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Stabilizing Periodic Orbits of a Class of Mechanical Systems with Impulse Effects: A Lyapunov Constraint Approach

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Abstract: This paper study the stabilization of mechanical system with impulse effects around a hybrid limit cycle, the proposed control approach is based on LaSalle's invariance principle for hybrid systems and Layounov constraint based method. Theorem 2 shows necessary and sufficient condition of the existence and the uniqueness of the developed controller which leads to a system of partial differential equations (PDE) whose solutions are the kinetic and potential energy of smooth Lyapunov function, furthermore Theorem 3 gave an alternative existence condition which states that the largest positively invariant set should be nowhere dense and closed and it is none other than the hybrid limit cycle itself.

Keywords: Compass-gait biped, control force, hybrid limit cycle, LaSalle's invariance principle, Lyapunov constraints, mechanical systems with impulse effects, nowhere dense set, PDE.

1. INTRODUCTION

Mechanical system with impulse effects is a class of hybrid dynamical systems governed by continuous and discrete dynamics, a broad range of its applications arise in passive dynamic walking [1, 2]. stability and stabilization of periodic orbits of such systems is of a big complexity because of their dynamic nature which is a combination of underactuation and impulse effect. Moreover Brockett's necessary condition [3] falls to provide any sign of smooth feedback existence, this render it struggling against standard control techniques [4].To overcome these problems researchers from control community have proposed a versatile set of new control strategies including hybrid zero dynamics [5], continuous time controllers [6], orbital stabilization via virtual constraints [7, 8], basically developed for bipedal robots. The most useful of them are those based on energetic approaches namely energy shaping methods including potential energy shaping [9–11], passivity-based control [12, 13] that have been considered as a prominent control design technique among other approaches, the way how it works is to transform the energy of an unstable mechanical system via feedback to a new one whose minimum coincides with the equilibium point, the main advantage is that it preserves the physical structure of the system and provides a large region of stability [14–16]. Mainly two fundamental approaches have dominated the area, the method of controlled Lagrangian [14, 16] and the method of Interconnection and Damping Assignment Passivity-Based Control IDA-PBC [17]. Both methods are equivalent [18] and lead to a system of nonlinear partial differential equations PDE whose solutions determine the energy shaping feedbacks, an approach to possibly solving these equations is the lambda-method [19], Gharsiferd et al [20] has studied systematically their solutions in the context of geometric theory of PDE.

Based on using Lyapunov conditions of asymptotic stability as an affine kinematic constraint, Grillo et al [21] have developed a new method for global asymptotic stabilization for underactuated mechanical systems, where it is proved that it is a generalization of the energy shaping techniques, its main feature is that it reduces the solvability complexity for the system of PDE whose solutions is the Lyapunov function. The idea of stabilizing mechanical systems by constraints back to Marle [22] and Shiriaev et al [8], the system in question is forced to achieve stability region by applying a set of kinematic constraints on its positions and velocities, and the corresponding control force takes the form of constraint force, an extended version of this approach for underactated systems with arbitrary number of actuators was developed in [23], furthermore a version for mechanical systems with impulse effect was given in [24]. The contribution of this paper is to design a controller which lead the trajectories of the system to converge asymptotically toward the hybrid limit cycle in the sense of orbital stability [25], in contrast to the Lyapunov constraints we assume that our system is al-

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ready controlled, more precisely by using LaSalle's invariance principle and in order to find such control force one has to solve a system of two PDE for kinetic and potential energy resulted from conditions of existence and uniqueness. Moreover we show an equivalent condition of existence which states that the control force exists if and only if the largest positively invariant set is nowhere dense and is none other than the set where the time derivatives of Lyapunov function vanishes, in our case this set is the hybrid limit cycle itself.

The organization of this paper is structured as follows, in Section 2 and 3 we develop the dynamic models of the system in Lagrangian framework as in [24] for more details about geometric mechanics and geometric control see [26, 27]. Section 4 presents the main contribution of the paper, it provides a theoretical formulation of controller design, followed by an illustrative example in section 5. Finally conclusion and perspectives about future works are given.

2. MATHEMATICAL PRELIMINARIES

Let Q be the configuration manifold of mechanical system with coordinates $q=(q_1,q_2,...,q_n)$, its tangent bundle is denoted by TQ, a second order tangent bundle is $T(TQ)$. For every $(q,\dot{q})\in TQ$ the Riemannian metric $\mathfrak{g}$ defines the kinetic energy of the system by $K:TQ\rightarrow\mathbb{R}, K=\frac{1}{2}\mathfrak{g}_{ij}\dot{q}_i\dot{q}_j$ for the potential energy $\mathbb{V}(q)$, the Lagrangian function takes the form $L=K-\mathbb{V}(q)$. A set of vector fields on Q and TQ are $\Gamma(Q)$ and $\Gamma(TQ)$ respectively, a vector field $X\in\Gamma(TQ)$ is vertical if $d\pi(X)=0$ and is special if $d\pi(X)=\dot{q}$, where π is a canonical projection $(\pi:TQ\rightarrow Q, \pi(q,\dot{q})=q)$ and $d\pi$ its tangent map$(d\pi:T(TQ)\rightarrow TQ)$. The space of vertical and special vector fields are noted by $\mathbf{V}_{(q,\dot{q})}$ and $\mathbf{S}_{(q,\dot{q})}$.

Two important geometrical objects are associated to the tangent bundle TQ [27], the Liouville vector fields $\Delta: T_qQ\rightarrow T_{(q,\dot{q})}TQ$ that generating dilations along the fibres and the vertical endomorphism $\Lambda: T_{(q,\dot{q})}TQ\rightarrow T_{(q,\dot{q})}TQ$ that is a $(1,1)$ tensor field which annihilates vertical vector fields. In terms of local coordinates $(q_1,,q_n,\dot{q}_1,,\dot{q}_n)$, Δ and Λ have the form $\Delta=\dot{q}_i\frac{\partial}{\partial\dot{q}_i}$, $\Lambda=\frac{\partial}{\partial\dot{q}_i}\bigotimes dq_i$. Assume that the hessian of the matrix $(\frac{\partial^2L}{\partial\dot{q}_i\dot{q}_j})$ is non-singular, one can construct a 1-form $\theta_L=\frac{\partial}{\partial\dot{q}_i}dq_i$, a Lagrangian 2-form

$$\omega_L=\frac{\partial^2L}{\partial q_j\dot{q}_i}dq_i\wedge dq_j+\frac{\partial^2L}{\partial\dot{q}_i\dot{q}_j}dq_i\wedge d\dot{q}_j,$$

and an energy function $E_L=\dot{q}_i\frac{\partial}{\partial\dot{q}_i}-L$.

