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Quasi-min-max Output-feedback Model Predictive Control for LPV Systems with Input Saturation

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Abstract: In the research field of model predictive control (MPC), an output-feedback-type MPC method is consistently required for controlling a wide range of constrained systems. In this paper, we propose a two-stage control strategy for polytopic linear parameter varying (LPV) systems subject to input constraints. This strategy consists of a modified quasi-min-max output-feedback MPC method and a novel terminal output-feedback robust control technique. The proposed control mechanism involves the system states to be first controlled via the MPC method to be driven into a prescribed neighborhood of the origin, and then, the terminal output-feedback robust control method guaranteeing the input constraints is applied to make such states converge to the origin. It is also verified that our control method guarantees the closed-loop stability and feasibility in the presence of model uncertainties and input constraints. Finally, a numerical example is given to demonstrate its effectiveness.

Keywords: Constrained systems, linear matrix inequality, linear parameter varying systems, model predictive control, quasi-min-max optimization.

1. INTRODUCTION

Model predictive control (MPC) is one of the most promising control approaches because of its ability to handle control problems for constrained systems (see [1, 2]). Model quality plays a vital role in this control scheme; however, in practical applications, there always exist modeling errors, which may significantly degrade certain system performances. Thus, a number of advanced MPC algorithms, which are robust against model uncertainties, have been investigated by considering the explicit model uncertainty descriptions (see, for example, [3–10]). On the other hand, most existing MPC schemes are based on state-feedback techniques. Furthermore, system states are assumed to be exactly measurable. However, such MPC methodologies and assumptions may be restrictive because in practical implementations only the output signals of the systems are usually available. Hence, an output-feedback-type MPC method is consistently required for controlling a wide range of constrained systems.

With regard to an output-feedback MPC, Park *et al.* [11] recently proposed a quasi-min-max MPC method for linear parameter varying (LPV) systems subject to input constraints. Note that LPV system can be described as a polytopic system model which is effective for uncertainty modeling of both linear time-varying (LTV) and linear time-invariant (LTI) systems. Their MPC approach mainly consists of a robust state observer designed using an off-line process, and a state-feedback predictive control that works on-line iteratively. Additionally, they claimed that for this control system structure the proposed MPC scheme can guarantee recursive feasibility and robust stability of closed-loop constrained LPV systems. However, as stated recently by Su and Tan [12], such a simple combination of a stable state observer and stable state feedback controller might not guarantee a closed-loop stability because the separation principle does not hold for nonlinear systems. From the above viewpoint, the quasi-min-max output-feedback MPC algorithm proposed by Park *et al.* [11] should be improved in a simple and direct way in order to rigorously stabilize closed-loop LPV systems subject to input constraints. This is the main focus of this paper.

In this paper, we first present discrete-time LPV systems subject to input constraints followed by the control aim in Section 2.1 Next, the optimization problem for Park *et al.*'s output-feedback MPC [11], which was derived based on a quasi-min-max algorithm, is summarized in Section 2.2 It is noteworthy that their MPC method introduced an auxiliary condition on the cost value to stabilize the controlled system; however, it undermines the recursive feasibility of the developed optimization prob-

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lem [12]. Consequently, the intended closed-loop robust stability may not be guaranteed, which could be fatally counterproductive. To overcome such a problem, we propose a two-stage control strategy in Section 3 It consists of a quasi-min-max MPC scheme, which is a modification of Park *et al.*'s MPC scheme [11], and a novel terminal output-feedback robust control scheme. This two-stage control mechanism involves the system states to be first controlled via the MPC method to be driven into a prescribed neighborhood of the origin, and then, the terminal output-feedback robust control method guaranteeing the input constraints is applied to make such states converge to the origin. A detailed off-line design method to determine a terminal output-feedback gain, which is used when the system states are in the robustly invariant ellipsoidal set, is described in this section. The proposed control scheme can guarantee the robust stability of the closed-loop LPV systems subject to input constraints, which are thoroughly analyzed. The effectiveness of the proposed method is illustrated through a numerical example in Section 4 Finally, in Section 5, the conclusion is presented.

Notation: The symbol $*$ will be used for convenience to denote

$$\begin{bmatrix} M & * \\ N & H \end{bmatrix} := \begin{bmatrix} M & N^T \\ N & H \end{bmatrix}.$$

We denote the transpose and inverse of a matrix M by M^T and M^{-1}, respectively. $M \succ 0$ ($M \succeq 0$) implies that M is a symmetric positive (semi-) definite matrix. The ith entry of a vector x is denoted by x_i. A diagonal matrix is denoted as diag$(\cdot)$.

2. PROBLEM STATEMENT

In this section, the discrete-time constrained LPV systems to be controlled is described first. Next, the optimization problem for Park *et al.*'s output-feedback MPC [11] is summarized. Finally, the infeasibility of their MPC formulation and resulting closed-loop instability shown by Su and Tan [12] are briefly presented.

