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A Simple Method to Design Robust Fractional-order Lead Compensator

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Abstract: This paper proposes a generalized and simple analytical method to design robust fractional-order lead compensator (FOLC). The aim of the proposed fractional-order compensator is to adjust the system's Bode phase curve to achieve the required phase margin at a specified frequency. The structure selected in this paper is more generalized and novel. It is easy to implement for a real world application. The method proposed is frequency domain and parameters of fractional compensator are selected from the plant information and specifications. This FOLC satisfies the specifications on static error constant, K_{ss}, gain crossover frequency, ω_c and phase margin, φ_m. The applicability of the proposed method is demonstrated with illustrative examples. From the simulation results obtained, it is observed that FOLC gives robust and stable performance as compared to existing FOLC and integer-order lead compensator (IOLC).

Keywords: Fractional calculus, fractional-order lead compensator, fractional-order system, integer-order lead compensator, robust performance.

1. INTRODUCTION

Fractional calculus is a three centuries old field of applied mathematics. It is a generalization of the ordinary differentiation and integration to non-integer orders (including complex-orders) [1]. It can be considered as the super set of integer-order calculus. Fractional calculus is also known as "Generalized Integral and Differential calculus" and sometimes "Calculus of Arbitrary Order". Research community in most of the fields of science, engineering and mathematics found worth to give importance to fractional calculus [2–7]. Fractional derivatives are best suitable for understanding an increasing number of physical, biological, economical, social phenomena and other fields. It is extensively used for modeling and analysis in the areas of viscoelasticity, electrical engineering, electrochemistry, biology, biophysics and bioengineering, signal and image processing, mechanics, mechatronics, physics, and control theory. It has been proved that the use of fractional calculus in modeling provides detailed description and deeper insight of natural and man-made systems. The extension of non-integer order controllers to fractional order shows improved performance [8–12].
The control system theory can include either the FO dynamic system to be controlled or the FO control or both [4, 11]. Flexibility in the design and guaranteed robust closed-loop performance, are its attractive features. The introduction of transient and frequency response of the non-integer integral/derivative and its application to control is reported in [2]. Various structures of FO controllers are reported in the literature [8, 10]. A $PI^{\lambda}D^{\mu}$ controller, involving an integrator of order λ and a differentiation of order μ $(\lambda, \mu \in \mathbb{R})$ is proposed in [9] and shown that this FO controller gives better performance as compared to classical PID controller for the given FO system. FO control algorithms and performance superiority of the *CRONE*(Commaande Robuste d'Ordre Non Entier) method over the *PID* controller is presented in [13]. A frequency domain approach of designing fractional-order *PID* controllers is also studied in [14]. Research is progressing to develop new tuning rules for fractional-order controller (FOC) [10]. Four representatives of FOC in the literature [11] are, TID (tilt-integral-derivative) controller [15], *CRONE* generation controller [13], $PI^{\lambda}D^{\mu}$ controller [9] and fractional-order lead-lag compensator [10]. From application point of view lead-lag compensator is known as controller next to PID.

This paper presents simple design methodologies for FOLC. In the literature, it is observed that compensator design methods are categorized as graphical which are trial-error methods and analytical methods. Graphical approaches are popular due to their simplicity, particularly when plant models are unknown and only experimental data is available [16]. Graphical methods modify the mag-

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nitude curve of the Bode plot, moving the gain crossover frequency, ω_c to the right and resulting in decrease in the obtained phase margin. In order to maintain the specified phase margin, this phase lag must be compensated by increasing phase angle. The designed compensator should be able to provide more phase. Another approach is analytical and can be carried out by computer programs which provides the basis for a more sophisticated design and performance analysis. A simple method for finding analytical solutions to lag/lead cascade and general second-order compensation design problems in the frequency domain are given in [12,16,17] for integer-order (IO) system. Various analytical and graphical methods for the synthesis of lead-lag compensators that achieve design specifications on the phase margin and the gain crossover frequency are given in [10, 18, 19]. Analytical methods can be used for the design of IO compensator in order to guarantee desired φ_m at specified ω_c. The FOLC structure proposed by [10] is difficult for real time implementation and only few methods are available to obtain its integer-order approximation [3]. The highlights of this paper are:

- A simple method is proposed for generalized IO and FO plant structures.
- The method is flexible and parameters of FOLC are selected from the plant information and specifications.
- The FOLC structure considered is generalized and novel.
- There are several approximation techniques available for the term s^α in the proposed FOLC and it has been successfully implemented using various common hardware platforms like DSP, FPGA, Microcontroller, PLC, etc.
- Simulation results obtained, shows that proposed FOLC gives robust and stable performance as compared with existing FOLC and integer-order lead compensator (IOLC).

The organization of this paper is as follows: Some preliminary concept of linear fractional-order system and its stability are introduced in Section 2. In Section 3, FOLC problem formulation and the design steps are discussed. The applicability of the proposed method is demonstrated by illustrative design examples and simulation results in Section 4. Conclusion is given in Section 5.

2. PRELIMINARY

The history of fractional calculus (also known as non-integer calculus) begin at the end of the 17^{th} century [8]. From the last couple of decades, the non-integer calculus has emerged as a powerful mathematical tool and very popular in the research community of almost all the fields of science and engineering [2–7]. It is generalization of the conventional calculus to real or complex orders [33].

Formally the real order generalization is introduced as follows:

$$D^\alpha = \begin{cases} \frac{d^\alpha}{dt^\alpha} & \alpha > 0 \\ 1 & \alpha = 0 \\ \int_a^t (d\tau)^{-\alpha} & \alpha < 0 \end{cases}$$

with $\alpha \in \mathbb{R}$. Some common definitions of fractional derivative and integrals are listed as follows :

(i) Riemann-Liouville:
Integral:

$$J_c^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_c^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}}\, d\tau, \tag{1}$$

where $\alpha \in \mathbb{R}^+$ and $\Gamma(\alpha)$ is Gamma of α.
Derivative:

$$D^\alpha f(t) = \frac{d^m}{dt^m}\left[\frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau\right], \tag{2}$$

where, $m \in \mathbb{Z}^+$, m is an integer, $\alpha \in \mathbb{R}^+$ and $(m-1) < \alpha < m$.