Let F be a covector of non-conservative forces acting on the system, it includes control forces.Then dynamic equation of the system is written by the flow of the special vector field $X\in\mathbf{S}_{(q,\dot{q})}$, it is determined from the following equation:

$$\omega_L(X,Y)=dE_L(Y)+F,\forall Y\in T_{(q,\dot{q})}TQ, \tag{1}$$

the vector field X is considered also as a second order differential equation (SODE), and is the solution of (1)

$$X=X_L+X_F. \tag{2}$$

X_F is a vertical vector field corresponding to forces acting on the systems in our case is the control force imposed by the actuators of the system, and X_L a vector field corresponding to $dE_L(Y)$.

The system is underactuated iff $X_F\in\mathcal{W}$, where $\mathcal{W}$ is a subset of $\mathbf{V}$. $\mathcal{W}$ is a vertical distribution spanned by $\{\frac{\partial}{\partial\dot{q}^i}\}$ $i=1,2,...,m$ in terms of coordinates X_L and X_F takes respectively the following forms:

$$X_L=\dot{q}^l\frac{\partial}{\partial q^l}+\left(\mathfrak{g}^{lj}\frac{\partial\mathbb{V}(q)}{\partial q^j}-\Gamma^l_{jk}\dot{q}^j\dot{q}^k\right)\frac{\partial}{\partial\dot{q}^l},$$
$$l=1,2,...,n, \tag{3a}$$

$$X_F=\lambda_i\frac{\partial}{\partial\dot{q}^i}, i=1,2,...,m. \tag{3b}$$

λ_i is the Lagrange multiplier, it takes the form $\mathfrak{g}^{ij}u_i$ where u_i is the control law corresponding to the i^{th} actuator, Γ^l_{jk} is the Christoffel symbols and $\mathfrak{g}^{lj}$ the inverse of $\mathfrak{g}_{lj}$.

Note that $\|.\|$ is the Riemannian distance on Q, if we work in usual Euclidean space, like $\mathbb{R}$, we use Euclidean norm. Through the paper, for simplicity $(q,\dot{q})$ is replaced by x.

Remark 1: standard equation of motion [26] is given by: for $(q,\dot{q})\in TQ$; $M(q)\ddot{q}+C(q,\dot{q})\dot{q}+N(q)=\tau$, where $M(q)\in\mathbb{R}^{n\times n}$ is the positive definite inertia matrix, $C(q,\dot{q})$ is the time derivatives of inertia matrix, it regroups centrifugal and Coriolis forces, $N(q)=\frac{\partial M(q)}{\partial q}$ is the potential forces and τ the generalized torque (control forces). This equation could be rewritten in state space form

$$\begin{cases}\dot{q}=v,\\ \ddot{q}=-M^{-1}(q)\left(C(q,\dot{q})\dot{q}+N(q)-\tau\right),\end{cases}$$

one can see the similarity between this form and the SODE in equation (2), where X_L is equivalent to $[v,-M^{-1}(q)(C(q,\dot{q})\dot{q}+N(q))]^T$, and X_F equivalent to $[0_{n\times1},M^{-1}(q)\tau]^T$.

3. MECHANICAL SYSTEM WITH IMPULSE EFFECTS

A mechanical system with impulse effects is a simple mechanical system which interacts with the surrounding environment via kinematic constraints (holonomic, nonholonomic) [28–30], such interactions induce a discontinuous jumps on its velocities. Thus the system will evolve by two set of dynamics, continuous and discrete.

A mechanical system with impulse effect is defined by

$$\begin{cases}X=X_L+X_F & x\in\mathcal{M}\backslash\mathcal{S}.\\ x^+=P(x^-) & x\in\mathcal{S},\end{cases} \tag{4}$$

where $\mathcal{M} = \{(q,\dot{q}) \in TQ \mid h(q) \geqslant 0\}$ is a submanifold of feasible motion, and $\mathcal{S} = \{(q,\dot{q}) \in TQ \mid h(q) = 0$ and $\nabla h(q)\dot{q} < 0\}$ a distribution spanned by the instantaneous holonomic constraint

$$h : TQ \to \mathbb{R},$$
$$h(q) = 0.$$

$\mathcal{S}$ here is defined as the boundary of $\mathcal{M}$ it is a submanifold of co-dimension one consisting of all discrete states. $P : \mathcal{S} \to \mathcal{S}$ is the resetting map modeling inelastic impacts, it reinitializes the state x of the system after each impact. Moreover if $(q(t^-),\dot{q}(t^-))$ and $(q(t^+),\dot{q}(t^+))$ are respectively pre and post-impact states suppose that P is linear where $q(t^+) = q(t^-)$ and $\dot{q}(t^+) = \Pi(q(t^-))\dot{q}(t^-)$, with Π a smooth map.

For initial condition $x_0 \in \mathcal{M}, x : \mathbb{R} \to \mathcal{M}$; $x(t)$ is the solution of (4), it evolves according to continuous dynamics, when it reaches the surface $\mathcal{S}$ transversally it will be reinitialized by the reset map P. If there exists $T > 0$ such that $x(t) = x(t+T)$, then $x(t)$ is a T-periodic solution denoted by $x^*(t)$. Let $\phi_t(x)$ be the flow of the vector field X. A periodic orbit of solution $x(t)$ is the set $\mathcal{O} = \{x(t) \in \mathcal{M} : x(t) = x^*(t)\}$, any two distinct solutions lie on $\mathcal{O}$ cannot be asymptotically stable in the Lyapunov sense, but they can be orbitally stable.