2.1. System description

The discrete-time LPV system to be controlled is described as follows:

$$x(k+1) = A(p(k))x(k) + B(p(k))u(k), \tag{1}$$

$$y(k) = Cx(k), \tag{2}$$

where $x(k) \in \mathbb{R}^{n_x}$ is the state, $y(k) \in \mathbb{R}^{n_y}$ is the output, and $u(k) \in \mathbb{R}^{n_u}$ is the input. The system matrices A and B are affine functions of $p(k)$, which denotes the time-varying parameter, and $[A(p(k))|B(p(k))]$ belongs to a given polytope $\boldsymbol{\Omega}$ at all times k as

$$[A(p(k))|B(p(k))] \in \boldsymbol{\Omega}, \quad \forall k \geq 0, \tag{3}$$

with

$$\boldsymbol{\Omega} := Co\,\{[A_1|B_1], [A_2|B_2], \cdots, [A_{n_c}|B_{n_c}]\}, \tag{4}$$

where $Co\{\cdot\}$ denotes the convex hull and $[A_j|B_j]$, $j = 1, 2, \cdots, n_c$, are vertices of the convex hull. This implies that there exist n_c nonnegative coefficients $\alpha_j(k)$ such that

$$[A(p(k))|B(p(k))] = \sum_{j=1}^{n_c} \alpha_j(k)[A_j|B_j], \tag{5}$$

where

$$\sum_{j=1}^{n_c} \alpha_j(k) = 1, \quad 0 \leq \alpha_j(k) \leq 1. \tag{6}$$

It is assumed that $p(k)$ is measurable at each time instant k [9, 11]. Therefore, the current system matrices $[A(p(k))|B(p(k))]$ are accurately obtained. However, the subsequent ones, $[A(p(k+i))|B(p(k+i))]$, $i \geq 1$ are uncertain, but are known as varying inside a prescribed polytope $\boldsymbol{\Omega}$. It is also assumed that the LPV system (1)-(2) has a control input constraint that should be fulfilled at $k \geq 0$ such as

$$u(k) \in \boldsymbol{U}_c, \; \boldsymbol{U}_c := \{u \in \mathbb{R}^{n_u} : |u_\ell(k)| < u_{\ell,\max}, \; k \geq 0\}, \tag{7}$$

where $u_{\ell,\max}$, $\ell = 1, 2, \cdots, n_u$, denotes the given peak bound on the ℓth entry of a control input $u(k) \in \mathbb{R}^{n_u}$.

2.2. Stability issue: An output-feedback MPC algorithm by Park *et al.* [11]

Let $u_{k+i|k}$ denote the future control input for time step $k+i$ estimated at time instant k. Then, the robust output-feedback MPC scheme is described by

Optimization Problem for MPC Algorithm:

$$\min_{u_{k|k},\, Y(k),\, Q(k),\, X(k)} \gamma(k), \tag{8}$$

where $\gamma(k)$ is a suitable nonnegative variable to be minimized, subject to the following constraint conditions (C1)-(C5):

(C1) The prediction model used to estimate the future state behavior of the system (1): For $i \geq 1$,

$$\hat{z}_{k+1+i|k} = A(p(k+i))\hat{z}_{k+i|k} + B(p(k+i))u_{k+i|k}, \tag{9}$$

where $\hat{z}_{k+i|k} \in \mathbb{R}^{n_x}$ denotes the predicted value of $x(k+i)$ of the system (1) at time instant k, and $\hat{z}_{k+1|k} := \hat{x}(k+1)$. Here, $\hat{x}(k+1)$ is calculated using the state observer introduced later. We also set $\hat{z}_{k|k} := \hat{x}(k)$. For the future control input $u_{k+i|k} \in \mathbb{R}^{n_u}$ estimated at time instant k, the following control sequence U_0^∞ is adopted:

$$U_0^\infty(k) = [u_{k|k}, \; U_1^\infty(k)], \tag{10}$$

with

$$U_1^\infty(k) := \{u_{k+i|k} \in \mathbb{R}^{n_u} : u_{k+i|k} = F(k)\hat{z}_{k+i|k},\ i \geq 1\}, \tag{11}$$

where $u_{k|k}$ is the first control implemented on the plant (1); *i.e.*, $u(k) = u_{k|k}$ whereas the future control sequence $u_{k+i|k}$, $i \geq 1$, is calculated using the feedback gain $F(k)$, which is determined as $F(k) := Y(k)Q^{-1}(k)$ at each time instant k.

(C2) The robust state observer used to estimate the uncertain states of the system (1):

$$\begin{aligned}\hat{x}(k+1) = &A(p(k))\hat{x}(k) + B(p(k))u(k) \\ &+ L_p(y(k) - C\hat{x}(k)),\end{aligned} \tag{12}$$

where $\hat{x}(k) \in \mathbb{R}^{n_x}$ denotes the estimate of $x(k)$. It is assumed that the initial estimate $\hat{x}_0 := \hat{x}(0)$ is given. Further, the output $y(k)$ in (2) is assumed to be exactly measurable at all time instants $k \geq 0$. Note that since one knows the current system matrices $[A(p(k))|B(p(k))]$, the output $y(k)$, and the state estimate $\hat{x}(k)$ at the current time instant k, $\hat{x}(t+1)$ can be calculated based on (12), whereas $\hat{x}(t+i)$, $i \geq 2$, is uncertain. In (12), L_p denotes the observer gain and is determined in terms of LMI with regard to off-line processing; i.e., $L_p := P_e^{-1}Y_e$ with $P_e \succ 0$ and Y_e satisfying

