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Containment Control for Directed Networks Multi-agent System with Nonlinear Dynamics and Communication Time-delays

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Abstract: In this paper, containment control problem in directed networks for second-order multi-agent systems with inherent nonlinear dynamics and time-delays are investigated. A distributed control protocol is proposed for each follower using the relative states among neighboring agents. Based on Lyapunov-Razumikhin theorem, some sufficient conditions in terms of linear matrix inequalities (LMIs) are derived to ensure that all followers asymptotically converge to the convex hull spanned by the dynamic leaders. Finally, simulation results are presented to illustrate the effectiveness of the conclusion.

Keywords: Containment control, directed networks, multi-agent systems, nonlinear dynamics, time-delays.

1. INTRODUCTION

In the past decades, distributed cooperative control problem of a group of mobile autonomous agents have received considerable attention in a variety of research fields including biology, physics, robotics, sensor networks, artificial intelligence and control engineering. An important feature in distributed cooperative control of multiple agents is that each agent updates its own state based on the information from itself and its local neighbors. Therefore, the consensus is one of the key fundamental problems. The study of consensus algorithms can be dated back to Reynolds [1] and Vicsek [2], Jadbabaie [3]. Detailed information about the recent study of consensus algorithms in cooperative control can be found in Moreau [4] and Ren, Beard [5]. Recently, the leader-following consensus under directed or undirected graph was investigated. The leader-following consensus means a leader is designated and all the followers track the leader using the consensus algorithm. In some circumstances, the systems have multiple leaders that the containment control belongs to.

The containment control is that all followers will ultimately converge to the convex hull formed by multiple static or dynamic leaders. The initial study of containment can be found in Ji [6]. Then the research team which led by the Ren proposed enormous interesting results for the containment control problem from different perspectives [7–12] in recent years. Such as finite-time attitude containment control, containment control with multiple dynamic leaders, containment control in fixed and switching directed networks, adaptive coordination for multiple Lagrangian systems and containment control of multi-agent systems with general linear dynamics. Other study of containment can also be found in [13–16]. Due to the limited communication speed, the time required by sensor and computation, time-delays appears in almost all the real systems. Since the delays may degrade a system's performance or even destroy a system's stability, the investigations have been extensively conducted in this direction. Saber had studied the first-order continuous-time system with time-delays in [17, 18]. Tian [19, 20] and Lin [21, 22] also have rich achievements in this direction. When the delays are constant, the frequency-domain approach has been used to give the consensus conditions [23–26]. For the discrete-time multi-agent systems, some study can be found in [27, 28]. The single leader case and/or the leaderless case have been considered in these works. But in the study of the containment control problem, there have few works been done for the multi-agent systems with time-delays. To the best of our knowledge, only the containment control problem of the linear time-invariant singular swarm systems with a constant delay was investigated in [29]. Note that physical systems in practice are inherently nonlinear. Liu *et al*. researched coordinative control of multi-agent systems using distributed nonlinear output regulation in [30]. Yu *et al*. addressed second-order consensus of nonlinear multi-agent systems under fixed network topology in [31]. Formation control and containment control of nonlinear multi-agent systems was inves-

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tigated in [32, 33], respectively. Consensus of second-order multi-agents systems with nonlinear dynamics and time-delays was researched in [34]. Recently a paper [35] investigated containment of multi-agents systems with nonlinear dynamics and hybrid time-delays.

Motivated by the above analysis, in this paper, we investigate the distributed containment control problem for multi-agent systems with nonlinear dynamics and fixed communication time-delays under fixed directed network topology. The main contribution of this paper is twofold: first, the sufficient conditions in terms of linear matrix inequalities (LMIs) are given can deal with containment control problem of multi-agent systems with nonlinear dynamics and fixed communication time-delays. Second, the interaction topology among the followers is supposed to be directed, which is more common in reality.

This paper is organized as follows: In Section 2, some preliminaries are given, and the problem to be solved is formulated. In Section 3, containment control problem for nonlinear multi-agent system with fixed communication time-delays is investigated, respectively. Numerical simulations and conclusion are given in Sections 4 and 5, respectively.