Definition 1 [25,31]**:** Consider the mechanical system (4), let $\mathcal{B}_\varepsilon(\mathcal{O})$ denote the neighborhood of its periodic orbit $\mathcal{O}$ consisting of all trajectories $x(t) \in \mathcal{M}$ with initial condition $x(t_0)$ such that$\|x(t) - \mathcal{O}\| < \varepsilon$. The orbit $\mathcal{O}$ is:

- *Orbitally stable* if for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $x(t_0) \in \mathcal{B}_\delta(\mathcal{O})$, then $\phi_t(x) \in \mathcal{B}_\varepsilon(\mathcal{O})$ for all $t \geq t_0$.
- *Asymptotically orbitally* stable if it is orbitally stable and there exists $\delta > 0$ such that if $x(t_0) \in \mathcal{B}_\delta(\mathcal{O})$, then $\|\phi_t(x) - \mathcal{O}\| \to 0$ as $t \to \infty$.

4. CONTROLLER DESIGN

The main motivation of our work is to design a control force X_F which steer the trajectories of the system $x(t)$ toward the attraction region of stable limit cycle, the proposed approach is based on LaSalle's invariance principle. Let us first introduce the notion of *ω-limit set* of periodic orbits.

Definition 2: a point $p \in \mathcal{M}$ is said to be an *ω-limit point* of $x \in \mathcal{M}$ if there exists an increasing sequence of time $\{t_n\}_{n=0,\ldots,\infty}$ with $t_n \to \infty$ as $n \to \infty$ such that $p = \lim_{n\to\infty} \phi_{t_n}(x)$, the set of all *limit points* of x is called *ω-limit set* and is denoted by $\omega(x)$.

A *limit cylce* is a closed orbit γ for which there exists a point $x \notin \gamma$ such that $\omega(x) = \gamma$. Since γ includes all of its limit points then it is closed, and when $\phi_t(x)$ is bounded, it is nonempty, compact, and positively invariant with respect to (4). Moreover $\omega(x)$ is the smallest closed set that $\phi_t(x)$ approaches as $t \to \infty$ i.e. $\lim_{t\to\infty} \|\phi_t(x) - \omega(x)\| = 0$.

Theorem 1: Consider the mechanical system with impulse effect (4), let $\mathcal{C} \subset \mathcal{M}$ be a compact positively invariant set under the flow $\phi_t(x)$, assume that there exists a C^1 function $V : \mathcal{C} \to \mathbb{R}$ and positive semi-definite function $\mu : \mathcal{C} \to \mathbb{R}^+$ such that

$$\begin{cases} \langle dV(x), X_L + X_F \rangle = -\mu(x) & x \in \mathcal{M}\backslash\mathcal{S}, \\ V(x^+) - V(x^-) = -\mu(x^-) & x \in \mathcal{S}, \end{cases} \tag{5}$$

let $\mathcal{N} = \{x \in \mathcal{C}\backslash\mathcal{S}, \langle dV(x), X_L + X_F \rangle = 0\} \cup \{x \in \mathcal{S}, V(x^+) - V(x^-) = 0\}$ be a subset of $\mathcal{C}$ for which the time derivatives of V vanishes, and let Ω be the largest positively invariant set in $\mathcal{N}$, then for initial condition $x_0 \in \mathcal{C}$, the solution $x(t)$ approaches Ω as $t \to \infty$. Furthermore when $\Omega = \omega(x)$, the limit cycle γ is asymptotically orbitally stable i.e., $\lim_{t\to\infty} \|\phi_t(x) - \omega(x)\| = 0$.

Proof: Since $\mathcal{C}$ is positively invariant under the flow $\phi_t(x)$ then any solution $x(t)$ of (4) starting in $\mathcal{C}$ will remain there for any $t \geqslant t_0$. $V(x(t))$ is a decreasing function of t because

$$\begin{cases} \langle dV(x), X_L + X_F \rangle = -\mu(x) & x \in \mathcal{M}\backslash\mathcal{S}, \\ V(x^+) - V(x^-) = -\mu(x^-) & x \in \mathcal{S}, \end{cases}$$

for a positive semi-definite function μ. Since V is continuously differentiable then $\lim_{t\to\infty} V(x(t)) = a$ and $\lim_{t\to\infty} V(x^+(t)) = \lim_{t\to\infty} V(x^-(t))$ at the impact. Let $\mathcal{L}^+$ the *ω-limit set* ($\mathcal{L}^+ = \omega(x)$) it resides in $\mathcal{C}$ because of its closeness. For any $p \in \mathcal{L}^+$ the limit point of $x(t_n)$ as $t_n \to \infty$ then $V(p) = a$ and $V(p^+) = V(p^-)$ lies on $\mathcal{L}^+$ (p^-, p^+ are limit points before and after impact). Since $\mathcal{L}^+$ is invariant we have $\dot{V}(p) = 0$ and $V(p^+) - V(p^-) = 0$ for any p, p^+, $p^- \in \mathcal{L}^+$ (Lyapunov function has constant values along the limit cycle). Thus $\mathcal{L}^+ \subset \Omega \subset \mathcal{N} \subset \mathcal{C}$ since $x(t)$ is bounded because of the compactness of $\mathcal{C}$, then $x(t)$ approaches $\mathcal{L}^+$ as $t \to \infty$. Consequently $x(t)$ approaches Ω as $t \to \infty$. □

This theorem is an extension of LaSalle invariance principle [32] to systems exhibiting impacts on their dynamics was firstly given by [33, 34] and for mechanical systems with impacts was discussed in [30], it is considered as a powerful tool for stability analysis beside Poincaré map approach [1, 35], recently different stability certificates have been proposed and are based on computational approaches, see for example [36]. Unlike Lyapunov direct method, asymptotic stability criteria doesn't require function V to be positive definite. The control force X_F constraints the trajectories to enter to the region of attraction of limit cycle γ without leaving it as time increases, to establish the theorem which guarantee existence and uniqueness of such force we have to suppose the following assumptions.

Definition 3: Let $V : \mathcal{C} \to \mathbb{R}^+$ be C^1 smooth Lyapunov function, assume that it is simple and takes the following form:

$$V = \frac{1}{2}H_{ij}(q)\dot{q}^i\dot{q}^j + v(q).$$

H is a Riemannian metric, v is potential energy. Any point on the stable limit cycle has constant energy i.e. for all $x \in \gamma$, $V(x) = c$, $L_X V(x) = 0$ (Lie derivative of function V along the vector field X [26]) and $V(x^+) - V(x^-) = 0$, whereas if $x \notin \gamma$ its energy will decrease along the trajectory, $L_X V(x) < 0$ and $V(x^+) - V(x^-) < 0$.