$$\begin{bmatrix} \rho^2 P_e - L_e & * \\ P_e A_j - Y_e C & P_e \end{bmatrix} > 0,\ j = 1, 2, \cdots, n_c, \tag{13}$$

where ρ $(0 < \rho < 1)$ is the decay rate and L_e is a suitable weighting matrix, which are set by the designer in advance. Note that the above L_p guarantees the stability of error dynamics $e(k+1) = (A(p(k)) - L_pC)e(k)$, $e(k) := x(k) - \hat{x}(k)$, for any $[A(p(k))|B(p(k))] \in \mathbf{\Omega}$. Thus, the estimated state $\hat{x}(k)$ converges to the system state $x(k)$; i.e., $\hat{x}(k) \to x(k)$ as $k \to \infty$.

(C3) The stability condition related on a quadratic function $V(\hat{z}_{k+i|k}) = \hat{z}_{k+i|k}^T P(k)\hat{z}_{k+i|k}$ where $P(k) := \gamma(k)Q^{-1}(k) \succ 0$: For $j = 1, 2, \cdots, n_c$,

$$\begin{bmatrix} Q(k) & * & * & * \\ A_jQ(k) + B_jY(k) & Q(k) & * & * \\ L^{\frac{1}{2}}Q(k) & 0 & \gamma(k)I & * \\ R^{\frac{1}{2}}Y(k) & 0 & 0 & \gamma(k)I \end{bmatrix} > 0, \tag{14}$$

where $L \succ 0$ and $R \succ 0$ are suitable weighting matrices. Note that (14) guarantees the following inequality condition at each step k:

$$\begin{aligned}&V(\hat{z}_{k+1+i|k}) - V(\hat{z}_{k+i|k}) \\ &\qquad < -(\hat{z}_{k+i|k}^T L\hat{z}_{k+i|k} + u_{k+i|k}^T Ru_{k+i|k}).\end{aligned} \tag{15}$$

(C4) The constraint of the system's performance:

$$\begin{bmatrix} 1 & * & * & * \\ T(k) & Q(k) & * & * \\ L^{\frac{1}{2}}\hat{x}(k) & 0 & \gamma(k)I & * \\ R^{\frac{1}{2}}u(k) & 0 & 0 & \gamma(k)I \end{bmatrix} > 0, \tag{16}$$

where $T(k) := A(p(k))\hat{x}(k) + B(p(k))u(k) + L_p(y(k) - C\hat{x}(k))$, which is derived from the following condition:

$$\hat{z}_{k|k}^T L\hat{z}_{k|k} + u_{k|k}^T Ru_{k|k} + \hat{z}_{k+1|k}^T P(k)\hat{z}_{k+1|k} < \gamma(k). \tag{17}$$

The proofs and details for (C3)-(C4) can be found in Park *et al*. [11].

(C5) The input constraints: For $\ell = 1, 2, \cdots, n_u$,

$$\begin{bmatrix} u_\ell(k) & -u_{\ell,\max} \\ -u_{\ell,\max} & -u_\ell(k) \end{bmatrix} < 0, \tag{18}$$

$$\begin{bmatrix} X(k) & * \\ Y^T(k) & Q(k) \end{bmatrix} > 0 \text{ with } X_{\ell\ell} \leq u_{\ell,\max}^2. \tag{19}$$

If a symmetric matrix $X(k)$ satisfying (18)-(19) exists, then the input constraint, $|u_{\ell,k+i|k}| < u_{\ell,\max}$, is guaranteed, which was proved in Park *et al*. [11].

It is important to note that, in finding $u(k)$ via the minimization problem (8) subject to constraints (C1)-(C5), Park *et al*. [11] introduced an auxiliary condition, $\gamma(k) < \gamma(k-1)$, to guarantee the robust stability (see condition (18) in Park *et al*. [11]). However, this additional condition undermines the recursive feasibility of the optimization problem as recently proved by Su and Tan [12]. Consequently, the intended closed-loop robust stability may not be guaranteed, which could be fatally counterproductive.

In order to overcome the difficulties involved with guaranteeing the system stability, we utilize the terminal output-feedback control scheme, which is combined with the aforementioned on-line quasi-min-max output-feedback MPC method.

3. OUTPUT-FEEDBACK MPC COMBINED WITH TERMINAL OUTPUT-FEEDBACK ROBUST CONTROL

As described in the previous section, Park *et al*.'s quasi-min-max output-feedback MPC formulation [11] can cause a closed-loop instability problem due to infeasibility. In order to overcome such a problem, we propose a two-stage control strategy in this section. This strategy consists of a modified quasi-min-max MPC scheme and a novel terminal output-feedback robust control technique. Note that our MPC scheme does not adopt auxiliary conditions on the cost values as in Park *et al*.' scheme (i.e., the conditions (18) and (19) in Park *et al*. [11] are excluded.). A detailed off-line design method to determine a terminal output-feedback gain, which is used when the system states are in the robustly invariant ellipsoidal set, is described later.