2. PROBLEM FORMULATIONS AND PRELIMINARIES

Let $G(\bar{V},\varepsilon,A)$ be a directed graph of order n, where $\bar{V}=\{\bar{v}_1,\bar{v}_2,\ldots,\bar{v}_n\}$ is the set of nodes, $\varepsilon\in\bar{V}\times\bar{V}$ is the set of directed edges, and $A=[a_{ij}]\in\mathbb{R}^{n\times n}$ with nonnegative adjacency elements a_{ij} is adjacency matrix. A directed edge ε_{ij} in G is denoted by the ordered pair of node $(\bar{v}_j,\bar{v}_i)$, where $\bar{v}_i$ is defined as the parent node and $\bar{v}_j$ is defined as the child node, which means that node $\bar{v}_j$ can receive information from node$\bar{v}_i$. The adjacency elements associated with the edges are positive, that is, $\varepsilon_{ij}\in\varepsilon\Leftrightarrow a_{ij}>0$. Moreover, we assume $a_{ii}=0$ for all $\bar{v}_i\in\bar{V}$. In this paper, We use $N_i(t)$ to denote the neighbor set of agent $i$at the time t. For the multi-agent system, an agent is called a leader if the agent has no neighbor. An agent is called a follower if the agent has a neighbor.

Correspondingly, the Laplacian matrix $L=[l_{ij}]\in\mathbb{R}^{n\times n}$ of the directed graph is defined as

$$l_{ij}=\begin{cases}\sum_{j=1}^{n}a_{ij}, & i=j\\ -a_{ij}, & i\neq j.\end{cases}$$

Definition 1 [12]**:** The containment control is achieved for the second-order system under a certain control input if the position and velocity states of the followers asymptotically converge to the convex hull formed by those of the leaders.

Assumption 1 [10]**:** For each follower, there exists at least one leader that has a directed path to the follower.

Assumption 2 [33]**:** Given $\eta_1,\eta_2,\ldots\eta_{n-m}$ with $\sum_{i=1}^{n-m}\eta_i=1$, and $\eta_i\geq 0,i=1,2,\ldots n-m$. There exist nonnegative constants $\rho_1$and ρ_2 such that nonlinear function f satisfies

$$\begin{aligned}&\left\|f(t,x,v)-\textstyle\sum_{i=1}^{n-m}\eta_i f(t,r_i,s_i)\right\|\\ &\quad\leq\rho_1\left\|x-\textstyle\sum_{i=1}^{n-m}\eta_i r_i\right\|+\rho_2\left\|v-\textstyle\sum_{i=1}^{n-m}\eta_i s_i\right\|\end{aligned}$$

for any $x,v,r_i,s_i\in\mathbb{R}^m, i=1,2,\ldots n-m$, and $t\geq 0$.

Remark 1: The condition given in **Assumption 2** can guarantee that the leader-following network achieves the containment control objective in the presence of multiple leaders. It applies to all linear functions and some nonlinear functions. Furthermore, note that, if **Assumption 2** holds, then the Lipschitz condition is satisfied, but not vice versa.

Definition 2 [36]**:** Let Q be a set in a real vector space $W\subseteq\mathbb{R}^n$, The set Q is called convex if, for any $\tilde{x}$ and $\tilde{y}$ in Q, the point $(1-\theta)\tilde{x}+\theta\tilde{y}$ is in Q for any $\theta\in[0,1)$. The convex hull for a set of points $\tilde{X}=\{\tilde{x}_1,\tilde{x}_2,\ldots,\tilde{x}_n\}$ in W is the minimal convex set containing all points in $\tilde{X}$. We use $\mathrm{Co}(\tilde{X})$ to denote the convex hull of $\tilde{X}$. In particular,

$$\mathrm{Co}(\tilde{X})=\{\sum_{i=1}^{n}\tilde{a}_i\tilde{x}_i|\tilde{x}_i\in\tilde{X},\tilde{a}_i\in\mathbb{R}\geq 0,\sum_{i=1}^{n}\tilde{a}_i=1\}.$$

Assume that there are $m$leaders, where $m<n$, and $n-m$ followers. In this paper, we use $\Re$ and F to denote, respectively, the leader set and the follower set. Without loss of generality, we assume that agents 1 to $n-m(m<n)$ are followers and agents $n-m+1$ to n are leaders. Accordingly, Lcan be partitioned as

$$L=\begin{bmatrix}L_1 & L_2\\ \mathbf{0}_{m\times n-m} & \mathbf{0}_{m\times m}\end{bmatrix},$$

where $L_1\in\mathbb{R}^{(n-m)\times(n-m)}$, $L_2\in\mathbb{R}^{(n-m)\times m}$.