(ii) Grunwald-Letnikov:
Integral:

$$D^{-\alpha} = \lim_{h \longrightarrow 0} h^\alpha \sum_{m=0}^{(t-a)/h} \frac{\Gamma(\alpha+m)}{m!\Gamma(\alpha)} f(t-mh). \tag{3}$$

Derivative:

$$D^\alpha = \lim_{h \longrightarrow 0} \frac{1}{h^\alpha} \sum_{m=0}^{(t-a)/h} (-1)^m \frac{\Gamma(\alpha+1)}{m!\Gamma(\alpha-m+1)} f(t-mh). \tag{4}$$

(iii) Caputo:

$$D^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^m(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau, \tag{5}$$

where $(m-1) < \alpha < m$, and $f^m(.)$ is the m^{th} derivative of $f(t)$ with respect to time.

2.1. Fractional-order system

The fractional calculus has been successfully applied for most of the engineering problems related to electronic circuits, chaotic, hyperchaotic systems, biological system and economics [5, 6]. With the help of today's high-performance computing in this powerful tool has solved many problems in electrical circuits, filters, oscillators, memristor elements, Liu chaotic and hyperchaotic systems, chaos control, and the synchronization of fractional-order chaotic systems [37]. The design of FO controller has been of interest to many researchers. In fact classical integer-order PID controllers represent a special case of

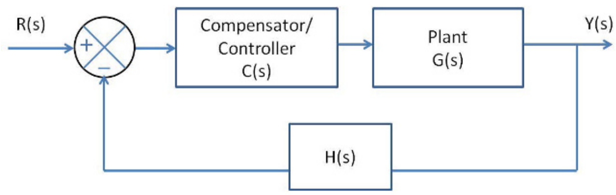

Fig. 1. Block diagram of closed-loop FO system.

fractional-order PID. Most of the classical control strategies are generalised and some new novel control strategies are also developed. FO controllers have been applied to a variety of processes which further enhance the robustness and performance.

The continuous-time transfer function of fractional-order (FO) system is of the form:

$$\frac{Y(s)}{R(s)} = \frac{b_m s^{\beta_m} + b_{m-1} s^{\beta_{m-1}} + \ldots + b_0 s^{\beta_0}}{a_n s^{\alpha_n} + a_{n-1} s^{\alpha_{n-1}} + \ldots + a_0 s^{\alpha_0}}, \tag{6}$$

where where, $\alpha_n \geqslant \beta_m$, $\alpha_n \in \mathbb{R}$ and $\beta_m \in \mathbb{R}$.

Fig. 1 represents the block diagram of a general closed-loop system. $Y(s)$ and $R(s)$ are the Laplace transforms of the output $y(t)$ and input $r(t)$. $G(s)$ is the system transfer function and $C(s)$ is the controller/compensator with unity feedback component $H(s) = 1$. Polynomials in (6) are pseudo-polynomials with fractional-orders [21].

2.2. Stability of FO system using Riemann surfaces

The stability of FO systems can be studied by obtaining the solution of FO differential equations that characterize the system [4]. The characteristic equation is given by,

$$a_n s^{\alpha_n} + a_{n-1} s^{\alpha_n - 1} + \ldots + a_0 s^{\alpha_0} = 0. \tag{7}$$

The pseudo-polynomials in (7) is a multi-valued function whose domain can be seen as a Riemann surface with number of sheets. The principal sheet is defined by $-\pi < arg(s) < \pi$. In the case of $\alpha_i \in Q^+$, that is $\alpha_i = 1/v$, v being a positive integer and there are v sheets of the Riemann surface. The angels of Riemann surface are calculated by (7),

$$s = \mid s \mid e^{j\phi}, (2k+1)\pi < \phi < (2k+3)\pi, \tag{8}$$

$$k = -1, 0, \ldots, v-2,$$

where $j = \sqrt{-1}$ and $k = -1$ is the principal Riemann sheet. These sheets are transformed to another plane called w-plane with the relation $w = s^{\alpha}$ [28]. The regions of these sheets on the w-plane [4, 36, 37] are,

$$w = \mid w \mid e^{j\theta}, \alpha(2k+1)\pi < \theta < \alpha(2k+3)\pi, \tag{9}$$

$$\text{where, } k = -1, 0, \ldots, v-2.$$

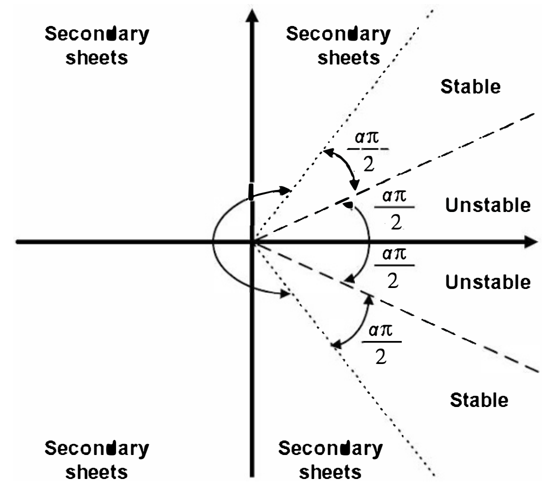

Fig. 2. Stability regions in w-plane [20].

In general (7) is not a polynomial and it will have an infinite number of roots and only a finite number of roots will be on the principal sheet of the Riemann surface. The roots which are in the secondary sheets of the Riemann surface are iso-damped as shown in Fig. 2. The only roots that are in the principal sheet of the Riemann surface are responsible for a dynamic behaviour like damped oscillation, oscillation of constant amplitude and oscillation of increasing amplitude. For the case of systems, whose characteristics equation is a polynomial of the complex variable $w = s^{\alpha}$ the stability condition is expressed as [8],

$$\mid arg(w_i) \mid > \frac{\alpha\pi}{2}, \tag{10}$$

where w_i are the roots of the characteristic polynomial in w. For the particular case of $\alpha = 1$, the well known stability condition for linear time-invariant systems is recovered,

$$\mid arg(w_i) \mid > \frac{\pi}{2}. \tag{11}$$

2.3. Compensator structure

The primary function of a compensator is to reshape phase curve of a system and provide phase lead angle to achieve the required phase margin at a specified frequency [23].

The available lead compensator structures are as follows:

Integer-order Lead Compensator (IOLC) : It consists of a gain, one pole and one zero. The traditional structure given by [23] is:

$$C(s) = k_c \left(\frac{s + \frac{1}{\lambda}}{s + \frac{1}{x\lambda}} \right) = k_c x \left(\frac{\lambda s + 1}{x\lambda s + 1} \right). \tag{12}$$

Here, k_c is the gain of compensator, $\frac{1}{\lambda}$ - is the zero frequency, $\frac{1}{x\lambda}$ - is the pole frequency, and $x < 1$ to achieve the phase lead.