It is first assumed that a given initial state estimate $\hat{x}_0$ of $x(0)$ satisfies

$$e_0 := x(0) - \hat{x}_0 \in \mathbf{\Lambda}(0), \tag{20}$$

with

$$\mathbf{\Lambda}(k) := \{e(k) \in \mathbb{R}^{n_x} |\ e^T(k)P_e e(k) \le \rho^{2k} e_{\max}\}, \tag{21}$$

where $e_{\max}$ is given. Note that $e(k+i)$, $\forall i \ge 0$, satisfies $e(k+i) \in \mathbf{\Lambda}(k+i) \subseteq \mathbf{\Lambda}(k)$. This fact can easily be verified from (20)-(21) and the following quadratic condition:

$$\begin{aligned} &E(e(k+1+i)) - \rho^2 E(e(k+i)) < -e^T(k+i)L_e e(k+i), \\ &E(e(k)) := e^T(k)P_e e(k), \end{aligned} \tag{22}$$

which was introduced to derive an LMI condition (13) for the design of a stable state observer (see Park *et al.* [11]).

Next, the following terminal set $\boldsymbol{X}_f$ is introduced to guarantee the robust stability of the closed-loop LPV system subject to input constraint:

$$\boldsymbol{X}_f := \{\hat{x}(k) \in \mathbb{R}^{n_x} |\ \hat{x}^T(k) P_f^{-1} \hat{x}(k) \le 1\}, \tag{23}$$

where $P_f = P_f^T \succ 0$ is set by the designer in advance. In the following, we derive LMIs to determine a terminal feedback gain K_f in $u(k) = K_f \hat{x}(k)$, which guarantees that for $\forall \hat{x}(k) \in \boldsymbol{X}_f$ and $\forall e(k) \in \mathbf{\Lambda}(k)$, the next state $\hat{x}(k+1)$ of (12) satisfies $\hat{x}(k+1) \in \boldsymbol{X}_f$ without violating the given constraint on the control input (7).

Theorem 1: For some constant $N \ge 0$, suppose that there exist τ_{f_1}, τ_{f_2}, η_ℓ $(\ell = 1,2)$ and K_f satisfying $0 \le \tau_{f_1} < 1$, $\tau_{f_2} \ge 0$, $\eta_\ell \ge 0$ and

$$S_f := \begin{bmatrix} \tau_{f_1} P_f^{-1} & * & * \\ 0 & \tau_{f_2} P_e & * \\ -(A_j + B_j K_f)^T & -(L_p C)^T & P_f^{-1} \end{bmatrix} \ge 0, \tag{24}$$

$$\tau_{f_1} + \tau_{f_2} \rho^{2N} e_{\max} \le 1, \tag{25}$$

$$\begin{bmatrix} \eta_\ell P_f^{-1} & * \\ -g_\ell^T K_f & 2h - \eta_\ell \end{bmatrix} \ge 0, \tag{26}$$

where $h := [u_{1,\max}, u_{2,\max}, \cdots, u_{n_u,\max}]^T$, $g_1 := I \in \mathbb{R}^{n_u \times n_u}$, $g_2 := -I \in \mathbb{R}^{n_u \times n_u}$ and $j = 1,2,\cdots,n_c$. Then, if $\hat{x}(k_N) \in \boldsymbol{X}_f$ for $k_N \ge N$, the control $u(k) = K_f \hat{x}(k)$ guarantees the following properties: For $k \ge k_N$ and $\forall i \ge 0$,

(i) $A(p(k+i)) + B(p(k+i))K_f$ is Hurwitz,

(ii) $\hat{x}(k+i) \in \boldsymbol{X}_f$,

(iii) $u(k+i) \in \boldsymbol{U}_c$.

Proof: In order to prove Theorem 1, the following lemmas will be used (for details and proofs, refer to [13] and [14]).

Lemma 1: Let $S_0, S_1, \cdots, S_m$ be quadratic functions of the variable $\chi \in \mathbb{R}^n$:

$$S_i(\chi) := \chi^T H_i \chi + 2\zeta_i^T \chi + \xi_i, \tag{27}$$

where $i = 1,2,\cdots,m$ and $H_i = H_i^T$. Then, if there exist $\tau_1 \ge 0,\ \tau_2 \ge 0, \cdots,\ \tau_m \ge 0$ such that for all χ

$$S_0(\chi) - \sum_{i=1}^{m} \tau_i S_i(\chi) \le 0, \tag{28}$$

then $S_0(\chi) \le 0$ for all χ such that $S_i(\chi) \le 0$ where $i = 1,2,\cdots,m$. If $m = 1$ and there is some χ_0 such that $S_1(\chi_0) < 0$, then the above condition is necessary and sufficient.

The above lemma denotes the S-procedure for quadratic functions. The following result describes how to represent LMIs from quadratic form constraints.

