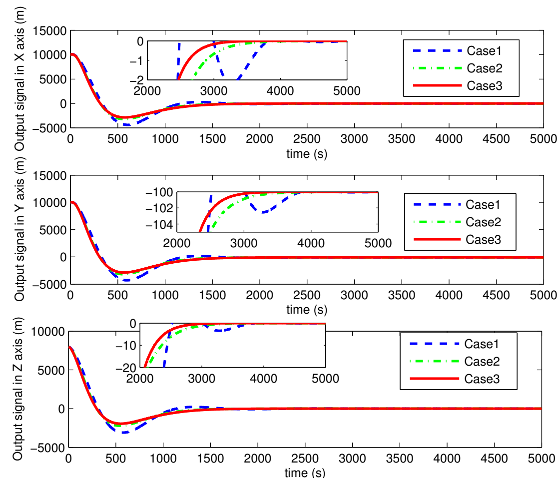

Fig. 9. The output signal $y_p(t)$ with $N = 100$, where Case 1 denotes $h_i = 0.5$, Case 2 denotes $h_i = 0.2$ and Case 3 denotes $h_i = 0$, $i = X, Y, Z$.

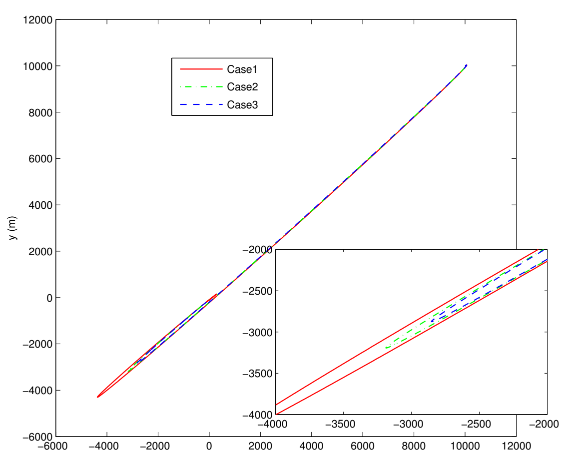

Fig. 8. The state trajectories with $N = 100$, where Case 1 denotes $h_i = 0.5$, Case 2 denotes $h_i = 0.2$ and Case 3 denotes $h_i = 0$, $i = X, Y$.

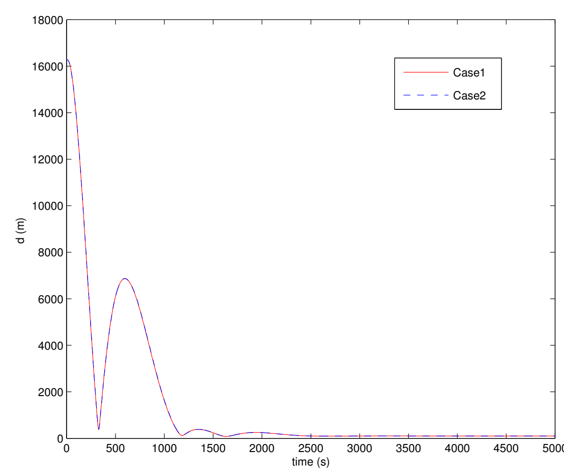

Fig. 10. Relative distance, where $d = \sqrt{x^2+y^2+z^2}$, Case 1 denotes the exponential growth method, Case 2 denotes a linear growth method.

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