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Barrier Lyapunov Functions-based Adaptive Control for Nonlinear Pure-feedback Systems with Time-varying Full State Constraints

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Abstract: This paper studies the problem of controller design for pure-feedback nonlinear systems with asymmetric time-varying full state constraints. The mean value theorem is employed to transform a pure-feedback system into a strict-feedback structure with non-affine terms. For the transformed system, a time-varying asymmetric Barrier Lyapunov Function (ABLF) with the error variables is employed to ensure the time-varying constraints satisfaction. By allowing the barriers to vary with the desired trajectory in time, the initial condition requirements are relaxed efficiently. The presented control scheme can guarantee that all signals in the closed-loop system are ultimately bounded. It is also proved that the tracking error converges to an adjustable neighborhood of the origin even in the presence of disturbance. The performance of the ABLF-based control are illustrated through two examples.

Keywords: ABLF, backstepping, full state constraints, nonlinear pure-feedback system, time-varying.

1. INTRODUCTION

Constraints occurring in various engineering systems are sources of instability and often cause undesirable performance. Driven by practical requirements and theoretical challenges, the research of constrained problem such as the nonlinear saturation, physical stoppages, as well as performance and safety specifications has become an important research topic [1–6]. Many fruitful results on constraint-handling methods have been generated, such as the model predictive control [2, 6], the set invariance notions [7, 8] and reference governors [9, 10]. Additionally, Barrier Lyapunov Function (BLF) has been employed to handle constraints for systems in the Brunovsky form [11], strict-feedback form [12–16] and pure-feedback form [17, 18]. The basic difference between BLF and the traditional Lyapunov function lies in the fact that the value of BLF approaches infinity whenever its arguments tend to some limits.

There have been extensive research efforts in strict-feedback nonlinear systems such as [12–16] and [19–26]. Several versions of the BLF-based control design for nonlinear systems have been studied. The work in [12] presented control designs for strict-feedback nonlinear systems with a constant output constraint. [13, 14] further extend the results to strict-feedback nonlinear systems with time-varying output constraint. For the strict-feedback nonlinear systems with state constraints have been studied in [19–22]. Both the problem of output constraint and state constraints mentioned in the above works were tackled by using BLF. However, the approach on state constraints reported in above mentioned works limits its application to a class of strict-feedback nonlinear systems. The pure-feedback system represents a more general class of triangular systems which have no affine appearance of the variables to be used as virtual controls. For the pure-feedback system with state constraints a good tracking performance was achieved in [17] without violating the constant states constraints. Note that the method of [17] can not solve the problem of asymmetric time-varying state constraints. One more interesting work is to consider the pure-feedback system with asymmetric time-varying state constraints. Therefore, the main difficulty of handling this class of system is to deal with time-varying full state constrained unknown non-affine nonlinearities.

Motivated by the above observations, in this paper, we employ the time-varying ABLF-based adaptive control to handle a pure-feedback system with time-varying full state constraints. The main contributions of this paper are summarized as follows:

1) The mean value theorem is employed to transform the pure-feedback system into the strict-feedback structure

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with non-affine terms. For the transformed system, a time-varying ABLF-based backstepping design is proposed to prevent the violation of the full state constraints.

2) By constructing a new ABLF, the control scheme is able to handle the state constraints that are both time-varying and asymmetric. The cases of symmetric state constraints considered in [17] and [20] are some special cases of our scheme.

3) A new adaptive controller is designed to handle parametric uncertainties. Furthermore, the time-varying ABLF is used, the states can start from anywhere within the initial states constrained space. As a result, the design scheme can add flexibility of controller and relax the restriction on initial conditions.

The rest of the paper is organized as follows. In Section 2, some mathematical preliminaries and statement of the problem are provided. The BLF-based control design for uncertain system is developed, and addresses robustness to disturbance in Section 3. Two simulation examples are given in Section 4 to illustrate the obtained results. Section 5 concludes this paper.

2. PROBLEM STATEMENT

Consider the following class of pure-feedback nonlinear systems with time-varying full state constraints

$$\begin{aligned} &\dot{x}_i(t) = f_i(\bar{x}_{i+1}(t)),\ i = 1,2,\cdots,n-1,\\ &\dot{x}_n(t) = f_n(\bar{x}_n(t),u(t)),\\ &y(t) = x_1(t), \end{aligned} \tag{1}$$

where $\bar{x}_i(t) = (x_1(t),x_2(t),\cdots,x_i(t))^T \in R^i$, $i = 1,2,\cdots,n$, $u \in R$ and $y \in R$ are the states, control input and output, respectively. $f_i(\bar{x}_{i+1}(t))(i = 1,\cdots,n-1)$ and $f_n(\bar{x}_n(t),u(t))$ are uncertain nonlinear smooth functions. According to the mean value theorem in [27], there must exist variables x^0_{i+1} and u^0 such that

$$\begin{aligned} &f_i(\bar{x}_{i+1}) = f_i(\bar{x}_i,0) + g_i(\bar{x}_i,x^0_{i+1})x_{i+1},\\ &f_n(\bar{x}_n,u) = f_n(\bar{x}_n,0) + g_n(\bar{x}_n,u^0)u, \end{aligned} \tag{2}$$

where x^0_{i+1} is some point between zero and x_{i+1}, u^0 is some point between zero and u, $g_i(\bar{x}_{i+1}) = \partial f_i(\bar{x}_{i+1})/\partial x_{i+1}$, $g_n(\bar{x}_n,u) = \partial f_n(\bar{x}_n,u)/\partial u$. Inspired by [17], we assume that the uncertain terms satisfy: $f_i(\bar{x}_i,0) = \theta_i^T h_i(\bar{x}_i)$ with $\theta_i \in R^m$ is uncertain constant vector and $h_i : R^i \to R^m$ is known continuous function vector.

Furthermore, in this paper all the states $x(t)$ are required to be constrained in a set as $x(t) \in \Omega_x$, where $\Omega_x := \{x \in R^n, \underline{k}_{c_i}(t) < x_i(t) < \bar{k}_{c_i}(t), i = 1,2,\cdots,n, \forall t > 0\} \subset R^n$, $\underline{k}_{c_i}(t) : R_+ \to R$ and $\bar{k}_{c_i}(t) : R_+ \to R$.

Remark 1: The state constraints $\underline{k}_{c_i}(t) < x_i(t) < \bar{k}_{c_i}(t)$ considered in this paper are based on the worst-case scenario. The designed constraint functions can be specified according to the requirements of practical problem. The constraints $\underline{k}_{c_i}(t), \bar{k}_{c_i}(t)$ should be guaranteed to satisfy $\underline{k}_{c_i}(t) < \alpha_{i-1}(t) < \bar{k}_{c_i}(t), i = 1,\cdots,n$, in which α_{i-1} are the virtual stabilizing functions to be designed.

The control objective of this paper is to design an adaptive state feedback control law $u(t)$ to ensure that the system output $y(t)$ tracks the reference signal $y_d(t)$. At the same time, we need guarantee that the time-varying full state constraints are not violated and all closed-loop signals are bounded. Towards this end, we make the following assumptions on system (1).