Lemma 2: Given a quadratic function defined by $S(\chi) := \chi^T H \chi + 2\zeta^T \chi + \xi$, the quadratic constraint $S(\chi) \le 0$ is satisfied for all χ, if and only if the following condition is satisfied:

$$\begin{bmatrix} H & \zeta \\ \zeta^T & \xi \end{bmatrix} \le 0. \tag{29}$$

Based on the above results, Theorem 1 will be proved. First, (i) will be proved. Pre- and post-multiplying S_f in (24) by $[I\ 0\ I]$ and its transpose implies that

$$\begin{bmatrix} -\tau_{f_1} P_f^{-1} & (A_j + B_j K_f)^T \\ * & -P_f^{-1} \end{bmatrix} \le 0. \tag{30}$$

A Schur complement technique shows that (30) is equivalent to

$$(A_j + B_j K_f)^T P_f^{-1} (A_j + B_j K_f) - \tau_{f_1} P_f^{-1} \le 0. \tag{31}$$

Then, it follows from $0 \le \tau_{f_1} < 1$ and $P_f \succ 0$ that

$$(A_j + B_j K_f)^T P_f^{-1} (A_j + B_j K_f) - P_f^{-1} < 0. \tag{32}$$

Because (32) is satisfied and $[A(p(k)) | B(p(k))] \in \mathbf{\Omega}$, it holds that

$$\begin{aligned} &(A(p(k)) + B(p(k))K_f)^T P_f^{-1} (A(p(k)) + B(p(k))K_f) \\ &\quad - P_f^{-1} < 0. \end{aligned} \tag{33}$$

Thus, the matrix P_f^{-1} can be seen as a solution to the discrete Lyapunov equation proving that $A(p(k)) + B(p(k))K_f$ is Hurwitz.

Next, we will prove (ii). Under the control $u = K_f \hat{x}$, we obtain from (12) that

$$\hat{x}(k+1) = (A(p(k)) + B(p(k))K_f)\hat{x}(k) + L_p C e(k). \tag{34}$$

Therefore, the requirement such as $\hat{x}(k+1) \in \boldsymbol{X}_f$ is equivalent to

$$S_0 := \begin{bmatrix} \hat{x}(k) \\ e(k) \end{bmatrix}^T \begin{bmatrix} (A(p(k))+B(p(k))K_f)^T \\ (L_pC)^T \end{bmatrix} P_f^{-1} \times \\ \begin{bmatrix} (A(p(k))+B(p(k))K_f) & L_pC \end{bmatrix} \begin{bmatrix} \hat{x}(k) \\ e(k) \end{bmatrix} - 1 \leq 0. \quad (35)$$

On the other hand, the conditions such as $\hat{x}(k) \in \boldsymbol{X}_f$ and $e(k) \in \boldsymbol{\Lambda}(k)$ can be expressed as quadratic functions of the vector variable $[\hat{x}^T(k)\ e^T(k)]^T$. The requirement that $\hat{x}(k) \in \boldsymbol{X}_f$ is equivalent to

$$S_1 := \begin{bmatrix} \hat{x}(k) \\ e(k) \end{bmatrix}^T \begin{bmatrix} I \\ 0 \end{bmatrix} P_f^{-1} \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} \hat{x}(k) \\ e(k) \end{bmatrix} - 1 \leq 0. \quad (36)$$

Note that it holds from (21) that

$$E(e(k)) = e^T(k)P_e e(k) < \rho^{2N} e_{\max}, \quad (37)$$

where $k \geq N$. Hence, the requirement that $e(k) \in \boldsymbol{\Lambda}(k) \subseteq \boldsymbol{\Lambda}(N)$ for $k \geq N$ is equivalent to

$$S_2 := \begin{bmatrix} \hat{x}(k) \\ e(k) \end{bmatrix}^T \begin{bmatrix} 0 \\ I \end{bmatrix} P_e \begin{bmatrix} 0 & I \end{bmatrix} \begin{bmatrix} \hat{x}(k) \\ e(k) \end{bmatrix} - \rho^{2N} e_{\max} \leq 0. \quad (38)$$

Therefore, from the S-procedure technique in Lemma 1, the requirement such that $S_0 \leq 0$ in (35) holds for all $\hat{x}(k) \in \boldsymbol{X}_f$ and $e(k) \in \boldsymbol{\Lambda}(k)$ is equivalent to the existence of $0 \leq \tau_{f_1} < 1$ and $\tau_{f_2} \geq 0$ satisfying

$$S_0 - \tau_{f_1} S_1 - \tau_{f_2} S_2 \leq 0. \quad (39)$$

Then, it follows from (39) that

$$\begin{bmatrix} \hat{x}(k) \\ e(k) \end{bmatrix}^T \left(\begin{bmatrix} (A(p(k))+B(p(k))K_f)^T \\ (L_pC)^T \end{bmatrix} P_f^{-1} \times \\ \begin{bmatrix} (A(p(k))+B(p(k))K_f) & L_pC \end{bmatrix} + \begin{bmatrix} -\tau_{f_1} P_f^{-1} & 0 \\ 0 & -\tau_{f_2} P_e \end{bmatrix} \right) \\ \times \begin{bmatrix} \hat{x}(k) \\ e(k) \end{bmatrix} - 1 + \tau_{f_1} + \tau_{f_2} \rho^{2N} e_{\max} \leq 0. \quad (40)$$