Fractional-order Lead Compensator (FOLC) : The conventional compensator structure is modified by introducing a new parameter α. FO compensator structures observed in the literature are:

FOLC proposed by [10] :

$$C(s) = k_c \left(\frac{s+1/\lambda}{s+1/x\lambda}\right)^{\alpha} \quad (13)$$

$$= k_c x^{\alpha} \left(\frac{\lambda s+1}{x\lambda s+1}\right)^{\alpha}. \quad (14)$$

Here α is the fractional term and indicates the order of compensator.

FOLC structure explored in this paper is :

$$C(s) = k_c \left(\frac{s^{\alpha}+1/\lambda}{s^{\alpha}+1/x\lambda}\right), \quad (15)$$

$$= k_c x \left(\frac{\lambda s^{\alpha}+1}{x\lambda s^{\alpha}+1}\right), \quad (16)$$

where, $0 < x < 1$ and $0 \leq \alpha \leq 2$.

The compensator structure (13) proposed by [8, 10] is difficult to realize and only few methods are available to obtain its IO approximation. The FOLC structure (15) is not extensively used in the literature. This (15) is more generalized and a factor s^{α} can be easily approximated by existing approximation techniques and implemented using common hardware's like DSP, FPGA, Micro-controller, PLC, etc. for real world applications.

This paper proposes a simple analytical method to design a more generalized FOLC as given in (15). This frequency domain method is flexible and compensator parameters are selected from the plant information and specifications. The specification like gain cross over frequency and phase margin can be provided easily as compared to compensator (13).

3. FOLC PROBLEM FORMULATION

A proposed method to design FOLC is discussed in this section. In the selected compensator structure (15), the value of the compensator gain $k' = k_c x$ can be set in order to fulfill the static error constant specification for the compensated system. For a general plant model of the form (system type n) [10],

$$G(s) = \frac{k\prod_i(\tau_i s+1)}{s^n\prod_j(\tau_j s+1)}, \quad (17)$$

where, $n \in \mathbb{R}$. The generic structure (17) of the system with n poles at origin is considered for design and analysis. The expression for static error constant k_{ss} is:

$$k_{ss} = \lim_{s\to 0}[s^n C(s)G(s)], \quad (18)$$

$$= \lim_{s\to 0}\left[s^n k'\left(\frac{\lambda s^{\alpha}+1}{x\lambda s^{\alpha}+1}\right)\frac{k\prod_i(\tau_i s+1)}{s^n\prod_j(\tau_j s+1)}\right], \quad (19)$$

$$= k'k. \quad (20)$$

That is,

$$k' = k_c x = \frac{k_{ss}}{k}. \quad (21)$$

By this much amount of gain i.e k', FOLC can fulfill the static error constant specification.
Now for a given phase margin at gain cross over frequency, the open loop transfer function becomes,

$$C(j\omega)G(j\omega)\mid_{\omega=\omega_c} = e^{j(-\pi+\phi_m)}. \quad (22)$$

Replacing $C(j\omega)$ given by (15) and evaluating at $\omega = \omega_c$ we get,

$$G(j\omega_c)k_c x\left(\frac{\lambda(j\omega_c)^{\alpha}+1}{x\lambda(j\omega_c)^{\alpha}+1}\right) = e^{j(-\pi+\varphi_m)}. \quad (23)$$

After substitution and rearranging (23), we get

$$\left(\frac{\lambda(j\omega_c)^{\alpha}+1}{x\lambda(j\omega_c)^{\alpha}+1}\right) = \frac{e^{j(-\pi+\varphi_m)}}{k_c x G(j\omega_c)}. \quad (24)$$

The right hand side of the (24) can be represented in complex form as,

$$\left(\frac{\lambda(j\omega_c)^{\alpha}+1}{x\lambda(j\omega_c)^{\alpha}+1}\right) = a+jb, \quad (25)$$

where a and b is real and imaginary part of right hand side of (24). Let,

$$T = \lambda, \qquad V = x\lambda. \quad (26)$$

Therefore, (25) becomes,

$$\frac{T(j\omega_c)^{\alpha}+1}{V(j\omega_c)^{\alpha}+1} = a+jb. \quad (27)$$

Replacing,

$$j^{\alpha} = cos(\alpha\frac{\pi}{2}) + j.sin(\alpha\frac{\pi}{2}), \quad (28)$$

and substituting in (27), we have,

$$\frac{1+T\omega_c^{\alpha}[cos(\alpha\pi/2)+jsin(\alpha\pi/2)]}{1+V\omega_c^{\alpha}[cos(\alpha\pi/2)+jsin(\alpha\pi/2)]} = a+jb. \quad (29)$$

Separating real and imaginary parts,

$$\frac{(1+T\omega_c^{\alpha}cos(\alpha\pi/2))+jT\omega_c^{\alpha}sin(\alpha\pi/2)}{(1+V\omega_c^{\alpha}cos(\alpha\pi/2))+jV\omega_c^{\alpha}.sin(\alpha\pi/2)} = a+jb. \quad (30)$$

After rationalization and by equating real and imaginary parts of (30), we get

$$a = \frac{1+TV\omega_c^{2\alpha}+(T+V)\omega_c^{\alpha}cos(\alpha\pi/2)}{1+V^2\omega_c^{2\alpha}+2V\omega_c^{\alpha}cos(\alpha\pi/2)}, \quad (31)$$

$$b = \frac{T\omega_c^{\alpha}sin(\alpha\pi/2)-V\omega_c^{\alpha}sin(\alpha\pi/2)}{1+V^2\omega_c^{2\alpha}+2V\omega_c^{\alpha}cos(\alpha\pi/2)}. \quad (32)$$

To find the value of T and V, this can be simplified as,

$$T\omega_c^{\alpha}cos(\alpha\pi/2)+TV\omega_c^{2\alpha}+V\omega_c^{\alpha}cos(\alpha\pi/2)(1-2a)$$

$$-aV^2\omega_c^{2\alpha}+1-a=0, \tag{33}$$

$$T\omega_c^{\alpha}sin(\alpha\pi/2)-V\omega_c^{\alpha}sin(\alpha\pi/2)-2bV\omega_c^{\alpha}cos(\alpha\pi/2) \\ -bV^2\omega_c^{2\alpha}-b=0. \tag{34}$$

This (33) and (34) are solved using MATLAB $R2013$ to obtain the values of T and V using different values of α. The values of λ and x are obtained using (26), by keeping necessary condition for lead compensator as,

$$0<x<1. \tag{35}$$

Therefore, starting from lowest value of α, find the values of λ and x and check whether $0<x<1$. If the condition is not satisfied, repeat with small increment in α till the condition (35) gets satisfied.