Assumption 1: The functions $g_i(\cdot)$ are bounded, i.e., there exist the constants $\bar{g}_i \geq \underline{g}_i > 0$ such that $\underline{g}_i \leq |g_i(\cdot)| \leq \bar{g}_i$. Without loss of generality, in this paper, we assume that $\underline{g}_i \leq g_i(\cdot) \leq \bar{g}_i (i = 1,\cdots,n)$.

Assumption 2: There exist functions $\bar{Y}_0(t) : R_+ \to R_+$ and $\underline{Y}_0(t) : R_+ \to R_+$ satisfy $\bar{Y}_0(t) < \bar{k}_{c_1}(t)$ and $\underline{Y}_0(t) > \underline{k}_{c_1}(t), \forall t \geq 0$. Furthermore, there exist positive constants $Y_i, i = 1,\cdots,n$, such that the reference signal $y_d(t)$ and its time derivatives satisfy $\underline{Y}_0(t) \leq y_d(t) \leq \bar{Y}_0(t)$ and $|y_d^{(i)}(t)| \leq Y_i$, $\forall t \geq 0$.

Assumption 3: There exist constants $\bar{K}_{c_i}, \underline{K}_{c_i}$, $\underline{d}_{c_{ij}}, \bar{d}_{c_{ij}}$ $i,j = 1,\cdots,n$, such that $\bar{k}_{c_i}(t) \leq \bar{K}_{c_i}, \underline{k}_{c_i}(t) \geq \underline{K}_{c_i}$ and their derivatives satisfy $|\underline{k}_{c_i}^{(j)}(t)| \leq \underline{d}_{c_{ij}}, |\bar{k}_{c_i}^{(j)}(t)| \leq \bar{d}_{c_{ij}}$.

To deal with the full state constraints, we present the following lemmas which play important roles in the coming feedback design and stability analysis.

Lemma 1 [28]**:** For any functions $k_{a_j}(t)$, $k_{b_j}(t)$, let $\bar{Z}_i := \{z_j(t) \in R : -k_{a_j}(t) < z_j(t) < k_{b_j}(t), j = 1,2,\ldots,i, t > 0\} \subset R^i$ be open set and $\mathcal{N} := R^l \times \bar{Z}_i \subset R^{l+i}$. Consider the system: $\dot{\eta}(t) = h(t,\eta(t))$, where $\eta(t) := [\omega(t),\bar{z}_i(t)]^T \in \mathcal{N}$, $\bar{z}_i = [z_1,z_2,\cdots,z_i]^T$. Note that $h : R_+ \times \mathcal{N} \to R^{l+i}$ is piecewise continuous with respect to t and locally Lipschitz about $\eta(t)$. Suppose there exist continuous differentiable and positive definite functions $U : R^l \to R_+$ and $V_i : \bar{Z}_i \to R_+$ in their respective domains, such that

$$\begin{aligned} &V_i(z_i(t)) \to \infty \quad as \quad z_j \to -k_{a_j}(t) \quad or \quad z_j \to k_{b_j}(t),\\ &\gamma_1(\|\omega(t)\|) \leq U(\omega(t)) \leq \gamma_2(\|\omega(t)\|), \end{aligned}$$

where $j = 1,\cdots,i$, γ_1 and γ_2 are class K_∞ functions. Let $V(\eta(t)) := \sum_{j=1}^i V_j(z_j(t)) + U(\omega(t))$ and $\bar{z}_i(0)$ belong to the set $\bar{Z}_i$. If the inequality holds

$$\dot{V} = \frac{\partial V}{\partial \eta} h \leq -cV + \upsilon, \tag{3}$$

with constants $c > 0, \upsilon > 0$ and $\eta \in \mathcal{N}$, then $\bar{z}_i(t)$ remain in the open set $\bar{Z}_i, \forall t \in [0,\infty)$.

Lemma 2 [14]**:** For all $|S_i| < k_{b_i}(t)$, the following inequality holds

$$\log \frac{k^2_{b_i}(t)}{k^2_{b_i}(t) - S^2_i} \leq \frac{S^2_i}{k^2_{b_i}(t) - S^2_i}. \tag{4}$$

3. TIME-VARYING ABLF-BASED CONTROL

In this section, we give the time-varying ABLF-based stability analysis for both system (1) and system with disturbances.

3.1. Control design for uncertain system

Step 1: Consider the x_1-subsystem. Denote $S_1 = x_1 - y_d$ and $S_2 = x_2 - \alpha_1$, where α_1 is a virtual control, then

$$\begin{aligned}\dot{S}_1(t) =& \theta_1^T h_1(x_1(t)) + g_1(x_1(t), x_2^0(t))(S_2(t)\\ &+ \alpha_1(t)) - \dot{y}_d(t).\end{aligned}$$

Since the state constraints are time-varying and asymmetric, we choose an ABLF as follows:

$$\begin{aligned}V_1 =& \frac{1-q(S_1)}{2p}\log\Big(\frac{k_{a_1}^{2p}(t)}{k_{a_1}^{2p}(t) - S_1^{2p}(t)}\Big)\\ &+ \frac{q(S_1)}{2p}\log\Big(\frac{k_{b_1}^{2p}(t)}{k_{b_1}^{2p}(t) - S_1^{2p}(t)}\Big) + \frac{1}{2}\tilde{\vartheta}_1^2(t), \qquad (5)\end{aligned}$$

where $2p \geq n$, $\tilde{\vartheta}_1(t) = \hat{\vartheta}_1(t) - \vartheta_1$ and $\hat{\vartheta}_1(t) > 0$ is the estimation of $\vartheta_1 = \|\theta_1\|^2$. The time-varying barriers are given by $k_{a_1}(t) := y_d(t) - \underline{k}_{c_1}(t), k_{b_1}(t) := \bar{k}_{c_1}(t) - y_d(t)$, and

$$q(\bullet) := \begin{cases} 1, & \text{if} \quad \bullet > 0,\\ 0, & \text{if} \quad \bullet < 0. \end{cases} \qquad (6)$$

Remark 2: The aim of p which is chosen as $2p \geq n$ is to ensure the differentiability of α_i for $i = 1, \cdots, n-1$.

Remark 3: The selection of $q(\bullet)$ in (6) is to ensure that the Lyapunov function in (5) can handle the case of asymmetric time-varying state constraints. Clearly the Lyapunov function in (5) can also handle both the cases of symmetric time-varying state constraints and the static constraints. Hence, the Lyapunov function in (5) is more general compared with that in [17] and [20].

Throughout this paper, for ease of notation, when on confusion arise the time and state dependence will be omitted unless otherwise stated. Due to Assumptions 2-3, there exist positive constants $\underline{K}_{b_1}, \bar{K}_{b_1}$, $\underline{K}_{a_1}$ and $\bar{K}_{a_1}$ such that $\underline{K}_{a_1} \leq k_{a_1}(t) \leq \bar{K}_{a_1}, \underline{K}_{b_1} \leq k_{b_1}(t) \leq \bar{K}_{b_1}$. Define a set $\Omega_s := \{S = (S_1, \cdots, S_n)^T \subset R^n, -k_{a_i}(t) < S_i(t) < k_{b_i}(t), i = 1, 2, \cdots, n, \forall t > 0\}$, in which $S_i, k_{a_i}(t), k_{b_i}(t)$ for $i = 2, \cdots, n$ will be specified later on and $\underline{K}_{a_i} \leq k_{a_i}(t) \leq \bar{K}_{a_i}, \underline{K}_{b_i} \leq k_{b_i}(t) \leq \bar{K}_{b_i}$.