Hence, from Lemma 2, the condition (40) can be written as negative semidefinite matrix constraints as follows:

$$\begin{bmatrix} -\tau_{f_1} P_f^{-1} & 0 \\ 0 & -\tau_{f_2} P_e \end{bmatrix} - \begin{bmatrix} (A(p(k))+B(p(k))K_f)^T \\ (L_pC)^T \end{bmatrix} \\ \times (-P_f^{-1}) \begin{bmatrix} (A(p(k))+B(p(k))K_f) & L_pC \end{bmatrix} \leq 0, \quad (41)$$

$$\tau_{f_1} + \tau_{f_2} \rho^{2N} e_{\max} \leq 1. \quad (42)$$

By using Schur complement, it is possible to represent the inequality (41) in the following form:

$$\begin{bmatrix} -\tau_{f_1} P_f^{-1} & 0 & (A(p(k))+B(p(k))K_f)^T \\ * & -\tau_{f_2} P_e & (L_pC)^T \\ * & * & -P_f^{-1} \end{bmatrix} \leq 0. \quad (43)$$

Since the inequality in (43) is affine in $[A(p(k))|B(p(k))]$, it is satisfied for all $[A(p(k))|B(p(k))] \in \boldsymbol{\Omega}$, if and only if there exist auxiliary variables $0 \leq \tau_{f_1} < 1$, $\tau_{f_2} \geq 0$ and a design variable K_f satisfying

$$\begin{bmatrix} -\tau_{f_1} P_f^{-1} & 0 & (A_j+B_jK_f)^T \\ * & -\tau_{f_2} P_e & (L_pC)^T \\ * & * & -P_f^{-1} \end{bmatrix} \leq 0, \quad (44)$$

which proves (ii).

In the following, (iii) will be proved. First, the following hyperplane constraints on the control input can be derived from (7):

$$g_\ell^T u(k) \leq h, \ \ \ell = 1,2, \ \ k \geq 0. \quad (45)$$

Substituting $u(k) = K_f \hat{x}(k)$ for $k \geq N$ into (45), we have

$$2g_\ell^T K_f \hat{x}(k) - 2h \leq 0, \ \ \ell = 1,2, \ \ k \geq N. \quad (46)$$

By using the S-procedure technique in Lemma 1, the condition (46) is satisfied for all $\hat{x}(k) \in \boldsymbol{X}_f$, if and only if there exists $\eta_\ell \geq 0$, $\ell = 1,2$, satisfying

$$2g_\ell^T K_f \hat{x}(k) - 2h - \eta_\ell(\hat{x}^T(k)P_f^{-1}\hat{x}(k) - 1) \leq 0. \quad (47)$$

Then, (47) can be rewritten as

$$\hat{x}^T(k)(-\eta_\ell P_f^{-1})\hat{x}(k) + 2g_\ell^T K_f \hat{x}(k) + \eta_\ell - 2h \leq 0. \quad (48)$$

Based on Lemma 2, (48) is expressed as a positive semidefinite matrix constraint as shown in (26). □

Note that Theorem 1 implies that $\boldsymbol{X}_f$ is a robustly invariant set under the control law $u = K_f \hat{x}$. Based on the above results, the two-stage control mechanism comprising quasi-min-max output-feedback MPC and terminal output-feedback robust control can be described as follows:

Output-feedback MPC algorithm with terminal output-feedback robust control technique:

Step 0: Find L_p presented in (C2) of Section 2.2 and K_f in Theorem 1, respectively.

Step 1: Initialize $k = 0$.

Step 2:. If $\hat{x}(k) \in \boldsymbol{X}_f$ for $k \geq N$, apply the control input $u(k) = K_f \hat{x}(k)$ to the plant and then, go to Step 4. Otherwise, go to Step 3.

Step 3: Solve the optimization problem for MPC in Section 2.2 based on $\hat{x}(k)$, $y(k)$, and $p(k)$ to calculate

$$u_{k+i|k} = \begin{cases} u_{k|k}, & i = 0 \\ F(k)\hat{z}_{k+i|k}, & i \geq 1. \end{cases} \quad (49)$$

Then, apply the control input $u(k) = u_{k|k}$ to the plant.

Step 4: Calculate $\hat{x}(k+1)$ in (12) based on the measurement $y(k)$ the control input $u(k)$, and $p(k)$.

Step 5: Set $k=k+1$ and go to Step 2.

Note that the above two-stage control mechanism involves the system states to be first controlled via the MPC method to be driven into a prescribed terminal set $\boldsymbol{X}_f$, and then, the terminal output-feedback robust control method guaranteeing the input constraints is applied to make such states converge to the origin.

The proposed control scheme can guarantee the robust stability of the output-feedback LPV systems subject to input constraints, which are thoroughly analyzed in the following. Here, the feasibility means that there exists u, which satisfies the constraints (C1)-(C5) of the optimization problem (8) with a finite value of γ. The following lemma concerning feasibility is a key result to prove the stability.