It is clear that lower the value of α or x longer is the distance between the zero and pole of the compensator and vice versa. The zero-pole distance will be the maximum possible (minimum value of parameter x) and the phase curve of the compensator is the flattest possible and also variations in a frequency range centered at ω_c will not produce a significant phase change as in integer-order cases, thus improving the robustness of the system.

Design steps:

The procedure for designing a FOLC involves following steps:

1. Select the desired design specifications $(k_{ss}, \varphi_m, \omega_c)$.
2. Select the structure (15) :

$$C(s)=k'\left(\frac{\lambda s^{\alpha}+1}{x\lambda s^{\alpha}+1}\right). \tag{36}$$

3. From (21), determine gain k' to satisfy the requirement on the given static error constant,

$$k'=k_cx=\frac{k_{ss}}{k}. \tag{37}$$

4. Find the values of a and b using (24),

$$a+jb=\frac{e^{j(-\pi+\varphi_m)}}{k'G(j\omega_c)}. \tag{38}$$

5. Starting from the lowest value of α, find values T and V by solving (33) and (34). This also gives values of λ and x from (26).
6. Check the condition (35),

$$0<x<1. \tag{39}$$

7. If condition is satisfied, terminate the algorithm. If not satisfied then repeat step 5 with small increment in α.
8. Plug all the parameter value in compensator (36) and test its performance with plant.

4. DESIGN EXAMPLES

The applicability of the proposed method is presented with illustrative examples. The considered examples are popular control problems and well cited in the literature.

Example No. 1: Integer-order Plant Model

This section presents design of FOLC for a plant $G(s)$ a well known D.C. motor model given in [8, 10] ,

$$G(s)=\frac{2}{s(0.5s+1)}. \tag{40}$$

At the gain crossover frequency, ω_c = 10 rad/sec, plant has a magnitude of -28.1188 dB and a phase of -169.65 deg. Following design specifications are selected as given in [10],

- Velocity error constant, k_v = 20.
- Gain crossover frequency, ω_c = 10 rad/sec.
- Phase margin, φ_m = 0.277π = 50 deg.

Solution:

1. Selected structure of FO compensator (36).
2. The gain k' can be obtained by substituting in (21),

$$k' = \frac{k_v}{k}=\frac{20}{2}=10. \tag{41}$$

3. From (24) the values of a and b obtained are,

$$a+jb = \frac{e^{j(-\pi+0.277\pi)}}{10G(j10)}, \tag{42}$$

$$= 1.9900+j1.5937. \tag{43}$$

Therefore,

$$a=1.9900, \qquad b=1.5937. \tag{44}$$

4. Solving (33) and (34) for the different values of α with condition $(0<x<1)$ and it is satisfied for $\alpha=0.7$. The values of T and V for $\alpha=0.7$ are,

$$T=0.4125, \qquad V=0.019851. \tag{45}$$

(46)

From (26) we get,

$$\lambda=0.4125, \qquad x=0.0481. \tag{47}$$

5. Finally the generalised structure FOLC is obtained as,

$$C_1(s)=10\left(\frac{0.4125s^{0.7}+1}{0.019851s^{0.7}+1}\right). \tag{48}$$

For the same plant FOLC compensator cited in the literature [10] is:

$$C_2(s)=10\left(\frac{0.6957s+1}{0.002123s+1}\right)^{0.48}. \tag{49}$$

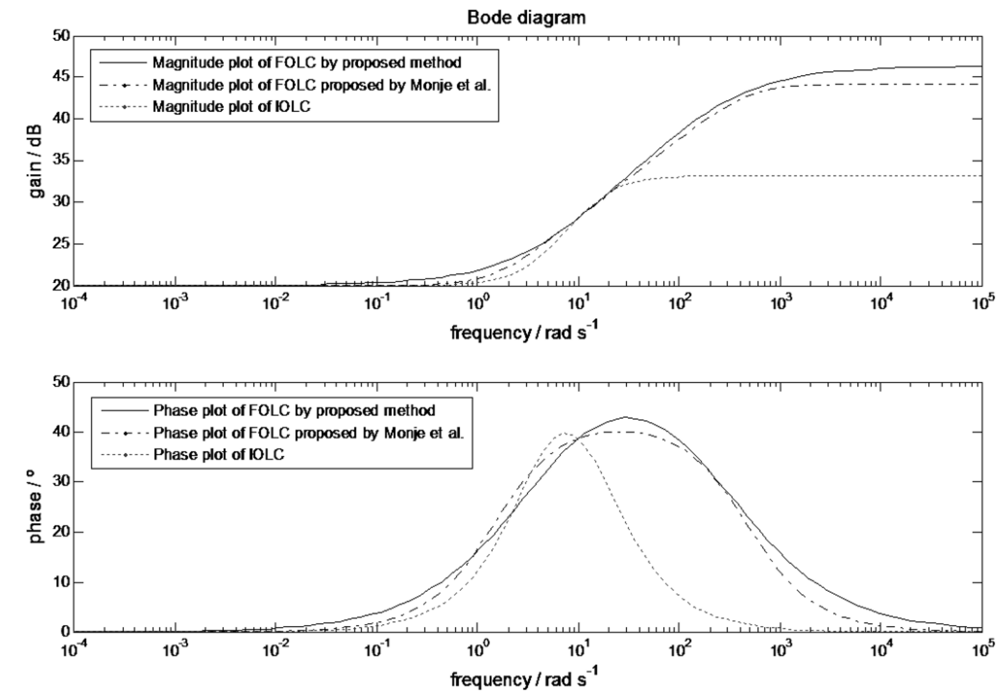

Fig. 3. Bode plot of compensators for Example-1.