In the set Ω_s, V_1 is piecewise smooth and continuously differentiable in terms with the fact that $\lim_{S_1 \to 0^+}(dV_1/dS_1) = \lim_{S_1 \to 0^-}(dV_1/dS_1) = 0$. Thus V_1 is a valid Lyapunov function. The time derivative of V_1 is given by

$$\begin{aligned}\dot{V}_1 =& S_1^{2p-1}K_{s_1}[\theta_1^T h_1 + g_1(S_2 + \alpha_1) - \dot{y}_d \qquad (7)\\ &+ (1-q(S_1))\frac{\dot{k}_{a_1}(t)}{k_{a_1}(t)}S_1 + q(S_1)\frac{\dot{k}_{b_1}(t)}{k_{b_1}(t)}S_1] + \tilde{\vartheta}_1\dot{\hat{\vartheta}}_1,\end{aligned}$$

where

$$K_{s_i} = \frac{1-q(S_i)}{k_{a_i}^{2p}(t) - S_i^{2p}} + \frac{q(S_i)}{k_{b_i}^{2p}(t) - S_i^{2p}}, \quad i = 1, \cdots, n. \qquad (8)$$

Design the adaptive law for $\hat{\vartheta}_1$ as well as the stabilising function α_1 in the form of

$$\dot{\hat{\vartheta}}_1 = -\hat{\vartheta}_1 + \frac{\underline{g}_1 K_{s_1}^2 S_1^{4p-2} h_1^2}{2\delta_1^2}, \qquad (9)$$

$$\begin{aligned}\alpha_1 =& -(K_1 + \bar{k}_1(t))S_1 - K_{s_1}S_1^{2p-1}\frac{(\dot{y}_d)^2}{2}\\ &- \frac{K_{s_1}S_1^{2p-1}h_1^2\hat{\vartheta}_1}{2\delta_1^2}, \qquad (10)\end{aligned}$$

where K_1, δ_1 are positive design parameters and the time-varying gain is given by

$$\bar{k}_1(t) = \frac{1}{\underline{g}_1}\sqrt{(1-q(S_1))(\frac{\dot{k}_{a_1}}{k_{a_1}})^2 + q(S_1)(\frac{\dot{k}_{b_1}}{k_{b_1}})^2 + \beta}. \qquad (11)$$

Note that β is a positive constant, then it guarantees that the time derivatives of α_1 are bounded even when $\dot{k}_{a_1}$ and $\dot{k}_{b_1}$ are both zero.

In view of $S_1 = x_1 - y_d$, from the formation of α_1 in (10), we know that α_1 is a continuously differentiable function of $x_1, y_d, \dot{y}_d$, $k_{a_1}, \dot{k}_{a_1}, k_{b_1}, \dot{k}_{b_1}$ and $\hat{\vartheta}_1$. From Assumption 2-3, y_d, $\bar{k}_{c_1}(t), \underline{k}_{c_1}(t)$ and their derivatives are all bounded. Since $k_{a_1}(t) = y_d(t) - \underline{k}_{c_1}(t), k_{b_1}(t) = \bar{k}_{c_1}(t) - y_d(t)$, so k_{a_1}, $\dot{k}_{a_1}$, k_{b_1} and $\dot{k}_{b_1}$ are bounded. For the last term of (9) is bounded, then $\hat{\vartheta}_1$ is bounded which can be verified by Lemma C.5 in [29]. Because of the boundedness of $x_1, y_d, \dot{y}_d$, $k_{a_1}, \dot{k}_{a_1}, k_{b_1}, \dot{k}_{b_1}$ and $\hat{\vartheta}_1$, we conclude that α_1 is bounded and express it as $|\alpha_1(t)| \leq \bar{\alpha}_1$.

Let $\varsigma_1 = [k_{a_1}, k_{b_1}]^T$, one has

$$\begin{aligned}\dot{\alpha}_1 =& \frac{\partial \alpha_1}{\partial x_1}(\theta_1^T h_1 + g_1 x_2) + \sum_{j=0}^{1}\frac{\partial \alpha_1}{\partial y_d^{(j)}}y_d^{(j+1)}\\ &+ \sum_{j=0}^{1}\frac{\partial \alpha_1}{\partial \varsigma_1^{(j)}}\varsigma_1^{(j+1)} + \frac{\partial \alpha_1}{\partial \hat{\vartheta}_1}\dot{\hat{\vartheta}}_1. \qquad (12)\end{aligned}$$

The boundedness of $\dot{\alpha}_1$ also can be verified. Using the Young's inequality, we have

$$S_1^{2p-1}K_{s_1}\theta_1^T h_1 \leq \frac{\underline{g}_1 K_{s_1}^2 S_1^{4p-2}\vartheta_1 h_1^2}{2\delta_1^2} + \frac{\delta_1^2}{2\underline{g}_1},$$

$$K_{s_1}g_1 S_1^{2p-1}S_2 \leq \frac{\underline{g}_2 K_{s_1}^2 S_1^{4p-2}S_2^2}{2} + \frac{\bar{g}_1^2}{2\underline{g}_2},$$

$$-K_{s_1}S_1^{2p-1}\dot{y}_d \leq \frac{\underline{g}_1 K_{s_1}^2 S_1^{4p-2}(\dot{y}_d)^2}{2} + \frac{1}{2\underline{g}_1}.$$

Consider α_1 in (10), the following inequality holds

$$K_{s_1}g_1 S_1^{2p-1}\alpha_1 \leq -(K_1 + \bar{k}_1(t))\underline{g}_1 K_{s_1}S_1^{2p}$$

$$-\frac{\underline{g}_1 K_{s_1}^2 S_1^{4p-2}(\dot{y}_d)^2}{2}-\frac{\underline{g}_1 K_{s_1}^2 h_1^2 S_1^{4p-2}\hat{\vartheta}_1}{2\delta_1^2}.$$

Substituting (9)and (10) into (7) and using the above four inequalities, we have

$$\dot{V}_1 \le \tilde{\vartheta}_1[\dot{\hat{\vartheta}}_1-\frac{\underline{g}_1 K_{s_1}^2 S_1^{4p-2}h_1^2}{2\delta_1^2}]-K_1\underline{g}_1 K_{s_1}S_1^{2p} +\frac{\underline{g}_2 K_{s_1}^2 S_1^{4p-2}S_2^2}{2}+\frac{\delta_1^2}{2\underline{g}_1}+\frac{\bar{g}_1^2}{2\underline{g}_2}+\frac{1}{2\underline{g}_1}. \tag{13}$$

For the term of $\tilde{\vartheta}_1[\dot{\hat{\vartheta}}_1-\frac{\underline{g}_1 K_{s_1}^2 S_1^{4p-2}h_1^2}{2\delta_1^2}]$, with the help of (9), it can be dealt with as follows:

$$\tilde{\vartheta}_1[\dot{\hat{\vartheta}}_1-\frac{\underline{g}_1 K_{s_1}^2 S_1^{4p-2}h_1^2}{2\delta_1^2}]\le -\frac{1}{2}\tilde{\vartheta}_1^2+\frac{1}{2}\vartheta_1^2. \tag{14}$$

Then,

$$\dot{V}_1 \le -K_1\underline{g}_1 K_{s_1}S_1^{2p}-\frac{1}{2}\tilde{\vartheta}_1^2+\mu_{c_1}+\frac{\underline{g}_2 K_{s_1}^2 S_1^{4p-2}S_2^2}{2}, \tag{15}$$

where $\mu_{c_1}=\frac{\delta_1^2+1}{2\underline{g}_1}+\frac{1}{2}\vartheta_1^2+\frac{\bar{g}_1^2}{2\underline{g}_2}$. It is noted that the last term of inequality (15) can be canceled in the subsequent step.