Lemma 3: Suppose that the optimization problem for MPC given in Section 2.2 is feasible at the current time instant k. Let $u^*_{k+i|k}$ denote the optimal solution determined at time instant k. Then, the control input

$$u_{k+i|k+1}=u^*_{k+i|k},\ \ i\geq 1, \tag{50}$$

is one of the feasible solutions of the optimization problem for MPC at the next time instant $k+1$.

Proof: Feasibility at time instant $k+1$ requires that the control input in (50) guarantees $|u_{\ell,k+i|k+1}|\leq u_{\ell,\max}$, $\ell=1,2,\cdots,n_u$. Note that when (19) is satisfied at time instant k, $u_{k+i|k}$, $i\geq 1$, is a feasible solution satisfying the above constraint on the control input. Therefore, it is clear that the condition (18) is satisfied by $u_{k+i|k+1}=u^*_{k+i|k}$, $i\geq 1$, at time instant $k+1$. □

Lemma 3 states that if the optimization problem for MPC is feasible at the time instant k, then it is also feasible at the next time instant. The following theorem describes the stability property of the proposed two-stage control scheme, which is analyzed based on Theorem 1 and Lemma 3. Note that the terminal feedback gain K_f is determined off-line.

Theorem 2: Suppose that the optimization problem for MPC given in Section 2.2 is feasible at $k=0$, i.e., there exists u minimizing (8) subject to (C1)-(C5) at $k=0$. Then, the proposed MPC method guarantees the following properties:

(i) The optimization problem for MPC is feasible at time instant $k\geq 1$.

(ii) The state $x(k)$ of the real system (1) converges to the origin as $k\to\infty$.

Proof: First, (i) will be proved by induction. The optimization problem is feasible at $k=0$ by assumption. We assume now that it is feasible at each time instant $k=i$ ($i=1,2,\cdots,N-2$). Then, since Lemma 3 shows that the control input of (50) is feasible at $k=i+1$, (i) is proved. Next, (ii) will be proved. As mentioned in Section 2.2, the developed state observer guarantees that $\hat{x}(k)\to x(k)$ as $k\to\infty$, which implies that $e(k)\to 0$ as $k\to\infty$. On the other hand, it follows from (12) and $u(k)=K_f\hat{x}(k)$ for $\forall\hat{x}(t)\in\boldsymbol{X}_f$ that

$$\hat{x}(k+1)=(A(p(k))+B(p(k))K_f)\hat{x}(k)+L_pCe(k). \tag{51}$$

Note that since $e(k)\to 0$ as $k\to\infty$ and $A(p(k))+B(p(k))K_f$ is Hurwitz as shown in Theorem 1, it holds that $\hat{x}(k)\to 0$ as $k\to\infty$. Based on the above results, we can see that $x(k)\to 0$ as $k\to\infty$ is guaranteed under the proposed two-stage output-feedback control algorithm. □

In the following section, a numerical example is presented to demonstrate the effectiveness of the proposed scheme.

4. NUMERICAL EXAMPLE

Consider the LPV system described as (1)-(2) with

$$A(p(k))=\begin{bmatrix}1 & 0.1\\ 0 & 1-0.1\beta(k)\end{bmatrix},\ \ B(p(k))=\begin{bmatrix}0\\ 0.1\kappa\end{bmatrix}, \tag{52}$$

$C=[1\ 0]$ and $\kappa=0.787$, which is a discretized angular positioning system reported in Kothare *et al.* [7] and Wan and Kothare [15]. The initial states of the above real system and the state observer are $x_0=(0.05,0)^T$ and $\hat{x}_0=(0.01,-0.05)^T$, respectively. We assume that the uncertain parameter $\beta(k)$ belongs to the set $\beta(k)\in[0.1,\ 10]$. Thus, the above LPV system belongs to the following polytope formed by the two local discrete models

$$\boldsymbol{\Omega}=Co\ \{[A_1|B],\ [A_2|B]\}, \tag{53}$$

where

$$A_1=\begin{bmatrix}1 & 0.1\\ 0 & 0.99\end{bmatrix},\ A_2=\begin{bmatrix}1 & 0.1\\ 0 & 0\end{bmatrix}. \tag{54}$$

The input constraint is given as $|u(k)|\leq 0.2$, which should be fulfilled at all time instants $k\geq 0$.

Based on the design scheme of state observer using a decay rate of $\rho=\sqrt{0.9}$ and the weighting matrix $L_e=\mathrm{diag}(0.01,0.01)$, the observer gain L_p is determined as $L_p=(0.8632,\ 0.4030)^T$. Next, a terminal feedback gain K_f will be determined. We first introduce the following terminal ellipsoidal set:

$$\boldsymbol{X}_f:=\{\hat{x}\in\mathbb{R}^2|\ \hat{x}^TP_f^{-1}\hat{x}\leq 1.0\times 10^{-5}\}, \tag{55}$$

with $P_f=\mathrm{diag}(1,1)$. Using the above variables, the feedback gain is determined based on Theorem 2 as $K_f=$