Assumption 1: Suppose that the system is actuated by one actuator [21] at almost all points of $\mathcal{C}\backslash\mathcal{S}$ i.e. the torque is defined as follows $\tau = (u_f, 0_{(n-1)\times 1})^T$, then $X_F = \mathfrak{g}^{11}u_f\frac{\partial}{\partial\dot{q}^1} + \mathfrak{g}^{21}u_f\frac{\partial}{\partial\dot{q}^2} + \ldots + \mathfrak{g}^{n1}u_f\frac{\partial}{\partial\dot{q}^n}$ and $\mathcal{W}$ is of dimension n. Define the state dependent vector field $\xi(q) = \mathfrak{g}^{11}\frac{\partial}{\partial\dot{q}^1} + \mathfrak{g}^{21}\frac{\partial}{\partial\dot{q}^2} + .. + \mathfrak{g}^{n1}\frac{\partial}{\partial\dot{q}^n}$ and assume that it doesn't vanish anywhere on $\mathcal{C}\backslash\mathcal{S}$.
A realistic example of this assumption is the compass-gait biped illustrated in section 5, where its dynamic is actuated by one actuator.

Assumption 2: For a simple Lyapunov function $V(x)$ defined in definition 3 and a vertical vector field $\xi(q)$, we assume that $\mu(x) = \alpha\langle dV(x), \xi(q)\rangle^2$, where $x \in \mathcal{C}$ and α is a positive constant.

Solving equation $\langle dV(x), X_L + X_F\rangle = -\mu(x)$ for unknown u_f, one get $u_f(x) = -\frac{\mu(x)+\langle dV(x),X_L\rangle}{\langle dV(x),\xi(q)\rangle}$ for all $x \in \mathcal{C}\backslash\mathcal{S}$. It is easy to observe that the existence and uniqueness of the obtained solution depends on the zeros of denominator, it is related to $\mathfrak{D} = \{x \in \mathcal{C}\backslash\mathcal{S}, \langle dV(x), \xi(q)\rangle = 0\} \cup \{x \in \mathcal{S}, V(x^+) - V(x^-) = 0\}$.

Lemma 1: The subset $\mathfrak{D}$ is closed and nowhere dense, it contains only $\omega(x)$ i.e., $\mathfrak{D} = \omega(x)$.

Proof: Solutions of $\langle dV(x), \xi(q)\rangle = 0$ and $V(x^+) - V(x^-) = 0$ are when $dV(x) = 0$ and $V(x) = c$, where c is a constant, and this holds only if x lies on the limit cycle γ then $\mathfrak{D} = \omega(x)$. Let $\phi_t(x)$ be a bounded trajectory defined on $\mathfrak{D} \subset \mathcal{C}$, according to Theorem 1 any trajectory starts from $\mathfrak{D}$ will remain on $\mathfrak{D}$ as $t \to \infty$ then it is closed, the following two properties [37] cl($\mathfrak{D}$)= $\omega(x)$ (the closure of $\mathfrak{D}$) and int(cl($\mathfrak{D}$)) (the interior of the closure of $\mathfrak{D}$) includes only the closed set $\omega(x)$ (contains no open set) imply that $\mathfrak{D}$ is nowhere dense. □

Remark 2: When the system is controlled by l actuator $\tau = (u_1, \ldots, u_l)^T$ one can study equation $\langle dV(x), X_L + X_F\rangle = -\mu(x)$ for each 1-dimensional distribution $\langle\frac{\partial}{\partial\dot{q}^i}\rangle$ [21] a detailed study of system with arbitrary number of actuators was given in [23, 38], in our situation we have chosen a one dimensional actuating system case, because working on an arbitrary number of actuators requires extra assumptions to work with it.

Necessary and sufficient condition of existence and uniqueness is given in the following theorem.

Theorem 2: Given a mechanical system (4) defined by a simple Lagrangian L and a vertical distribution $\mathcal{W}$ spanned by $\{\frac{\partial}{\partial\dot{q}^i}\}_{i=1,\ldots,n}$, there exists a unique solution $X_F \in \mathcal{W}$ for V simple and μ positive semi-definite, iff

$$\begin{cases}\langle dV(x), X_L\rangle = 0 & x \in \mathfrak{D}\backslash\mathcal{S},\\ V(x^+) - V(x^-) = 0 & x \in \mathcal{S},\end{cases} \tag{6}$$

and

$$-\frac{\mu(x) + \langle dV(x), X_L\rangle}{\langle dV(x), \xi(q)\rangle} \in C^\infty(\mathcal{C}\backslash\mathcal{S}), \tag{7}$$

where $\mathfrak{D} = \{x \in \mathcal{C}\backslash\mathcal{S}, \langle dV(x), \xi(q)\rangle = 0\} \cup \{x \in \mathcal{S}, V(x^+) - V(x^-) = 0\}$.
The solution X_F is given by

$$X_F = -\sum_{i=1}^{n}\mathfrak{g}^{i1}\left(\frac{\mu(x) + \langle dV(x), X_L\rangle}{\langle dV(x), \xi(q)\rangle}\right)\frac{\partial}{\partial\dot{q}^i}. \tag{8}$$

In local coordinates continuous dynamics of condition (6) takes the following form

$$\left(\frac{1}{2}\frac{\partial H_{ij}(q)}{\partial q^l} + H_{ik}(q)\Gamma^k_{jl}\right)\dot{q}^i\dot{q}^j\dot{q}^l = 0, \tag{9}$$

$$\left(\frac{\partial v(q)}{\partial q^i} + \mathfrak{g}^{lj}\frac{\partial\mathbb{V}(q)}{\partial q^j}H_{il}\right)\dot{q}^i = 0. \tag{10}$$