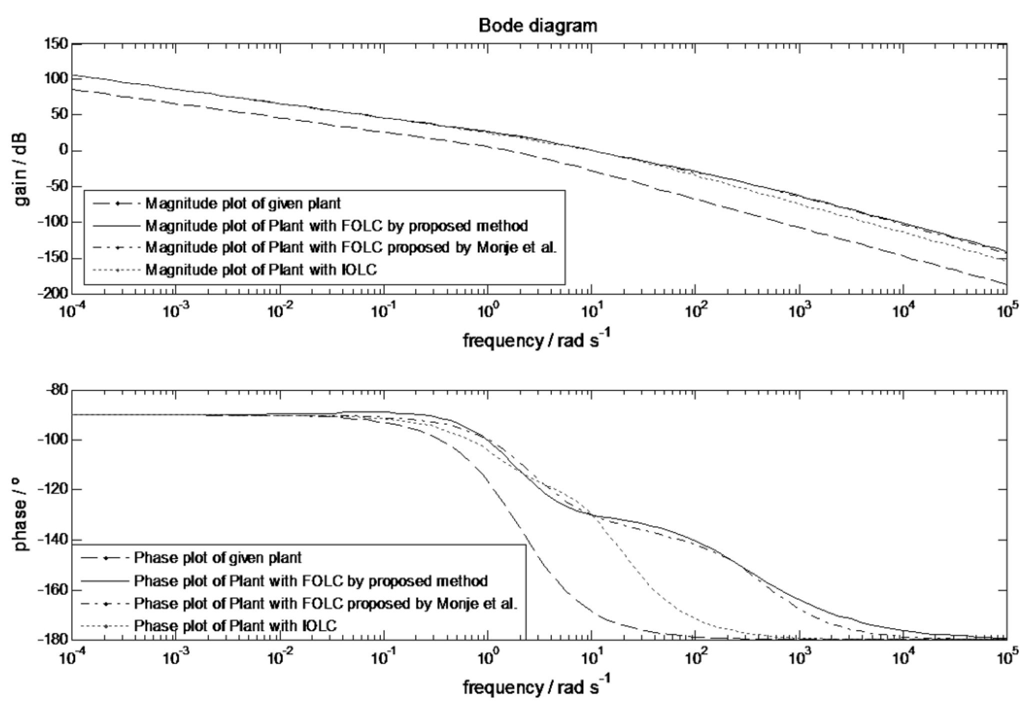

Fig. 4. Bode plot of plant with compensators for Example-1.

Again for same example, if the complete procedure is repeated with α =1, the integer-order compensator obtained is,

$$C_3(s) = 10\left(\frac{0.2829s+1}{0.06211s+1}\right). \tag{50}$$

The frequency response of compensators and compensated system is discussed here. The Bode plot of IOLC ((50)- $C_3(s)$-dotted line) and generalised FOLC ((48) - $C_1(s)$-continuous line) and FOLC ((49)- $C_2(s)$- dotted-continuous line) [10] are shown in Fig. 3. The Bode plot of the compensated systems are shown in Fig. 4. It is clear that both the structures given in (13) and (15) satisfies the

Table 1. Comparison of phase angle (ϕ deg) and phase margin (φ_m deg) of considered plant (40) with proposed FOLC, FOLC proposed by [10]. and with IOLC.

Freq. (rad/sec)	Plant with $C_1(s)$		Plant with $C_2(s)$		Plant with $C_3(s)$	
	ϕ	φ_m	ϕ	φ_m	ϕ	φ_m
5	-125.533	54.467	-122.989	57.011	-120.704	59.296
10	-130.000	50.000	-130.000	50.000	-130.000	50.000
15	-131.303	48.697	-132.708	47.292	-138.639	41.361
20	-132.085	47.915	-134.230	45.770	-145.478	34.522
25	-132.737	47.263	-135.264	44.736	-150.693	29.307
30	-133.348	46.652	-136.052	43.948	-154.686	25.315
35	-133.939	46.061	-136.698	43.302	-157.795	22.205
40	-134.517	45.483	-137.255	42.745	-160.264	19.737
45	-135.083	44.917	-137.753	42.247	-162.261	17.739
50	-135.637	44.363	-138.208	41.792	-163.905	16.095

condition and achieve the design specifications. However, it must be noted that the IOLC ($C_3(s)$) gives overphase though the desired specifications are achieved. With the fractional-order structure ((48) - $C_1(s)$), phase curve of the compensator is more flatter and variations in a frequency range centered at ω_c will not produce a significant phase change. From Table 1 and Fig. 4 it should be noted that phase variation (near gain crossover frequency) of plant (40) with compensator ($C_1(s)$ - (48)) is lesser than plant with compensator ($C_2(s)$ - (49)) and ($C_3(s)$ - (50)). Flatter frequency response at ω_c, results in the improved robust performance.

Example No. 2: FO Model given by I. Podlubny

The proposed design method is also tested for FO model of reheating-furnace system given by [3]:

$$G(s) = \frac{1}{0.8s^{2.2} + 0.5s^{0.9} + 1}. \tag{51}$$

At the gain crossover frequency, ω_c= 1.7028 rad/sec the phase margin of the plant is 3.5977 deg. Following design specifications are selected,

- Gain crossover frequency, ω_c = 7.3 rad/sec.
- Phase margin, φ_m = 0.305π = 55 deg.

Solution:

1. Same procedure is followed with plant (51) and compensator (36).
2. As there is no static error constant specification provided, for both IOLC and FOLC,
$$k' = 1 \tag{52}$$
3. From (24) the obtained values of a and b are,
$$a = 20.1283, \qquad b = 57.7756. \tag{53}$$
4. By solving (33) and (34), the values of T and V for $\alpha = 0.8$ are
$$T = 12.431, \qquad V = 0.00119. \tag{54}$$
From (26),
$$\lambda = 12.431, \qquad x = 0.0000104. \tag{55}$$
5. Necessary condition of lead compensator (35) is satisfied for $\alpha_{min} = 0.8$.
6. The resulting fractional compensator (36) is,
$$C_1(s) = \left(\frac{12.431s^{0.8} + 1}{0.00119s^{0.8} + 1}\right), \tag{56}$$

and integer-order compensator obtained is,

$$C_2(s) = \left(\frac{8.8274s + 1}{0.04535s + 1}\right). \tag{57}$$

Bode plot of FOLC (56) and IOLC (57) are shown in Fig. 5. It can be observed from Fig. 6 and Table 2 that phase curve of the plant (51) with compensator ($C_1(s)$ -(56)) is flatter than plant with ($C_2(s)$ -(57)). Variations in the frequency range near ω_c will not produce a significant phase change and improves the robustness of the system. The gain margin measured for plant without any compensator is 2.46 dB and with compensator is 103 dB. Fig. 7 (A)- show root locus plot of plant (51), Fig. 7 (B)- show root locus plot of plant with IOLC ((57)- $C_2(s)$) and Fig. 7 (C)- show root locus plot of plant with FOLC ((56)- $C_1(s)$). From the Fig. 7 (C) it is clear that plant with FOLC ($C_1(s)$) is relatively more stable (stable for more amount of gain) as compared to plant with IOLC ($C_2(s)$).