Step *i* ($2\le i\le n-1$)**:** Denote $S_{i+1}=x_{i+1}-\alpha_i$ and $\varsigma_l=[k_{a_l},k_{b_l}]^T$. The time derivative of S_i is

$$\dot{S}_i(t)=\theta_i^T h_i(\bar{x}_i(t))+g_i(\bar{x}_i(t),x_{i+1}^0(t))(S_{i+1}(t)+\alpha_i) -\dot{\alpha}_{i-1}(t).$$

with

$$\dot{\alpha}_{i-1}=\sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}(\theta_j^T h_j+g_j x_{j+1})+\sum_{j=0}^{i-1}\frac{\partial\alpha_{i-1}}{\partial y_d^{(j)}}y_d^{(j+1)} +\sum_{l=1}^{i-1}\sum_{j=0}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\varsigma_l^{(j)}}\varsigma_l^{(j+1)}+\sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\hat{\vartheta}_j}\dot{\hat{\vartheta}}_j. \tag{16}$$

Choose the Lyapunov function

$$V_i=V_{i-1}+\frac{1-q(S_i)}{2p}\log(\frac{k_{a_i}^{2p}(t)}{k_{a_i}^{2p}(t)-S_i^{2p}}) +\frac{q(S_i)}{2p}\log(\frac{k_{b_i}^{2p}(t)}{k_{b_i}^{2p}(t)-S_i^{2p}})+\frac{1}{2}\tilde{\vartheta}_i^2, \tag{17}$$

with $k_{a_i}(t):=\alpha_{i-1}(t)-\underline{k}_{c_i}(t)$, $k_{b_i}(t):=\bar{k}_{c_i}(t)-\alpha_{i-1}(t)$, $\tilde{\vartheta}_i=\hat{\vartheta}_i-\vartheta_i$ and $\hat{\vartheta}_i$ is the estimation of $\vartheta_i=\max_{1\le j\le i}\{\|\theta_j\|^2\}$. As the analysis in Step 1, there also exist a positive constant $\bar{\alpha}_{i-1}$ such that $|\alpha_{i-1}|\le\bar{\alpha}_{i-1}$. The time derivative of V_i is given by

$$\begin{aligned}\dot{V}_i=&\dot{V}_{i-1}+S_i^{2p-1}K_{s_i}[\theta_i^T h_i+g_i(S_{i+1}+\alpha_i)\\ &-\tilde{k}_i(\sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}(\theta_j^T h_j+g_j x_{j+1})+\sum_{j=0}^{i-1}\frac{\partial\alpha_{i-1}}{\partial y_d^{(j)}}y_d^{(j+1)}\\ &+\sum_{l=1}^{i-1}\sum_{j=0}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\varsigma_l^{(j)}}\varsigma_l^{(j+1)}+\sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\hat{\vartheta}_j}\dot{\hat{\vartheta}}_j)\\ &-(1-q(S_i))\frac{\dot{\underline{k}}_{c_i}(t)}{k_{a_i}(t)}S_i+q(S_i)\frac{\dot{\bar{k}}_{c_i}(t)}{k_{b_i}(t)}S_i]+\tilde{\vartheta}_i\dot{\hat{\vartheta}}_i,\end{aligned} \tag{18}$$

with $\tilde{k}_i=1-\frac{(1-q(S_i))S_i}{k_{a_i}}+\frac{q(S_i)S_i}{k_{b_i}}$. Similar with Step 1, the ith virtual controller α_i is designed as

$$\alpha_i=-(K_i+\bar{k}_i(t))S_i-K_{s_i}\tilde{k}_i^2 S_i^{2p-1}\frac{\psi_i}{2} -\frac{K_{s_i}S_i^{2p-1}\hat{\vartheta}_i}{2\delta_i^2}\gamma_i-\frac{K_{s_{i-1}}^2 S_{i-1}^{4p-2}}{2K_{s_i}S_i^{2p-3}}, \tag{19}$$

where K_i,δ_i are positive constants and

$$\bar{k}_i(t)=\frac{1}{\underline{g}_i}\sqrt{(1-q(S_i))(\frac{\dot{\underline{k}}_{c_i}}{k_{a_i}})^2+q(S_i)(\frac{\dot{\bar{k}}_{c_i}}{k_{b_i}})^2+\beta}, \tag{20}$$

$$\gamma_i=\|h_i(\bar{x}_i)\|^2+\|\tilde{k}_i\|^2\sum_{j=1}^{i-1}\|\frac{\partial\alpha_{i-1}}{\partial x_j}h_j\|^2, \tag{21}$$

$$\psi_i=\sum_{j=1}^{i-1}(\frac{\partial\alpha_{i-1}}{\partial x_j}x_{j+1})^2+\sum_{j=0}^{i-1}(\frac{\partial\alpha_{i-1}}{\partial y_d^{(j)}}y_d{}^{(j+1)})^2 +\sum_{l=0}^{i-1}\sum_{j=0}^{i-1}(\frac{\partial\alpha_{i-1}}{\partial\varsigma_l^{(j)}}\varsigma_l^{(j+1)})^2+\sum_{j=1}^{i-1}(\frac{\partial\alpha_{i-1}}{\partial\hat{\vartheta}_j}\dot{\hat{\vartheta}}_j)^2. \tag{22}$$

Remark 4: In Step i, we have defined $\bar{k}_i(t)$ for $i=2,\cdots n-1$ as (20), it is different from $\bar{k}_1(t)$ denoted in the first step. Since $k_{a_i}=\alpha_{i-1}-\underline{k}_{c_i}$, $k_{b_i}=\bar{k}_{c_i}-\alpha_{i-1}$, then $\dot{k}_{a_i}=\dot{\alpha}_{i-1}-\dot{\underline{k}}_{c_i}$, $\dot{k}_{b_i}=\dot{\bar{k}}_{c_i}-\dot{\alpha}_{i-1}$. While the term $\sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}(\theta_j^T h_j)$ in $\dot{\alpha}_{i-1}$ is uncertain, then it is not able to be used in virtual controller. So we separate it from the time-varied gain and define $\bar{k}_i(t)$ as (20).