Proof: Solution $X_F(x)$ of equation (5) exists if and only if $\langle dV(x), \xi(q)\rangle \neq 0$, $\forall x \in \mathcal{C}$. For $x \in \mathfrak{D}$ we get $\mu(x) + \langle dV(x), X_L\rangle = 0$ and $V(x^+) - V(x^-) = -\mu(x^-)$, since $\mu(x)$ vanishes on $\mathfrak{D}$ (from Assumption 2) then equation (6) holds. Subset $\mathfrak{D} = \{x \in \mathcal{C}\backslash\mathcal{S}, \langle dV(x), \xi(q)\rangle = 0\} \cup \{x \in \mathcal{S}, V(x^+) - V(x^-) = 0\}$ is closed nowhere dense (according to Lemma 1), its complement is an open dense subset of $\mathcal{C}$, then the value of u_f can be extended by continuity to be C^∞ function on $\mathcal{C}\backslash\mathcal{S}$. Therefore X_F exists and is unique, it is given by (8). Now we show why equations (9) and (10) should vanish simultaneously, so in local coordinates continuous dynamics of condition (6) $\langle dV(x), X_L\rangle = 0$ takes the following form $\frac{1}{2}\frac{\partial H_{ij}(q)}{\partial q^l}\dot{q}^i\dot{q}^j\dot{q}^l + \dot{q}^l\frac{\partial v(q)}{\partial q^l} + \mathfrak{g}^{lj}\frac{\partial\mathbb{V}(q)}{\partial q^j}H_{il}(q)\dot{q}^i + H_{il}(q)\Gamma^l_{jk}\dot{q}^i\dot{q}^j\dot{q}^k = 0$, after simplification it becomes $\left(\frac{1}{2}\frac{\partial H_{ij}(q)}{\partial q^l} + H_{ik}(q)\Gamma^k_{jl}\right)\dot{q}^i\dot{q}^j\dot{q}^l + \left(\frac{\partial v(q)}{\partial q^i} + \mathfrak{g}^{lj}\frac{\partial\mathbb{V}(q)}{\partial q^j}H_{il}\right)\dot{q}^i = 0$, it can be rewritten as $A(q)\dot{q}^3 + B(q)\dot{q} = 0$ where $A(q) = \left(\frac{1}{2}\frac{\partial H_{ij}(q)}{\partial q^l} + H_{ik}(q)\Gamma^k_{jl}\right)$ and $B(q) = \left(\frac{\partial v(q)}{\partial q^i} + \mathfrak{g}^{lj}\frac{\partial\mathbb{V}(q)}{\partial q^j}H_{il}\right)$, since this equation must be satisfied for all $(q, \dot{q}) \in \mathfrak{D}$ then it holds only if $A(q) = 0$ and $B(q) = 0$. □

Remark 3: Theorem 2 says that $X_F(x)$ exists if and only if equation (6) has solutions inside $\mathfrak{D}$ for unknown Lyapunov function V which lead to solve a system of 2 PDE: kinetic equations (9) and potential equations (10), where it was shown that are equivalent to the matching conditions corresponding to Chang's version of energy shaping [16, 23, 38]. Since $\mathfrak{D}$ is the limit cycle itself for which $\dot{V}$ vanishes then we can conclude that it is the unique largest positively invariant set that trajectories of the system could achieve under control force $X_F(x)$. Later, Lemma 2 and 3 show how subset $\mathfrak{D}$ could be the largest positively invariant set.

Lemma 2: The set $\dot{V}^{-1}(0)$ for which the time derivatives of V vanishes is equivalent to the subset $\mathfrak{D}$.

Proof: The time derivatives of Lyapunov function along the trajectories of system is given by

$$\begin{cases} \langle dV(x), X_L + X_F \rangle = -\mu(x) & x \in \mathcal{M} \backslash \mathcal{S}, \\ V(x^+) - V(x^-) = -\mu(x^-) & x \in \mathcal{S}, \end{cases}$$

so $\dot{V}$ vanishes when $\mu(x) = 0$, which means that $\dot{V}^{-1}(0) = \{x \in \mathfrak{D} \backslash \mathcal{S}, \langle dV(x), X_L + X_F \rangle = 0\} \cup \{x \in \mathcal{S}, V(x^+) - V(x^-) = 0\}$ also under existence condition of X_F we have $\langle dV(x), \xi(q) \rangle = 0$ for any $x \in \mathfrak{D} \backslash \mathcal{S}$ then the kernel of $\dot{V}$ is the subset $\mathfrak{D}$ □

Lemma 3: The largest positively invariant set in $\dot{V}^{-1}(0)$ is the limit cycle itself, it is closed and nowhere dense.

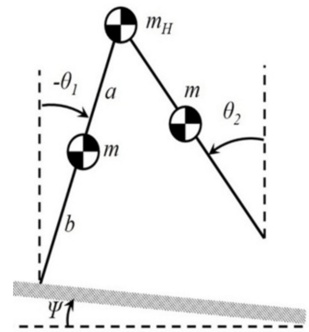
Proof: According to LaSalle theorem, the trajectories of the system achieve the largest positively invariant set included inside the surface $\dot{V}^{-1}(0)$ as $t \to \infty$, from Lemma 2 this surface is none other than the nowhere dense subset $\mathfrak{D}$ which is the closed *ω-limit set*. As a result $\dot{V}^{-1}(0)$ is considered as the largest positively invariant set that solutions $x(t)$ could reach in infinite time. □

Theorem 3: Given a mechanical system (4) defined by a simple Lagrangian L and a vertical distribution $\mathcal{W}$ spanned by $\{\frac{\partial}{\partial \dot{q}^i}\}_{i=1,\ldots,n}$, there exists a unique solution $X_F \in \mathcal{W}$ for V simple and μ positive semi-definite if and only if, the largest positively invariant set is closed and nowhere dense

Proof: Based on Theorem 2, X_F exists iff for any point x belongs to the nowhere dense subset $\mathfrak{D}$ equation (6) holds, also as we know from Lemma 3, $\mathfrak{D}$ is the only largest positively invariant set that solutions $x(t)$ could reach in an infinite time, then existence of X_F is guaranteed by the closeness and nowhere denseness of *ω-limit set*. □

5. ILLUSTRATIVE EXAMPLE

On this part we investigate the above approach to orbitally stabilize the motion of compass-gait biped [2, 39, 40].



m	5 kg
m_H	10 kg
a	0,5 m
b	0,5 m
l	$a+b$
g	9,8 m/s²

Fig. 1. Compass-gait biped.