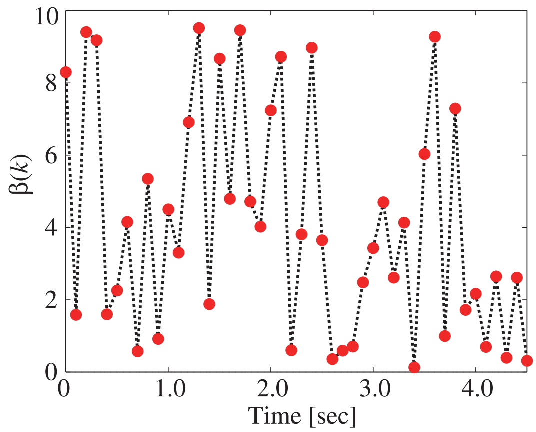

Fig. 1. Time plot of parameter $\beta(t)$.

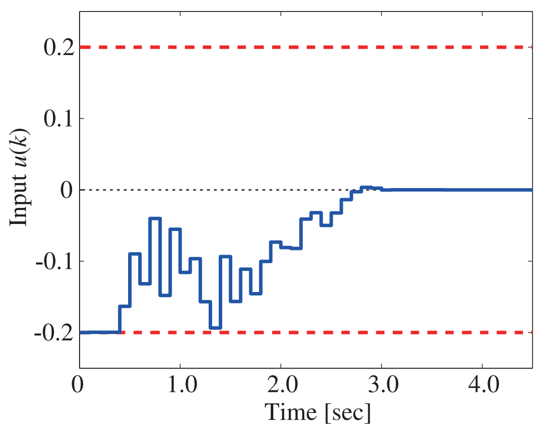

Fig. 2. Time plot of control input $u(t)$.

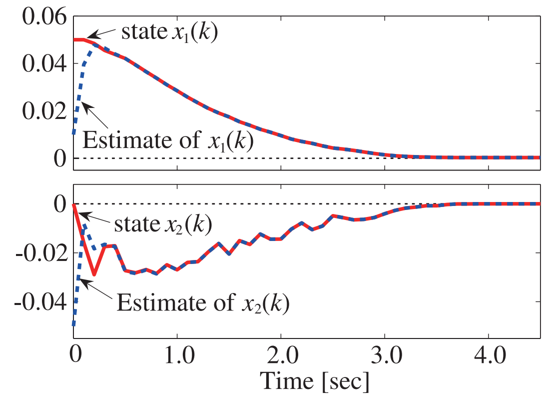

Fig. 3. Time plot of $x(t)$ and $\hat{x}(t)$.

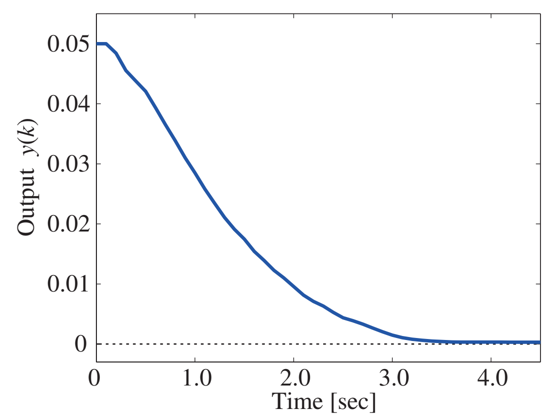

Fig. 4. Time plot of output $y(t)$.

$-(0.0314,\ 0.036)$ for $N = 30$. Then, the proposed MPC technique was implemented with $R = 0.005$ and $L = \text{diag}(1, 0.1)$. The time-varying parameter $\beta(k)$ is shown in Fig. 1.

In Fig. 2, the solid line shows the applied control input $u(k)$ and the dashed lines shows the upper and lower bounds on the control. It is evident that the control input $u(k)$ obtained by the proposed control method satisfies the given constraint. The real state $x_i(k)$ and estimated state $\hat{x}_i(k)$ are plotted in Fig. 3 by the solid line and the dashed line, respectively. This figure shows that the proposed robust state observer works well. The trajectory of the output $y(k)$ converging to zero as $k \to \infty$ is shown in Fig. 4, which verifies the feasibility and stability of the proposed output-feedback MPC algorithm combined with the terminal output-feedback robust control technique.

5. CONCLUSION

We presented a novel quasi-min-max output-feedback MPC algorithm combined with terminal output-feedback robust control technique for LPV systems subject to input constraints. The proposed control scheme overcomes the critical issue in Park *et al.*'s output-feedback MPC scheme [11] that the closed-loop robust stability may not be guaranteed theoretically. In our two-stage control mechanism, the system states are first controlled via the MPC method to be driven into a prescribed neighborhood of the origin, and then the terminal output-feedback robust control method guaranteeing the input constraints is applied to make such states converge to the origin. The robust stability of the output-feedback LPV systems subject to input constraints was confirmed in a rigorous manner. In addition, the effectiveness of the proposed control method was verified through a numerical experiment.

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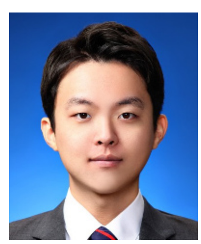

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