Example No. 3: FO Integrating type Plant Model

Design method is also extended to FO integrating plant:

$$G(s) = \frac{0.5}{s^{2.5} + 0.5s}. \tag{58}$$

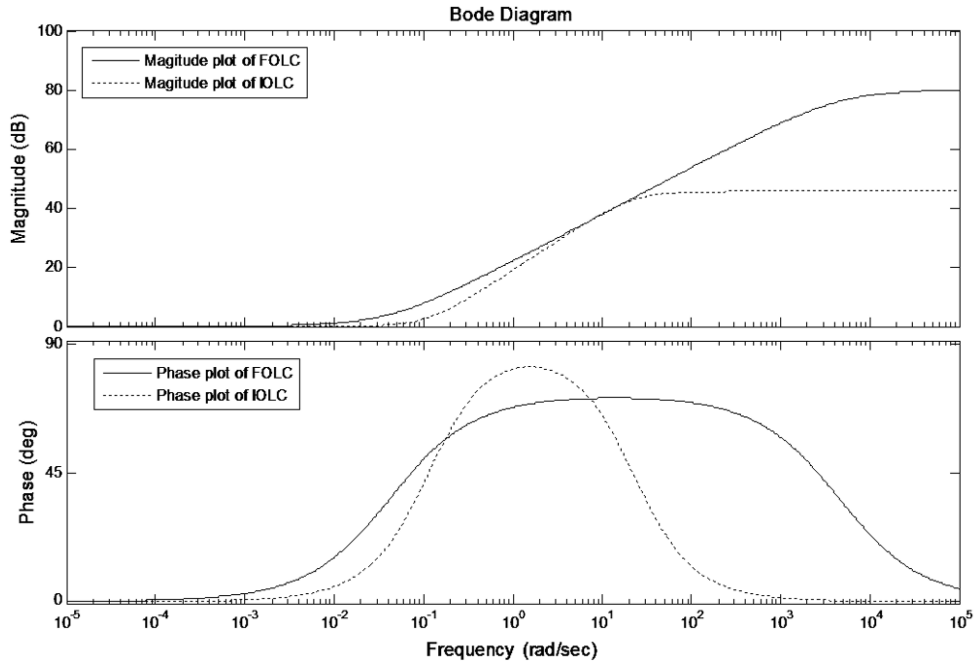

Fig. 5. Bode plot of compensators for Example-2.

Table 2. Comparison of phase angle (deg) and phase margin (deg) of given plant (51) with FOLC (56) and with IOLC (57).

Freq. (rad/sec)	Plant with FOLC $C_1(s)$		Plant with IOLC $C_2(s)$	
	Phase(ϕ)	margin(φ_m)	Phase(ϕ)	margin(φ_m)
5	-123.8992	56.1008	-118.5371	61.4629
7.3	-125.0000	55.0000	-125.0000	55.0000
10	-125.6101	54.3899	-131.5534	48.4466
15	-126.1700	53.8300	-141.7632	38.2368
20	-126.4865	53.5135	-149.9110	30.0890
25	-126.7139	53.2861	-156.3787	23.6213
30	-126.8989	53.1011	-161.5292	18.4708
35	-127.0603	52.9397	-165.6697	14.3303
40	-127.2070	52.7930	-169.0397	10.9603
45	-127.3437	52.6563	-171.8186	8.1814
50	-127.4732	52.5268	-174.1396	5.8604

For this plant at gain crossover frequency, ω_c = 0.8678 rad/sec, the phase margin is -7.1457 deg. Following design specifications are selected,

- Velocity error constant, k_v = 10.
- Gain crossover frequency, ω_c = 8 rad/sec.
- Phase margin, φ_m = 0.194π = 35 deg.

Solution:

1) Similar procedure is repeated for this plant.

2) For $k' = 10$,

$$a = 6.7456, \qquad b = 34.9985. \tag{59}$$

3) By solving (33) and (34) the values of T and V for $\alpha = 0.9$ are,

$$T = 5.464, \qquad V = 0.00088. \tag{60}$$

$$\lambda = 5.464, \qquad x = 0.000161. \tag{61}$$

4) Necessary condition (35) is satisfied for α_{min}= 0.9.

5) The final FOLC obtained is,

$$C_1(s) = 10\left(\frac{5.464s^{0.9}+1}{0.00088s^{0.9}+1}\right). \tag{62}$$

and IOLC is,

$$C_2(s) = 10\left(\frac{4.5132s+1}{0.0205s+1}\right). \tag{63}$$

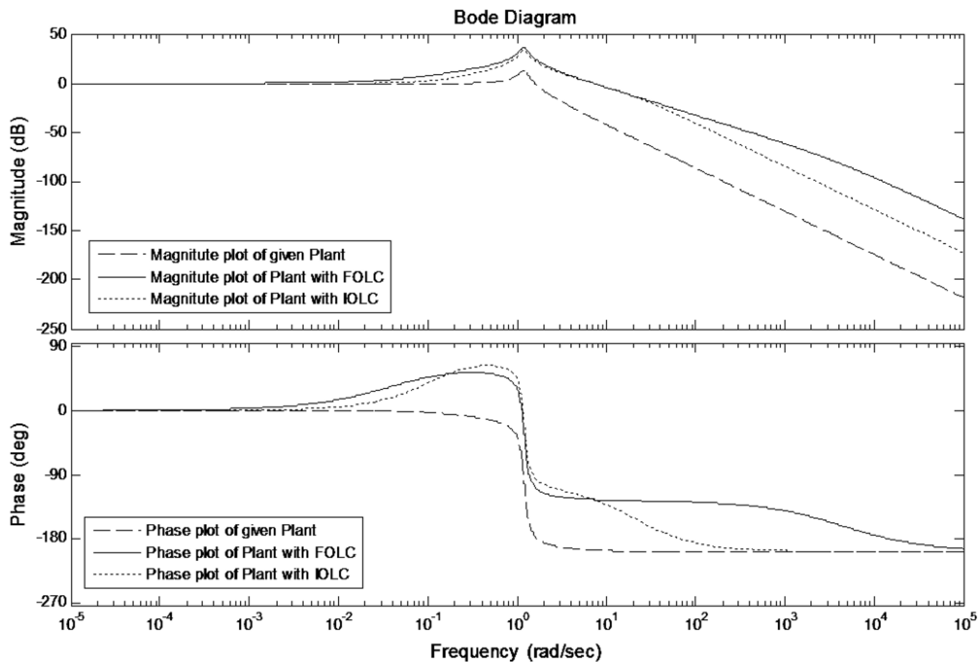

Fig. 6. Bode plot of plant with compensators for Example-2.