Lemma 1 [7]**:** All the eigenvalues of L_1 have positive real parts, each element of $-L_1^{-1}L_2$ is nonnegative, and the sum of each row of $-L_1^{-1}L_2$ is 1.

Lemma 2 [37]**:** Given a positive definite matrix $M\in\mathbb{R}^{n\times n}$, two constants γ_1 and γ_2 satisfying $\gamma_1<\gamma_2$, and a vector function $\omega:[\gamma_1,\gamma_2]\to\mathbb{R}^n$ such that the integrations concerned are well defined, the following inequality holds

$$\begin{aligned}&\left(\int_{\gamma_1}^{\gamma_2}\omega(s)ds\right)^{\mathrm{T}}M(\int_{\gamma_1}^{\gamma_2}\omega(s)ds)\\ &\quad\leq(\gamma_2-\gamma_1)\left(\int_{\gamma_1}^{\gamma_2}\omega^{\mathrm{T}}(s)M\omega(s)ds\right).\end{aligned}$$

Lemma 3 [38]**:** For any vectors p, $q\in\mathbb{R}^m$ and positive definite matrix $Z\in\mathbb{R}^{m\times m}$, the following matrix inequality holds: $2p^{\mathrm{T}}q\leq p^{\mathrm{T}}Zp+q^{\mathrm{T}}Z^{-1}q$.

3. MAIN RESULTS

Consider the second-order multi-agent system consisting of $n-m$ followers and m leaders. Each agent with nonlinear dynamics is represented as

$$\dot{x}_i(t) = v_i(t),$$
$$\dot{v}_i(t) = \begin{cases} f(t,x_i(t),v_i(t)), & i \in \Re \\ f(t,x_i(t),v_i(t)) + u_i(t), & i \in \mathcal{F}, \end{cases} \tag{1}$$

where $x_i(t)$, $v_i(t) \in \mathbb{R}^p$ and $u_i(t) \in \mathbb{R}^p$ are, respectively, the position, velocity and the control input of the ith agent. And $f(t,x_i(t),v_i(t))$ is the inherent nonlinear dynamics of the agents. In this paper, we consider $x_i(t)$ and $v_i(t)$ for $i = 1, 2, ..., n$ are in one dimension for simplicity. Then, we can generalize the results in one dimension to that in high dimensions.

Consider the time-delays is fixed, we use such containment control protocol for system (1)

$$u_i(t) = \sum_{j\in N_i(t)} a_{ij}[\alpha(x_j(t-\tau) - x_i(t-\tau)) + \beta(v_j(t-\tau) - v_i(t-\tau))]. \tag{2}$$

Let

$$X(t) = [\ x_1(t) \quad x_2(t) \quad \cdots \quad x_n(t)\ ]^{\mathrm{T}},$$
$$V(t) = [\ v_1(t) \quad v_2(t) \quad \cdots \quad v_n(t)\ ]^{\mathrm{T}},$$
$$F(t) = [\ f(t,x_1(t),v_1(t)) \quad \cdots \quad f(t,x_n(t),v_n(t))\ ]^{\mathrm{T}}.$$

Then the closed-loop system which is obtained by substituting (2) into (1) can be written as

$$\begin{bmatrix} \dot{X}(t) \\ \dot{V}(t) \end{bmatrix} = \begin{bmatrix} 0_{n\times n} & I_n \\ 0_{n\times n} & 0_{n\times n} \end{bmatrix} \begin{bmatrix} X(t) \\ V(t) \end{bmatrix} + \begin{bmatrix} 0_{n\times n} & 0_{n\times n} \\ -\alpha L & -\beta L \end{bmatrix} \begin{bmatrix} X(t-\tau) \\ V(t-\tau) \end{bmatrix} + \begin{bmatrix} 0_{n\times n} \\ I_n \end{bmatrix} F(t). \tag{3}$$

Let

$$X_{\mathrm{F}}(t) = [\ x_1(t) \quad x_2(t) \quad \cdots \quad x_{n-m}(t)\ ]^{\mathrm{T}},$$
$$V_{\mathrm{F}}(t) = [\ v_1(t) \quad v_2(t) \quad \cdots \quad v_{n-m}(t)\ ]^{\mathrm{T}}.$$