The adaptation law for $\hat{\vartheta}_i$ is given as

$$\dot{\hat{\vartheta}}_i=-\hat{\vartheta}_i+\frac{\underline{g}_i K_{S_i}^2 S_i^{4p-2}\gamma_i(\bar{x}_i)}{2\delta_i^2}. \tag{23}$$

Similar to (14) and using (23), we have

$$\tilde{\vartheta}_i[\dot{\hat{\vartheta}}_i-\frac{\underline{g}_i K_{s_i}^2 S_i^{4p-2}\gamma_i(\bar{x}_i)}{2\delta_i^2}]\le -\frac{1}{2}\tilde{\vartheta}_i^2+\frac{1}{2}\vartheta_i^2. \tag{24}$$

Substitute (19) and (23) into (18), with the help of Young's inequality as previous steps, it can be further shown that

$$\dot{V}_i\le -\sum_{j=1}^{i}K_j\underline{g}_j K_{s_j}S_j^{2p}-\frac{1}{2}\sum_{j=1}^{i}\tilde{\vartheta}_j^2 +\sum_{j=1}^{i}\mu_{c_j}+\frac{\underline{g}_{i+1}K_{s_i}^2 S_i^{4p-2}S_{i+1}^2}{2}, \tag{25}$$

where $\mu_{c_i} = \frac{i\delta_i^2+i^2+i-1}{2\underline{g}_i} + \frac{1}{2\underline{g}_i}\sum_{j=1}^{i-1}\bar{g}_j^2 + \frac{1}{2}\vartheta_i^2 + \frac{\bar{g}_i^2}{2\underline{g}_{i+1}}$.

Step n. As denoted in Step $n-1$, $S_n = x_n - \alpha_{n-1}$, then

$$\dot{S}_n(t) = \theta_n^T h_n(t) + g_n(\bar{x}_n, u^0(t))u(t) - \dot{\alpha}_{n-1}(t). \tag{26}$$

Define the Lyapunov function candidate

$$\begin{aligned} V_n =& V_{n-1} + \frac{1-q(S_n)}{2p}\log(\frac{k_{a_n}^{2p}(t)}{k_{a_n}^{2p}(t)-S_n^{2p}}) \\ &+ \frac{q(S_n)}{2p}\log(\frac{k_{b_n}^{2p}(t)}{k_{b_n}^{2p}(t)-S_n^{2p}}) + \frac{1}{2}\tilde{\vartheta}_n^2, \end{aligned} \tag{27}$$

where $k_{a_n}(t) := \alpha_{n-1}(t) - \underline{k}_{c_n}(t)$, $k_{b_n}(t) := \bar{k}_{c_n}(t) - \alpha_{n-1}(t)$, $\vartheta_n = \max_{1\le j\le n}\{\|\theta_j\|^2\}$, $\tilde{\vartheta}_n = \hat{\vartheta}_n - \vartheta_n$. Computing the time derivative of (27), we have

$$\begin{aligned} \dot{V}_n =& \dot{V}_{n-1} + S_n^{2p-1}K_{s_n}[\theta_n^T h_n + g_n u \\ &- \tilde{k}_n(\sum_{j=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial x_j}(\theta_j^T h_j + g_j x_{j+1}) + \sum_{j=0}^{n-1}\frac{\partial\alpha_{n-1}}{\partial y_d^{(j)}}y_d^{(j+1)} \\ &+ \sum_{l=1}^{n-1}\sum_{j=0}^{n-1}\frac{\partial\alpha_{n-1}}{\partial\varsigma_l^{(j)}}\varsigma_l^{(j+1)} + \sum_{j=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial\hat{\vartheta}_j}\dot{\hat{\vartheta}}_j) \\ &- (1-q(S_n))\frac{\dot{\underline{k}}_{c_n}(t)}{k_{a_n}(t)}S_n + q(S_n)\frac{\dot{\bar{k}}_{c_n}(t)}{k_{b_n}(t)}S_n] + \tilde{\vartheta}_n\dot{\hat{\vartheta}}_n, \end{aligned} \tag{28}$$

in which

$$\tilde{k}_n = 1 - \frac{(1-q(S_n))S_n}{k_{a_n}(t)} + \frac{q(S_n)S_n}{k_{b_n}(t)}. \tag{29}$$

The actual controller and the adaption law are chosen as

$$\begin{aligned} u(t) =& -(K_n + \bar{k}_n(t))S_n - K_{s_n}\tilde{k}_n^2 S_n^{2p-1}\frac{\psi_n}{2} \\ &- \frac{K_{s_n}S_n^{2p-1}\hat{\vartheta}_n(t)}{2\delta_n^2}\gamma_n - \frac{K_{s_{n-1}}^2 S_{n-1}^{4p-2}}{2K_{s_n}S_n^{2p-3}}, \end{aligned} \tag{30}$$

$$\dot{\hat{\vartheta}}_n(t) = -\hat{\vartheta}_n(t) + \frac{\underline{g}_n K_{s_n}^2 S_n^{4p-2}\gamma_n(\bar{x}_n)}{2\delta_n^2}, \tag{31}$$

where K_n, δ_n are positive constants and

$$\gamma_n = \|h_n(\bar{x}_n)\|^2 + \|\tilde{k}_n\|^2\sum_{j=1}^{n-1}\|\frac{\partial\alpha_{n-1}}{\partial x_j}h_j\|^2,$$

$$\begin{aligned} \psi_n =& \sum_{j=1}^{n-1}(\frac{\partial\alpha_{n-1}}{\partial x_j}x_{j+1})^2 + \sum_{j=0}^{n-1}(\frac{\partial\alpha_{n-1}}{\partial y_d^{(j)}}y_d^{(j+1)})^2 \\ &+ \sum_{l=1}^{n-1}\sum_{j=0}^{n-1}(\frac{\partial\alpha_{n-1}}{\partial\varsigma_l^{(j)}}\varsigma_l^{(j+1)})^2 + \sum_{j=1}^{n-1}(\frac{\partial\alpha_{n-1}}{\partial\hat{\vartheta}_j}\dot{\hat{\vartheta}}_j)^2. \end{aligned}$$

Using the Young's inequality as previous steps it can be verified that

$$\dot{V}_n \le -\sum_{j=1}^{n}K_j\underline{g}_j K_{s_j}S_j^{2p} - \frac{1}{2}\sum_{j=1}^{n}\tilde{\vartheta}_j^2 + \sum_{j=1}^{n}\mu_{c_j}, \tag{32}$$

with $\mu_{c_n} = \frac{n\delta_n^2+n^2+n-1}{2\underline{g}_n} + \frac{1}{2\underline{g}_n}\sum_{j=1}^{n-1}\bar{g}_j^2 + \frac{1}{2}\vartheta_i^2$. This completes the controller design procedure.

We state the main results in the following Theorem 1.

Theorem 1: Suppose the investigated system (1) satisfies Assumptions 1-3. The virtual controllers $\alpha_i(i = 1,2,\cdots,n-1)$ in (10) and (19), the actual controller u in (30), and the adaptation law in (9), (23) and (31) are constructed on the set Ω_s. Chosen appropriate positive design parameters $K_i, \delta_i, i = 1,\cdots,n$ and given $S(0) \in \Omega_s$, the resulting closed-loop system has the following properties:

(i) The error signals $S_i(t), i = 1,\cdots,n$ can converge to a bounded set: $-\underline{D}_{s_i} \le S_i(t) \le \bar{D}_{s_i}$, where the bounds $\underline{D}_{s_i} = \bar{K}_{a_i}\sqrt[2p]{1-e^{-\frac{2pc}{\rho}}}$, $\bar{D}_{s_i} = \bar{K}_{b_i}\sqrt[2p]{1-e^{-\frac{2pc}{\rho}}}$, $\rho = \min\{2p\bar{K}_i(i = 1,\cdots,n), 1\}$;

(ii) The full state constraints are not violated;

(iii) Signals in the closed-loop system are all bounded.