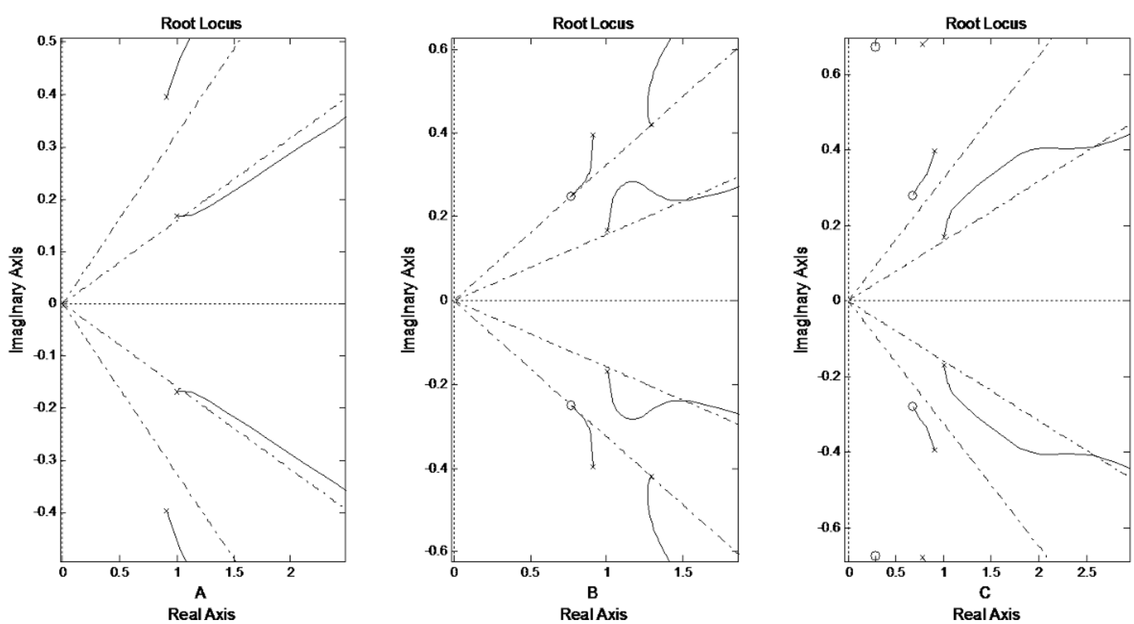

Fig. 7. Root locus plot of compensators for Example.2.

Bode plot of FOLC (62) and IOLC (63) are shown in Fig.8. It is observed that the phase curve of the plant (58) with FOLC ((62)-$C_1(s)$) is flatter than with IOLC ((63)-$C_2(s)$). The variations in a frequency range near ω_c will not produce a significant phase change (see Table 3 and Fig. 9). The gain margin obtained for plant without compensator is $-2.01\ dB$, plant with IOLC ($C_2(s)$) is 26.4 dB and with FOLC ($C_1(s)$) is 80 dB.

Fig. 10 (A) shows root locus of plant (58), Fig. 10 (B)-root locus plot of plant with IOLC ((63)-$C_2(s)$) and Fig. 10 (C)-root locus plot of plant with FOLC ((62)- $C_1(s)$). From Fig. 10 (C) it is clear that plant with FOLC (62) is relatively more stable (stable for a larger gain interval) as compared to plant with IOLC (63).

In the design specification, if required phase margin is increased beyond 44 deg then it fails to fulfill the phase margin specification using IOLC. If it is designed for $\varphi_m = 45$

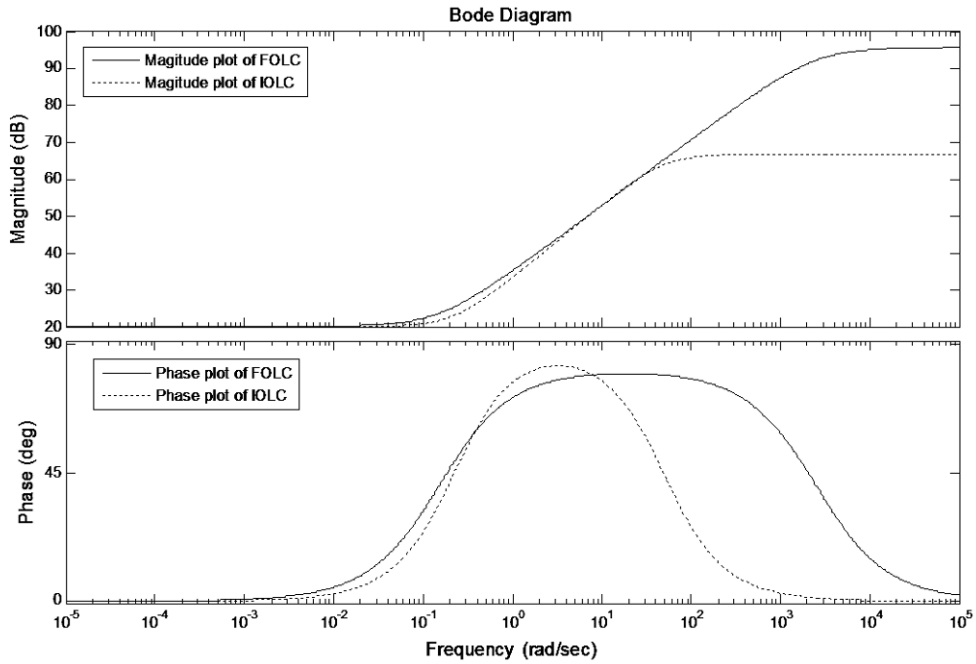

Fig. 8. Bode plot of compensators for Example-3.