Then we have

$$\begin{bmatrix} \dot{X}_{\mathrm{F}}(t) \\ \dot{V}_{\mathrm{F}}(t) \end{bmatrix} = \begin{bmatrix} 0_{(n-m)\times(n-m)} & I_{(n-m)\times(n-m)} \\ 0_{(n-m)\times(n-m)} & 0_{(n-m)\times(n-m)} \end{bmatrix} \begin{bmatrix} X_{\mathrm{F}}(t) \\ V_{\mathrm{F}}(t) \end{bmatrix} + \begin{bmatrix} 0_{(n-m)\times(n-m)} & 0_{(n-m)\times(n-m)} \\ -\alpha L_1 & -\beta L_1 \end{bmatrix} \begin{bmatrix} X_{\mathrm{F}}(t-\tau) \\ V_{\mathrm{F}}(t-\tau) \end{bmatrix} + \begin{bmatrix} 0_{(n-m)\times m} & 0_{(n-m)\times m} \\ -\alpha L_2 & -\beta L_2 \end{bmatrix} \begin{bmatrix} X_{\mathrm{L}}(t-\tau) \\ V_{\mathrm{L}}(t-\tau) \end{bmatrix} + \begin{bmatrix} 0_{(n-m)\times(n-m)} \\ I_{n-m} \end{bmatrix} F_{\mathrm{F}}(t). \tag{4}$$

Let

$$Y(t) = \begin{bmatrix} X(t) \\ V(t) \end{bmatrix},\ Y_{\mathrm{F}}(t) = \begin{bmatrix} X_{\mathrm{F}}(t) \\ V_{\mathrm{F}}(t) \end{bmatrix},\ Y_{\mathrm{L}}(t) = \begin{bmatrix} X_{\mathrm{L}}(t) \\ V_{\mathrm{L}}(t) \end{bmatrix},$$

$$\bar{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ B_{11} = \begin{bmatrix} 0 & 0 \\ -\alpha & -\beta \end{bmatrix},\ B_{21} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Then (4) can be rewritten as

$$\dot{Y}_{\mathrm{F}}(t) = (I_{n-m}\otimes\bar{A})Y_{\mathrm{F}}(t) + (L_1\otimes B_{11})Y_{\mathrm{F}}(t-\tau) + (L_2\otimes B_{11})Y_{\mathrm{L}}(t-\tau) + (I_{n-m}\otimes B_{21})F(t,Y_{\mathrm{F}}(t)) \tag{5}$$
$$\dot{Y}_{\mathrm{L}}(t) = (I_m\otimes\bar{A})Y_{\mathrm{L}}(t) + (I_m\otimes B_{21})F(t,Y_{\mathrm{L}}(t)). \tag{6}$$

By defining the error transformation

$$E(t) = Y_{\mathrm{F}}(t) + L_1^{-1}L_2Y_{\mathrm{L}}(t), \tag{7}$$

the following error system can be obtained

$$\begin{aligned} \dot{E}(t) =&\ \dot{Y}_{\mathrm{F}}(t) + L_1^{-1}L_2\dot{Y}_{\mathrm{L}}(t) \\ =&\ AY_{\mathrm{F}}(t) + (L_1\otimes B_{11})Y_{\mathrm{F}}(t-\tau) \\ &+ (L_2\otimes B_{11})Y_{\mathrm{L}}(t-\tau) + (I_{n-m}\otimes B_{21})F(t,Y_{\mathrm{F}}(t)) \\ &+ L_1^{-1}L_2[AY_{\mathrm{L}}(t) + (I_m\otimes B_{21})F(t,Y_{\mathrm{L}}(t))] \\ =&\ AE(t) + (L_1\otimes B_{11})E(t-\tau) + (I_{n-m}\otimes B_{21})\bar{F}(t) \\ =&\ AE(t) + B_1E(t-\tau) + B_2\bar{F}(t), \end{aligned} \tag{8}$$

where

$$A = I_{n-m}\otimes\bar{A},\ B_1 = L_1\otimes B_{11},\ B_2 = I_{n-m}\otimes B_{21},$$
$$\bar{F}(t) = F(t,Y_{\mathrm{F}}(t)) + L_1^{-1}L_2F(t,Y_{\mathrm{L}}(t).$$

Then the containment control problem of nonlinear multi-agent system (1) is asymptotically solved by protocol (2) if the error system (8) is globally asymptotically stable.