Proof: (i) Denote a change of gain parameters: $\bar{K}_i = K_i\underline{g}_i$, $c = \sum_{i=1}^{n}\mu_{c_i}$, then the inequality of (32) can be expressed as

$$\dot{V}_n \le -\sum_{i=1}^{n}\bar{K}_i K_{s_i}S_i^{2p} - \frac{1}{2}\sum_{i=1}^{n}\tilde{\vartheta}_i^2 + c. \tag{33}$$

From Lemma 2 and (33), the following inequality holds

$$\begin{aligned} \dot{V}_n \le& -\sum_{i=1}^{n}\bar{K}_i[(1-q(S_i))\log\frac{k_{a_i}^{2p}(t)}{k_{a_i}^{2p}(t)-S_i^{2p}} \\ &+ q(S_i)\log\frac{k_{b_i}^{2p}(t)}{k_{b_i}^{2p}(t)-S_i^{2p}}] - \frac{1}{2}\sum_{i=1}^{n}\tilde{\vartheta}_i^2 + c. \end{aligned} \tag{34}$$

Since $\rho = \min\{2p\bar{K}_i(i = 1,\cdots,n), 1\}$, it yields

$$\dot{V}_n \le -\rho V_n + c. \tag{35}$$

For $S(0) \in \Omega_s$, then from Lemma 1 we get that $S(t) \in \Omega_s, \forall t > 0$.

Multiplying both sides of (35) by $e^{\rho t}$, (35) can be rewritten as $\frac{d(e^{\rho t}V_n)}{dt} \le ce^{\rho t}$. Then integrating it over $[0,t]$, it has

$$V_n(t) \le (V_n(0) - \frac{c}{\rho})e^{-\rho t} + \frac{c}{\rho}. \tag{36}$$

Then, it is easy to obtain

$$\begin{aligned} &(1-q(S_i))\frac{k_{a_i}^{2p}}{k_{a_i}^{2p}-S_i^{2p}} + q(S_i)\frac{k_{b_i}^{2p}}{k_{b_i}^{2p}-S_i^{2p}} \\ &\le e^{2p[(V_n(0)-\frac{c}{\rho})e^{-\rho t}+\frac{c}{\rho}]}. \end{aligned}$$

Since $-k_{a_i}(t) \le S_i \le k_{b_i}(t)$, we have $k_{a_i}^{2p}(t) - S_i^{2p} > 0$ and $k_{b_i}^{2p}(t) - S_i^{2p} > 0$. When $S_i > 0$, $q = 1$. Multiplying both the sides by $k_{b_i}^{2p}(t) - S_i^{2p} > 0$ and applying manipulations, the following inequality holds

$$\frac{k_{b_i}^{2p}}{k_{b_i}^{2p}-S_i^{2p}} \le e^{2p[(V_n(0)-\frac{c}{\rho})e^{-\rho t}+\frac{c}{\rho}]}. \tag{37}$$

Then,

$$S_i(t) \le k_{b_i}(t) \sqrt[2p]{1-e^{-2p[(V_n(0)-\frac{c}{\rho})e^{-\rho t}+\frac{c}{\rho}]}}. \tag{38}$$

As $t \to \infty$, $S_i \le \bar{K}_{b_i} \sqrt[2p]{1-e^{-\frac{2pc}{\rho}}}$. Similarly, when $S_i \le 0$, $q=0$, we obtain $S_i \ge -\bar{K}_{a_i} \sqrt[2p]{1-e^{-\frac{2pc}{\rho}}}$. It could be concluded as $-\bar{K}_{a_i} \sqrt[2p]{1-e^{-\frac{2pc}{\rho}}} \le S_i \le \bar{K}_{b_i} \sqrt[2p]{1-e^{-\frac{2pc}{\rho}}}$.

Remark 5: For getting the bounds of the error S_i as small as possible, we should provide small c and large ρ by selecting suitable parameters δ_i for $i=1,\cdots n$ and β. More precisely, we should let δ_i be as small as possible and β as large as possible.

(ii) Since $x_1 = S_1 + y_d$ and $S_1 \in \Omega_s$ which has been proved in (i), then $\underline{k}_{c_1}(t) \le x_1 \le \bar{k}_{c_1}(t)$ where $\underline{k}_{c_1}(t) = y_d(t) - k_{a_1}(t), \bar{k}_{c_1}(t) = k_{b_1}(t) + y_d(t)$. Because $x_i = S_i + \alpha_{i-1}, i=2,\cdots,n$, $S_i \in \Omega_s$, we can also prove that $\underline{k}_{c_i}(t) \le x_i \le \bar{k}_{c_i}(t)$ where $\underline{k}_{c_i}(t) = \alpha_{i-1}(t) - k_{a_i}(t)$ and $\bar{k}_{c_i}(t) = k_{b_i}(t) + \alpha_{i-1}(t)$ for $i=2,\cdots,n$. Thus we conclude that the full state constraints are not violated.

(iii)The error signals $S_i(t)$ and the states $x_i(t), i = 1,\cdots,n$ are all bounded, as shown in (i) and (ii). In Section 3 we have proved α_1 and $\dot{\alpha}_1$ are bounded, by signal chasing, we can progressively get that $\alpha_i, \dot{\alpha}_i, i = 2,\cdots,n-1, \hat{\vartheta}_n$ are bounded. Then with the help of Assumption 2-3 we derive $u(t)$ is bounded. Since S_1 and y_d are bounded, y is also bounded. Thus, we have all the closed-loop signals are bounded. □

3.2. Control design for system with disturbances

We also can modify the proposed control to handle system with bounded disturbances. Consider the plant (1) with disturbances

$$\begin{aligned}
&\dot{x}_i(t) = f_i(\bar{x}_{i+1}(t)) + d_i(t),\ i=1,2,\cdots,n-1,\\
&\dot{x}_n(t) = f_n(\bar{x}_n(t), u(t)) + d_n(t),\\
&y(t) = x_1(t),
\end{aligned} \tag{39}$$

where $|d_i(t)| \le D_i$. Note that $D_i, i=1,\cdots,n$ are constant disturbance bounds.

Based on the stabilizing function and controller we proposed in Subsection 3.1, we augment them with compensation terms as follows:

$$\begin{aligned}
\alpha_{1,d} &= \alpha_1 - \frac{\hat{D}_1}{\underline{g}_1}\tanh(\frac{\eta_1}{\lambda_1}),\\
\alpha_{i,d} &= \alpha_i + \dot{\alpha}_{i-1,d} - \dot{\alpha}_{i-1} - \frac{\hat{D}_i}{\underline{g}_i}\tanh(\frac{\eta_i}{\lambda_i}),\\
u_d &= u + \dot{\alpha}_{n-1,d} - \dot{\alpha}_{n-1} - \frac{\hat{D}_n}{\underline{g}_n}\tanh(\frac{\eta_n}{\lambda_n}).
\end{aligned} \tag{40}$$

The adaptive estimates $\hat{D}_i$ of the disturbance bounds is designed as

$$\dot{\hat{D}}_i = \omega_i(\eta_i \tanh(\frac{\eta_i}{\lambda_i}) - \sigma\hat{D}_i), \tag{41}$$

where $\eta_i = K_{s_i}S_i^{2p-1}$ and $\lambda_i, \omega_i, \sigma$ are positive constants.