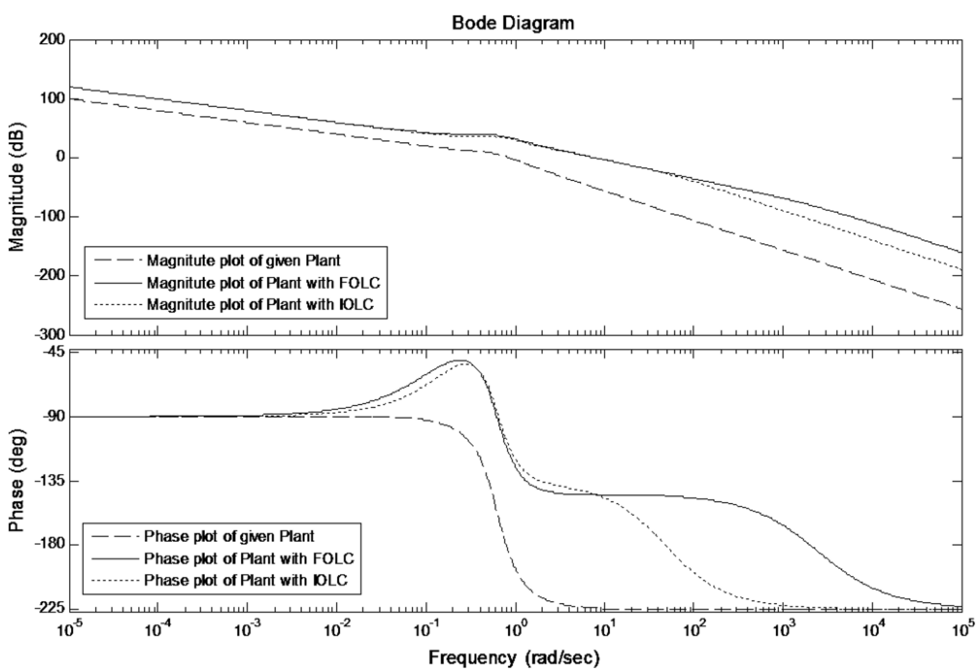

Fig. 9. Bode plot of plant with compensators for Example-3.

deg then it is unstable. The obtained IOLC for φ_m = 45 deg is,

$$IOLC = 10\left(\frac{4.4555s+1}{-0.0037s+1}\right). \tag{64}$$

Where as with the use of fractional structure it is possible to design a stable FOLC for φ_m = 45 deg. The FOLC obtained with same methodologies is,

$$FOLC = 10\left(\frac{3.5904s^{1.1}+1}{0.0124s^{1.1}+1}\right). \tag{65}$$

Discussion and Analysis

The paper presents a design method and simulation results of novel FOLC for generalized IO and FO plant structures. The contribution of this paper is proved using exten-

Table 3. Comparison of phase angle (deg) and phase margin (deg) of given plant (58) with FOLC (62)and with IOLC (63).

Freq.	Plant with FOLC $C_1(s)$		Plant with IOLC $C_2(s)$	
(rad/sec)	Phase(ϕ)	margin(φ_m)	Phase(ϕ)	margin(φ_m)
5	-144.7567	35.2433	-141.5253	38.4747
8	-145.0000	35.0000	-145.0000	35.0000
10	-145.0458	34.9542	-147.2181	32.7819
15	-145.1202	34.8798	-152.6045	27.3955
20	-145.2057	34.7943	-157.7214	22.2786
25	-145.3071	34.6929	-162.5041	17.4959
30	-145.4196	34.5804	-166.9171	13.0829
35	-145.5394	34.4606	-170.9517	9.0483
40	-145.6641	34.3359	-174.6175	5.3825
45	-145.7921	34.2079	-177.9355	2.0645
50	-145.9222	34.0778	-180.9900	-0.9900

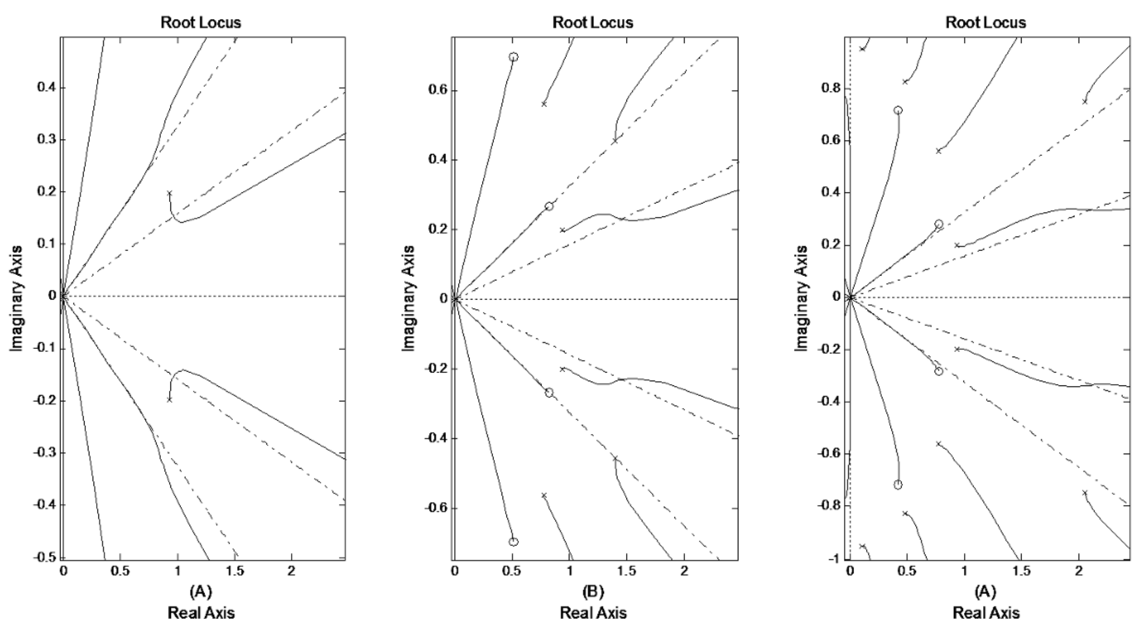

Fig. 10. Root locus plot of compensators for Example-3.

sive analysis and simulation exercise. The designed FOLC ensures minimum variation around the specified value of phase margin as compared to its IO counterparts. From Tables 1–3 it is observed that for a frequency range of $5-50$ rad/sec, the phase margin variation for Example.1 with proposed FOLC ($C_1(s)$) is 10.104 deg and with the FOLC ($C_2(s)$) is 15.219 deg. The phase variation for Example.1 with IOLC ($C_3(s)$) is 43.201 deg. This variation is especially very small if the plant has FO dynamics. For FO plant Example. 2 with FOLC the phase variation is 3.574 deg and with IOLC 55.6025 deg. Similarly for FO plant Example. 3 with FOLC the phase variation is 1.1655 deg and with IOLC 37.4847 deg. The root locus plots Fig. 7 and Fig. 10 shows that designed FOLC improves gain margin and hence stability as compared to IOLC.

5. CONCLUSION

A generalized and simple analytical method is proposed to design stable FOLC for generalized IO and FO plant structures. The FOLC structure is simple and easy to implement. The compensator parameter α in this generalized structure allows flexibility on the fulfillment of specifications of phase margin φ_m, gain crossover frequency ω_c and static error constant, k_{ss}. The simulation results proved that FOLC ensures minimum variation around the specified value of the phase margin φ_m as compared to its IO counterpart. The phase curve of the FOLC is flatter than IOLC. This phase variation is very small if the plant has FO dynamics. The performance of FOLC is robust and also improves the gain margin and hence the stability.

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