Lemma 4: Suppose that Assumption 1 holds, $\Phi = (A + B_1)$ is Hurwitz stable if and only if the feedback gains α and β satisfy

$$\frac{\beta^2}{\alpha} > \max_{1\le k\le n-m} \frac{\mathrm{Im}^2(\lambda_k)}{\mathrm{Re}(\lambda_k)\,\|\lambda_k\|^2}. \tag{9}$$

Proof: Consider the characteristic polynomial of Φ, we have

$$\begin{aligned} \det(sI_{2(n-m)} - \Phi) &= \det(sI_{2(n-m)} - A - B_1) \\ &= \prod_{k=1}^{n-m}(s^2 + (\alpha+\beta s)\lambda_k) \\ &= 0, \end{aligned}$$

where λ_k ($k = 1, 2, ..., n-m$) is the eigenvalues of L_1. Hence,

$$s_{k1} = \frac{-\beta\lambda_k + \sqrt{\beta^2\lambda_k^2 - 4\alpha\lambda_k}}{2},$$
$$s_{k2} = \frac{-\beta\lambda_k - \sqrt{\beta^2\lambda_k^2 - 4\alpha\lambda_k}}{2}, \quad k = 1,2,...,n-m.$$

The rest of the proof had been done in [24,33]. In order to avoid duplication, it is omitted.

Remark 2: The condition that $\Phi = (A+B_1)$ is Hurwitz stable implies that there exists a positive definite matrix P satisfying $P\Phi^{\mathrm{T}} + \Phi P < 0$.

Denote

$$-L_1^{-1}L_2 = [r_{ij}],\ \ i \in \mathcal{F},\ \ j = 1,2,...n-m.$$

From Lemma 1, we have

$$\sum_{j=1}^{n-m} r_{ij} = 1,\ \ r_{ij} \geq 0,\ \ \forall i \in \mathrm{F},\ \ j = 1,2,...n-m.$$

Thus, we get

$$\begin{aligned}\|\bar{F}(t)\| &= \left\|F(t,Y_{\mathrm{F}}(t)) + L_1^{-1}L_2F(t,Y_{\mathrm{L}}(t)\right\| \\ &= \left\|\begin{bmatrix} \|F(t,\mathrm{x}_1(t),\mathrm{v}_1(t)) \\ \quad -\sum_{j=1}^{N-M} r_{1j}F(t,\mathrm{x}_{M+j}(t),\mathrm{v}_{M+j}(t))\| \\ \cdots \\ \|F(t,\mathrm{x}_M(t),\mathrm{v}_M(t)) \\ \quad -\sum_{j=1}^{N-M} r_{Mj}F(t,\mathrm{x}_{M+j}(t),\mathrm{v}_{M+j}(t))\| \end{bmatrix}\right\| \\ &\leq \rho_1 \left\|\begin{bmatrix} \|\mathrm{x}_1(t) - \sum_{j=1}^{N-M} r_{1j}\mathrm{x}_{M+j}(t)\| \\ \cdots \\ \|\mathrm{x}_M(t) - \sum_{j=1}^{N-M} r_{Mj}\mathrm{x}_{M+j}(t)\| \end{bmatrix}\right\| \\ &\quad + \rho_2 \left\|\begin{bmatrix} \|\mathrm{v}_1(t) - \sum_{j=1}^{N-M} r_{1j}\mathrm{v}_{M+j}(t)\| \\ \cdots \\ \|\mathrm{v}_M(t)) - \sum_{j=1}^{N-M} r_{Mj}\mathrm{v}_{M+j}(t)\| \end{bmatrix}\right\| \\ &\leq (\rho_1+\rho_2)\left\|Y_{\mathrm{F}}(t) + L_1^{-1}L_2Y_{\mathrm{L}}(t)\right\| \\ &= (\rho_1+\rho_2)\left\|E(t)\right\|. \end{aligned} \tag{10}$$

Theorem 1: Suppose that Assumptions 1 and Assumptions 2 hold. Protocol (2) asymptotically solves the containment control problem for system (1) with time-delays, if the feedback gains α and β satisfy (9), and there exists a positive denite matrix P and positive scalar k, k_1, k_2, γ satisfying