5.1. Compass-gait biped model

A compass-gait biped is a passive walking biped, it consists of two symmetric links connected by a revolute joint at the hip. Let Θ be the configuration space with coordinates $q(t) = (\theta_1(t), \theta_2(t))^T$ where θ_1, θ_2 are respectively stance and swing leg angles with respect to the vertical. The walking dynamic of the system is divided into swing phase and transition phase, in the first one the stance leg is in contact with the ground, it is governed by Euler-Lagrange equation

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + N(q) = \tau, (q,\dot{q}) \notin S, \tag{11}$$

while in the second one when the swing leg touches down the walking surface, the dynamic of the system exhibits impacts, if $h(q)$ defines the height of the swing leg above the ground, then jumps on the states occurs if $h(q) = 0$ and $\nabla(q)\dot{q} < 0$, such that for pre and post-impact states $(q^-, \dot{q}^-)$, $(q^+, \dot{q}^+)$ the jump is modeled as follows:

$$\begin{aligned} \dot{q}^+ &= \Pi(q(t^-))\dot{q}^-, \\ q^+ &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} q^-, \end{aligned} \tag{12}$$

here $S = \{(q,\dot{q}) | h(q) = 0, \nabla(q)\dot{q} < 0\}$ is the switching surface, where $h(q) = l(\cos(\theta_1 + \psi) - \cos(\theta_2 + \psi))$.

The elements $M(q), C(q,\dot{q}), N(q)$ and τ are respectively the inertia matrix, Coriolis and centrifugal forces, the potential force and the generalized torques. Defining the state $x = (q,\dot{q})$ and the control input $\tau = (u,0)^T$, we can write the dynamic of the system (11), (12) in the form of equation (4):

$$\begin{cases} X = X_L + X_F & x \in T\Theta \backslash S, \\ x^+ = P(x^-) & x \in S. \end{cases} \tag{13}$$

5.2. Controller design by Lyapunov constraints

Assume that two control forces are governing the dynamics of the system, so $\tau = \tau_g + \tau_f$ where τ_g shapes the potential energy of the biped by adding a virtual gravitational force [11], which allows the robot to walk on an inclined surface, the other controller $\tau_f = (u_f, 0)^T$ is the one that we have to design using Lyapunov constraints based-method.

Let $\Phi_g(q) = (\theta_1 - \phi, \theta_2 - \phi)^T$ be the group action applied to the configuration $q(t) = (\theta_1(t), \theta_2(t))^T$ where ϕ is the virtual slope angle corresponding to the desired passive limit cycle, note that the kinetic energy of the system is invariant under the effect of this group, then the control input that makes the biped stable under the slope variation is given by

$$\tau_g = N(q) - N(\phi_g(q)) \tag{14}$$

by substituting it in (11), X_F will have the form of control vector field (8) but along $\xi(q) = M^{11}\frac{\partial}{\partial \dot{q}^1} + M^{21}\frac{\partial}{\partial \dot{q}^2}$ so:

$$X_F = M^{11} u_f \frac{\partial}{\partial \dot{q}^1} + M^{21} u_f \frac{\partial}{\partial \dot{q}^2}, \tag{15}$$

$$u_f = -\frac{\mu(x) + \langle dV(x), X_L\rangle}{\langle dV(x), \xi(q)\rangle}, \tag{16}$$

with $V : \Gamma \to \mathbb{R}$ a smooth Lyapunov function on a compact subset $\Gamma \subset T\Theta$ defined as follows, for positive constants e, k and functions $f(\theta_1, \theta_2)$, $v(\theta_1, \theta_2)$, such that $ek - f^2 > 0$

$$V(\theta_1, \theta_2, \dot\theta_1, \dot\theta_2) = \frac{1}{2}(\dot\theta_1, \dot\theta_2)\begin{bmatrix} e & f \\ f & k\end{bmatrix}\begin{pmatrix}\dot\theta_1 \\ \dot\theta_2\end{pmatrix} + v(\theta_1, \theta_2).$$

$\mu(x)$ is a positive semi-definite function. M^{11} and M^{21} are the elements of $M^{-1}(q)$ the inverse of inertia matrix $M(q)$. According to theorem 2, the existence of X_F is guaranteed by statements (6) and (7), first let us identify parameters e, k, f and v of function V satisfying

$$\begin{cases} \langle dV(x), X_L\rangle = 0 & x \in \mathfrak{D}\backslash S, \\ V(x^+) - V(x^-) = 0 & x \in S, \end{cases} \tag{17}$$

where $\mathfrak{D} = \{x \in \Gamma\backslash S, (eM^{11} + fM^{21})\dot\theta_1 + (fM^{11} + kM^{21})\dot\theta_2 = 0\} \cup \{x \in S, V(x^+) - V(x^-) = 0\}$, as it is stated in theorem 2 by writing condition $\langle dV(x), X_L\rangle = 0$ in local coordinates and replacing $\dot\theta_1 = \frac{(fM^{11}+kM^{21})}{(eM^{11}+fM^{21})}\dot\theta_2$, it is splited into two partial differential equations on $\mathfrak{D}\backslash S$

$$\begin{aligned}&(eM^{11} + fM^{21})(fM^{11} + kM^{21})^2 \frac{\partial f}{\partial \theta_1} \\ &\quad - (fM^{11} + kM^{21})(eM^{11} + fM^{21})^2 \frac{\partial f}{\partial \theta_2} \\ &\quad - (f^2 - ek)(fM^{11} + kM^{21})^2 mlb \sin(\theta_1 - \theta_2) = 0,\end{aligned} \tag{18}$$

$$\begin{aligned}&(eM^{11} + fM^{21})\frac{\partial v}{\partial \theta_2} - (fM^{11} + kM^{21})\frac{\partial v}{\partial \theta_1} \\ &\quad + \frac{(f^2 - ek)N_2(\theta_2 - \phi)}{\begin{pmatrix} mb^2(m_H + m)l^2 + m^2b^2a^2 \\ - m^2l^2b^2\cos(\theta_1 - \theta_2)^2\end{pmatrix}} = 0.\end{aligned} \tag{19}$$

$N_2(\theta_2 - \phi)$ is the component of the potential force vector $N(\theta_1, \theta_2)$.