Theorem 2: Consider the system with bounded disturbances (39) satisfies Assumptions 1-3. The augmented control in (40) and adaptive law for disturbance bounds in (41) are constructed on the set Ω_s. If $S_0 \in \Omega_s$, then the full state constraints are not violated and all the signals in the closed-loop system are bounded.

Proof: Construct the Lyapunov function as $V_d = V_n + \sum_{i=1}^n \frac{\tilde{D}_i^2}{2\omega_i}$, where V_n is defined in (27). Computing the time derivative of V_d, we have

$$\begin{aligned}
\dot{V}_d \le &-\sum_{j=1}^n K_j \underline{g}_j K_{s_j} S_j^{2p} - \frac{1}{2}\sum_{j=1}^n \tilde{\vartheta}_j^2 + \sum_{j=1}^n \mu_{c_j}\\
&+\sum_{i=1}^n D_i(|\eta_i| - \eta_i \tanh(\frac{\eta_i}{\lambda_i}))\\
&+\sum_{i=1}^n \tilde{D}_i(\omega_i^{-1}\dot{\hat{D}}_i - \eta_i \tanh(\frac{\eta_i}{\lambda_i})).
\end{aligned} \tag{42}$$

Using the identity $|\eta_i| - \eta_i \tanh(\frac{\eta_i}{\lambda_i}) \le 0.2785\lambda_i$, it yields

$$\dot{V}_d \le -\rho_1 V_d + c_1, \tag{43}$$

where $\rho_1 = \min\{\rho, \sigma\omega_i\}$ and $c_1 = c + \sum_{i=1}^n D_i(\sigma D_i/2 + 0.2785\lambda_i)$ are positive constants. Then, the rest of the proof process are the same with Theorem 1 and we omitted it here. We conclude that the full state constraints are not violated and all the signals in the closed-loop system are bounded. □

4. SIMULATION

To verify the effectiveness of the proposed ABLF-based control, in this section we provide two simulation examples. One is a three dimensional numerical nonlinear system with time-varying full state constraints, and the other is a single-link robot with constraints.

Example 1: Consider the following nonlinear systems

$$\begin{aligned}
&\dot{x}_1(t) = 0.1x_1^2(t) + x_2(t),\\
&\dot{x}_2(t) = 0.1x_1(t)x_2(t) - 0.2x_1(t) + (1+x_1^2(t))x_3(t),\\
&\dot{x}_3(t) = 0.1x_1(t)x_3(t) + 0.2u(t),\\
&y(t) = x_1(t)
\end{aligned} \tag{44}$$

with $\theta_1 = 0.1$, $\theta_2 = [0.1, -0.2]^T$, $\theta_3 = 0.1$, $g_1(x_1) = 1$, $g_2(\bar{x}_2) = 1 + x_1^2(t)$ and $g_3(\bar{x}_3) = 0.2$. The objective of $y(t)$ is to track the desired trajectory $y_d(t) = 0.5 * \sin(t)$. The asymmetric time-varying full state constraints are $\underline{k}_{c_1}(t) \le x_1(t) \le \bar{k}_{c_1}(t)$, $\underline{k}_{c_2}(t) \le x_2(t) \le \bar{k}_{c_2}(t)$ and $\underline{k}_{c_3}(t) \le x_3(t) \le \bar{k}_{c_3}(t)$ in which $\bar{k}_{c_1}(t) = 0.6 + \sin(t)$, $\underline{k}_{c_1}(t) = -0.6 + \sin(t)$, $\bar{k}_{c_2}(t) = 0.8 + 0.8\cos(t)$, $\underline{k}_{c_2}(t) = -1 - 0.5\sin(t)$, $\bar{k}_{c_3}(t) = 1 + 0.2\cos(t)$ and $\underline{k}_{c_3}(t) = -1 + 0.2\cos(t)$. The initial values of the states are $x_1(0) = 0.5, x_2(0) = -0.2, x_3(0) = 0.1$.

Based on the design procedure in Section 3, we choose ABLF for system (44) as follows:

$$V_3 = \sum_{i=1}^{3}\Big(\frac{1-q(S_i)}{2}\log\big(\frac{k_{a_i}^4(t)}{k_{a_i}^4(t)-S_i^4}\big) + \frac{q(S_i)}{2}\log\big(\frac{k_{b_i}^4(t)}{k_{b_i}^4(t)-S_i^4}\big)\Big) + \sum_{i=1}^{3}\frac{1}{2}\tilde{\vartheta}_i^2, \tag{45}$$

with $k_{a_1}(t) = y_d(t) - \underline{k}_{c_1}(t)$, $k_{b_1}(t) = \bar{k}_{c_1}(t) - y_d(t)$, $k_{a_2}(t) = \alpha_1(t) - \underline{k}_{c_2}(t)$, $k_{b_2}(t) = \bar{k}_{c_2}(t) - \alpha_1(t)$, $k_{a_3}(t) = \alpha_2(t) - \underline{k}_{c_3}(t)$ and $k_{b_3}(t) = \bar{k}_{c_3}(t) - \alpha_2(t)$. The virtual and actual controllers are designed as

$$\alpha_1 = -(K_1 + \bar{k}_1(t))S_1 - K_{s_1}S_1^3\frac{(\dot{y}_d)^2}{2} - \frac{K_{s_1}S_1^3\gamma_1\hat{\vartheta}_1}{2\delta_1^2},$$

$$\alpha_2 = -(K_2 + \bar{k}_2(t))S_2 - K_{s_2}\tilde{k}_2^2 S_2^3\frac{\psi_2}{2} - \frac{K_{s_2}S_2^3\gamma_2\hat{\vartheta}_2}{2\delta_2^2} - \frac{K_{s_1}^2 S_1^6}{2K_{s_2}S_2},$$

$$u = -(K_3 + \bar{k}_3(t))S_3 - K_{s_3}\tilde{k}_3^2 S_3^3\frac{\psi_3}{2} - \frac{K_{s_3}S_3^3\gamma_3\hat{\vartheta}_3}{2\delta_3^2} - \frac{K_{s_2}^2 S_2^6}{2K_{s_3}S_3},$$

in which $K_i, \delta_i (i = 1,2,3)$ are positive constants and all the other variables have been defined in Section 3. The adaption laws are given as

$$\dot{\hat{\vartheta}}_i = -\hat{\vartheta}_i + \frac{g_i K_{s_i}^2 S_i^3 \gamma_i}{2\delta_i^2}, \quad i = 1,2,3, \tag{46}$$

with the initial adaptive laws $\hat{\vartheta}_1(0) = 0.01$, $\hat{\vartheta}_2(0) = 0.02$ and $\hat{\vartheta}_3(0) = 0.01$.