The key idea to solve these two PDE is to look at the form of the potential energy of the system. As it is seen the potential force $N(\theta_1, \theta_2)$ has two independent components, one depends on θ_1 and the other on θ_2, so the potential energy of the Lyapunov function may take the same form. Let $v(\theta_1, \theta_2) = v_1(\theta_1) + v_2(\theta_2)$ replacing it in (19) we get

$$\begin{aligned}&(eM^{11} + fM^{21})\frac{\partial v_2}{\partial \theta_2} - (fM^{11} + kM^{21})\frac{\partial v_1}{\partial \theta_1} \\ &\quad + \frac{(f^2 - ek)N_2(\theta_2 - \phi)}{\begin{pmatrix} mb^2(m_H + m)l^2 + m^2b^2a^2 \\ - m^2l^2b^2\cos(\theta_1 - \theta_2)^2\end{pmatrix}} = 0,\end{aligned} \tag{20}$$

one of possible solutions of this PDE is when

$$\begin{aligned}(f^2 - ek) = -A^2\big(&mb^2(m_H + m)l^2 + m^2b^2a^2 \\ &- m^2l^2b^2\cos(\theta_1 - \theta_2)^2\big),\end{aligned} \tag{21}$$

where A is a positive constant, solving (21) for unknowns f and ek, as a result we get $f = -Amlb\cos(\theta_1 - \theta_2)$ and $ke = A^2\big(mb^2(m_H + m)l^2 + m^2b^2a^2\big)$. Finally we obtain:

- *The Lyapunov function*: for all $\theta_1, \theta_2, \dot\theta_1, \dot\theta_2 \in \Gamma$

$$\begin{aligned}&V(\theta_1, \theta_2, \dot\theta_1, \dot\theta_2) \\ &= \frac{1}{2}(\dot\theta_1, \dot\theta_2)\begin{bmatrix} \frac{k\left((m_H+m)l^2 + ma^2\right)}{mb^2} & -\frac{kl}{b}\cos(\theta_1 - \theta_2) \\ -\frac{kl}{b}\cos(\theta_1 - \theta_2) & k\end{bmatrix}\begin{pmatrix}\dot\theta_1 \\ \dot\theta_2\end{pmatrix} \\ &\quad + \frac{k}{mb^2}\big(mbg\cos(\theta_2) \\ &\quad - (m_H l + ma + ml)g\cos(\theta_1)\big).\end{aligned} \tag{22}$$

We observe that the Lyapunov function is the Lagrangian scaled by a constant gain so $V = \frac{k}{mb^2}L$.

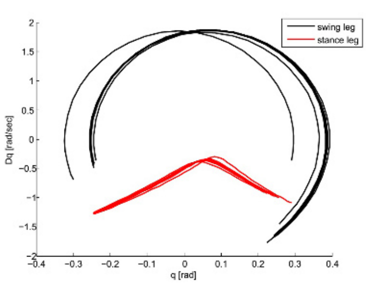
- *The control law*: for all $\theta_1, \theta_2, \dot\theta_1, \dot\theta_2 \in \Gamma\backslash S$ and $\mu(x) = \frac{c}{k}\big((eM^{11} + fM^{21})\dot\theta_1 + (fM^{11} + kM^{21})\dot\theta_2\big)^2$ with c is a positive constant, then

$$u_f = -\frac{\mu(x) + \langle dV(x), X_L\rangle}{(eM^{11} + fM^{21})\dot\theta_1 + (fM^{11} + kM^{21})\dot\theta_2}. \tag{23}$$

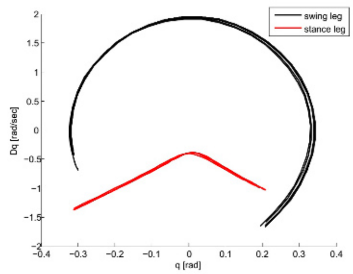
The parameter k is considered as an arbitrary positive constant, it is a common element between numerator and denominator of the controller, so it disappears by simplification.

5.3. Simulation results

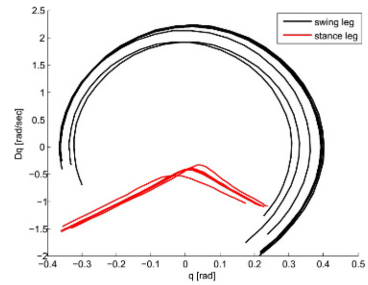
The results of simulation are shown in Fig. 2, Fig. 3 and Fig. 4, for initial conditions $q_0 = (0.2; -0.3)$, $\dot{q}_0 = (-1; -0.7)$ the control laws τ_g and $\tau_g + \tau_f$ generate a set of limit cycles on different ground slope angles $\psi = 0°$, $3°$, $4°$ and $5°$, it is observable that the convergence to the region of stable passive limit cycles is faster when the control force τ_f is added. For example in Fig. 2(a) by choosing $c = 0.002$, fixing ψ at $0°$ and the virtual slope angle



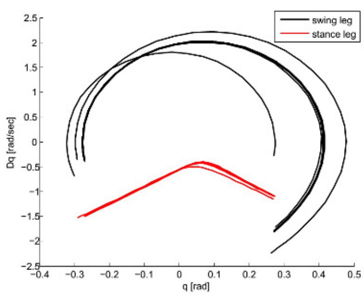
(a) $\psi = 0°$, $\phi = 3°$ and $u_f \neq 0$.



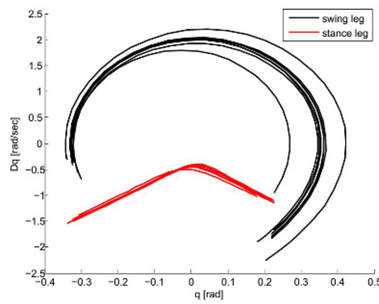
(b) $\psi = 3°$, $\phi = 3°$ and $u_f \neq 0$.