The simulation parameters are selected as $K_1 = 1.5$, $K_2 = 5$, $K_3 = 2$, $\delta_1 = \delta_2 = \delta_3 = 0.01$ and $\beta = 50$. Figs.1-3 illustrate the simulation results. Fig. 1 shows that the system output tracking performance y coincides with the desired trajectory $y_d(t)$. As a result a good tracking performance is achieved. Fig. 2 shows the trajectories of the states x_1, x_2 and x_3 respectively. From this figure we can see that the asymmetric time-varying full state constraints are not violated all the time. In accordance with Fig. 3, it is observed that the control input signal u is bounded and the peaks in the initial control input are due to the system uncertainties.

Example 2: Consider a single-link robot [17] whose dynamic equations are

$$M\ddot{q} + \frac{1}{2}mgl\sin q = u, \quad y = q, \tag{47}$$

where q is the angle, u the input torque, M the moment of inertia, g the acceleration due to gravity, m and l the mass and the length of the link. The manipulator angle and angle velocity are constrained. The robot parameters are $m = 1, l = 1, M = 0.5$ and $g = 9.8$ as in [17]. To be consistent with the notion of this article, the above dynamics (47) can be rewritten in the form

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= \frac{1}{M}(u - \frac{1}{2}mgl\sin x_1), \\ y &= x_1, \end{aligned} \tag{48}$$

where $x_1 = q$, $x_2 = \dot{q}$. The states are constrained in $\Omega_x = \{\underline{k}_{c_1}(t) < x_1(t) < \bar{k}_{c_1}(t), \underline{k}_{c_2}(t) < x_2(t) < \bar{k}_{c_2}(t)\}$, where; $\underline{k}_{c_1}(t) = -0.8 - 0.2\sin(0.5t)$, $\bar{k}_{c_1}(t) = 0.8 + 0.2\cos(0.5t)$, $\underline{k}_{c_2}(t) = -1 - 0.5\sin(0.5t)$, and $\bar{k}_{c_2}(t) =$

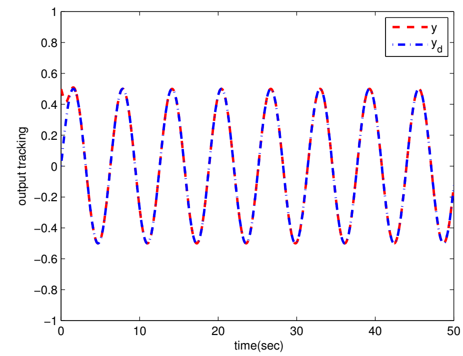

Fig. 1. The trajectories of y and y_d.

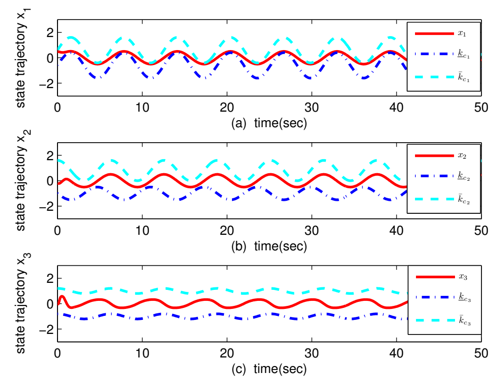

Fig. 2. The trajectories of x_1, x_2 and x_3.

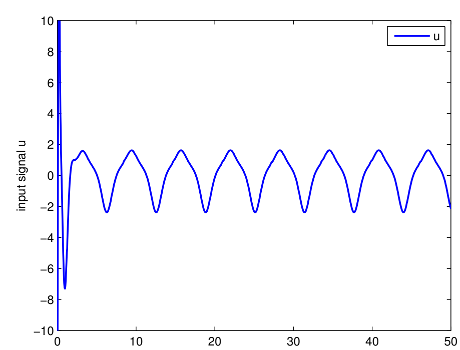

Fig. 3. The trajectories of control signal u .

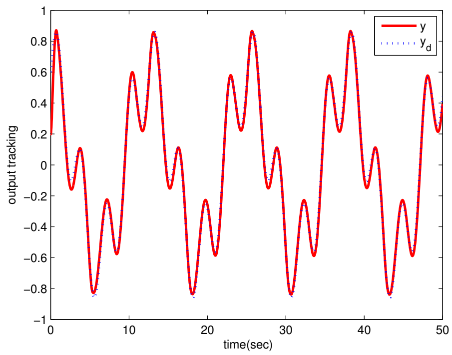

Fig. 4. The trajectories of y and y_d.

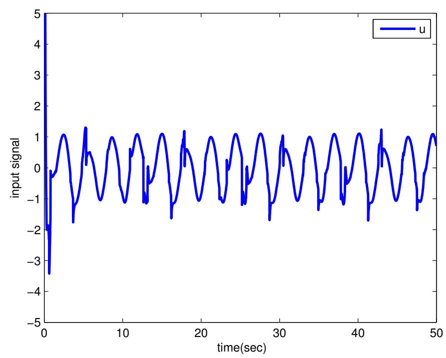

Fig. 5. The trajectories of control signal u .

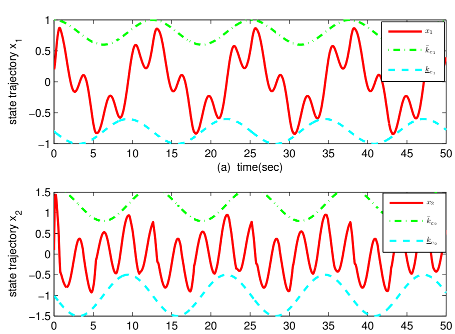

Fig. 6. The state trajectories of x_1 and x_2 with their constraints.

$1.2+0.4\cos(0.5t)$. The initial values of the states are $x_1(0)=0.2$, $x_2(0)=0.1$, the reference signal is $y_d(t)=0.3\sin(2t)+0.6\cos(0.5t)$.

Based on the design parameters $K_1=1.5$, $K_2=1.5$, $\delta_1=\delta_2=0.1$ and $\beta=10$, the simulation plots are shown in Figs. 4-6.

Remark 6: In this example, we use the same robot model with [17]. [17] has proved that all the signals in the closed-loop system are uniformly bounded. Moreover, a good tracking performance is achieved in [17] without violating the constant states constraints. Note that the method of [17] can not solve the problem of asymmetric time-varying state constraints. Compared with [17], we can solve the problem of not only the symmetric constant state constraints but also the asymmetric time-varying state constraints. It is worth mentioning that the reference signal in our example is multiple-frequency, compared with which in [17] only is a simple harmonic signal.

5. CONCLUSIONS

In this paper, a time-varying ABLF-based adaptive control has been developed for a class of nonlinear pure-feedback systems with asymmetric time-varying full state constraints. By employing the mean value theorem, the system is transformed into a strict-feedback structure with non-affine terms. Based on the transformed uncertain system, a modified backstepping design is constructed with the help of ABLF. The proposed control approach not only can guarantee that the time-varying full state constraints are not violated and all the closed loop signals remain bounded but also can deal with the unknown functions non-affine terms. It is shown that the tracking performance is propitious without violation of any state constraints. It could be concluded that the ABLF-based adaptive control is an effective method to deal with the problem of time-varying full state constraints. Two simulation examples demonstrate the effectiveness of the proposed method.

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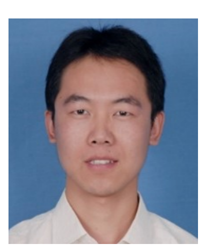

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