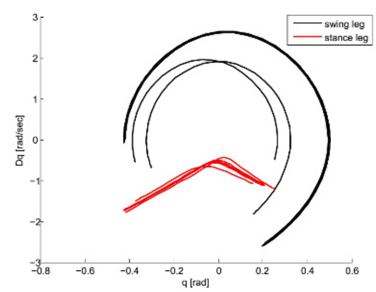
(c) $\psi = 4°$, $\phi = 4°$ and $u_f \neq 0$.



(d) $\psi = 0°$, $\phi = 3°$ and $u_f = 0$.



(e) $\psi = 3°$, $\phi = 3°$ and $u_f = 0$.



(f) $\psi = 5°$, $\phi = 5°$ and $u_f \neq 0$.

Fig. 2. Phase portraits.

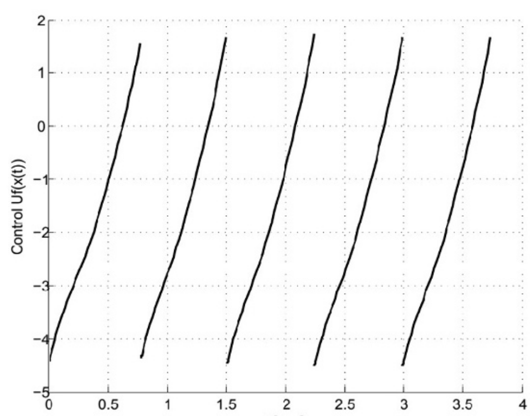


Fig. 3. Evolution of controller u_f for the case of Fig. 2(b).

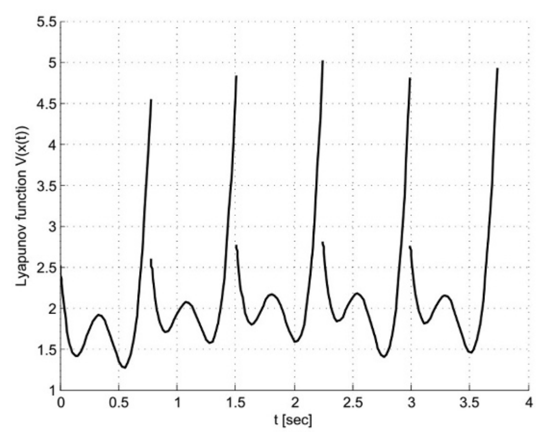


Fig. 4. Lyapunov function for the case of Fig. 2(b).

ϕ at 3° the control u_f force the trajectory to achieve stable limit cycles just after one step, whereas when $u_f = 0$ the system is controlled only by virtual gravitational force and it reaches the same region after two steps (Fig. 2(d)), by increasing the ground slope the rate of convergence changes (Fig. 2(c) and Fig. 2(f)), in the case when $\psi = 4°$ and 5° the region of stable limit cycles is reachable only when $u_f \neq 0$ where $c = 0.05$ and 0.009, the virtual slope angles ϕ are fixed at 4° and 5° respectively. Fig. 3 and Fig. 4 illustrates respectively the evolution of the controller u_f and Lyapunov function in time for the case of $\psi = 3°, \phi = 3°$. The parameters of the compass-gait biped are shown in Fig. 1.

6. CONCLUSION

In this paper we have given a constructive method to design a controller that brings the trajectories of the system to the attraction region of stable limit cycles, the approach is based on LaSalle's invariance principle applied to a controlled system. Theorem 2 has guaranteed the existence and uniqueness of the control law, and expresses those conditions in local coordinates which lead to a system of two partial differential equation for the kinetic and potential energy of Lyapunov function, solutions of those equations are the main block that constitute the control force. Whereas Theorem 3 concludes that the closeness and nowhere denseness of ω-limit set could be considered as an alternative of existence conditions from topological point of view. Finally we have applied the proposed approach for the case of compass-gait biped, it is proved that the obtained control law have generated a stable limit cycles in different slope grounds, note also that the obtained Lyapunov function has taken the form of Lagrangian scaled by a constant gain. Future works will be focused on robustness and designing new controllers using other classes of Lyapunov functions, and study insightfully the properties of solutions arises from those systems of PDE.

APPENDIX A

The inertia Matrix, the Coriolis and potential vectors of (11) are respectively [39]:

$$M(q) = \begin{bmatrix} (m_H+m)l^2+ma^2 & -mlb\cos(\theta_1-\theta_2) \\ -mlb\cos(\theta_1-\theta_2) & mb^2 \end{bmatrix}, \quad \text{(A.1)}$$

$$C(q,\dot{q}) = \begin{bmatrix} 0 & -mlb\sin(\theta_1-\theta_2)\dot{\theta}_2 \\ -mlb\sin(\theta_1-\theta_2)\dot{\theta}_1 & 0 \end{bmatrix}, \quad \text{(A.2)}$$

$$N(q) = \begin{bmatrix} -(m_H l+ma+ml)g\sin(\theta_1) \\ mbg\sin(\theta_2) \end{bmatrix}. \quad \text{(A.3)}$$

Under the following assumptions

- Impacts are perfectly inelastic.
- Transfer of support between swing and stance leg is instantaneous.
- There is no slipping at the foot/ground contact.

The resetting map which relates pre and post-impact velocities is [39]:

$$\Pi(q(t^-)) = \begin{bmatrix} p_{11}^+ & p_{12}^+ \\ p_{21}^+ & p_{22}^+ \end{bmatrix}^{-1} \begin{bmatrix} p_{11}^- & p_{12}^- \\ p_{21}^- & p_{22}^- \end{bmatrix}, \quad \text{(A.4)}$$

$$p_{11}^+ = ml\big(l-b\cos(\theta_1^- - \theta_2^-)\big)+ma^2+m_H l^2,$$
$$p_{12}^+ = mb\big(b-l\cos(\theta_1^- - \theta_2^-)\big),$$
$$p_{21}^+ =_m lb\cos(\theta_1^- - \theta_2^-),$$
$$p_{22}^+ = mb^2,$$
$$p_{11}^- = -mab+(m_H l^2+2mal)\cos(\theta_1^- - \theta_2^-),$$
$$p_{12}^- = p_{21}^- = -mab,$$
$$p_{22}^- = 0.$$

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