$$E_2 + \gamma\tau^2 F_2 < 0, \tag{11}$$

where

$$\begin{aligned} E_2 &= \begin{bmatrix} \Omega & -PB_1 \\ -B_1^{\mathrm{T}}P & -\gamma I_{2(n-m)} \end{bmatrix}, \\ \Omega &= P\Phi^{\mathrm{T}} + \Phi P + \frac{1}{k}PB_2B_2^{\mathrm{T}}P^{\mathrm{T}} \\ &\quad + k(\rho_1+\rho_2)^2 I_{2(n-m)}, \\ F_2 &= \begin{bmatrix} \Phi^{\mathrm{T}} \\ -B_1^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \Phi & -B_1 \end{bmatrix} \\ &\quad + \begin{bmatrix} \Omega_2 + \Omega_1 & 0 \\ 0 & \frac{1}{k_2}B_1^{\mathrm{T}}B_2B_2^{\mathrm{T}}B_1 \end{bmatrix}, \\ \Omega_1 &= \frac{1}{k_1}\Phi^{\mathrm{T}}B_2B_2^{\mathrm{T}}\Phi, \\ \Omega_2 &= (k_2+k_1+1)(\rho_1+\rho_2)^2 I_{2(n-m)}. \end{aligned}$$

Proof: Consider the following Lyapunov function candidate

$$\begin{aligned} V(t) =& E^{\mathrm{T}}(t)PE(t) \\ &+ \gamma\tau\int_{t-\tau}^{t}(s-t+\tau)\dot{E}^{\mathrm{T}}(s)\dot{E}(s)ds, \end{aligned} \tag{12}$$

where P is a positive denite matrix and γ, τ are positive scalar and fixed time-delay respectively.

The time derivative of this Lyapunov candidate along the trajectory of system (8) is

$$\begin{aligned} \dot{V}(t) =& 2E^{\mathrm{T}}(t)P\dot{E}(t) + \gamma\tau^2\dot{E}^{\mathrm{T}}\dot{E} \\ &- \gamma\tau\int_{t-\tau}^{t}\dot{E}^{\mathrm{T}}(s)\dot{E}(s)ds \\ =& 2E^{\mathrm{T}}(t)P[AE(t) + (B_1)E(t-\tau) \\ &+ (B_2)\bar{F}(t)] - \gamma\tau\int_{t-\tau}^{t}\dot{E}^{\mathrm{T}}(s)\dot{E}(s)ds \\ &+ \gamma\tau^2[AE(t) + (B_1)E(t-\tau) + (B_2)\bar{F}(t)]^{\mathrm{T}} \\ &\times [AE(t) + (B_1)E(t-\tau) + (B_2)\bar{F}(t)]. \end{aligned} \tag{13}$$

We define $\bar{E} = E(t) - E(t-\tau)$ and $\Phi = (A+B_1)$, then (13) can be rewritten as

$$\begin{aligned} \dot{V}(t) =& 2E^{\mathrm{T}}(t)P[(\Phi E(t) - B_1\bar{E} + B_2\bar{F}(t)] \\ &- \gamma\tau\int_{t-\tau}^{t}\dot{E}^{\mathrm{T}}(s)\dot{E}(s)ds \\ &+ \gamma\tau^2[\Phi E(t) - B_1\bar{E} + B_2\bar{F}(t)]^{\mathrm{T}} \\ &\times [\Phi E(t) - B_1\bar{E} + B_2\bar{F}(t)] \\ =& 2E^{\mathrm{T}}(t)P\Phi E(t) - 2E^{\mathrm{T}}(t)PB_1\bar{E} \\ &+ 2E^{\mathrm{T}}(t)PB_2\bar{F}(t)] - \gamma\tau\int_{t-\tau}^{t}\dot{E}^{\mathrm{T}}(s)\dot{E}(s)ds \\ &+ \gamma\tau^2[\Phi E(t) - B_1\bar{E} + B_2\bar{F}(t)]^{\mathrm{T}} \\ &\times [\Phi E(t) - B_1\bar{E} + B_2\bar{F}(t)]. \end{aligned} \tag{14}$$

We can know that from Lemma 2

$$\begin{aligned} &\bar{E}^{\mathrm{T}}(t)\bar{E}(t) \leq \tau\int_{t-\tau}^{t}\dot{E}^{\mathrm{T}}(s)\dot{E}(s)ds \\ &\Rightarrow -\gamma\tau\int_{t-\tau}^{t}\dot{E}^{\mathrm{T}}(s)\dot{E}(s)ds \leq -\gamma\bar{E}^{\mathrm{T}}(t)\bar{E}(t). \end{aligned}$$

So we have

$$\begin{aligned} \dot{V}(t) \leq& \begin{bmatrix} E(t) \\ \bar{E} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} P\Phi^{\mathrm{T}} + \Phi P & -PB_1 \\ -B_1^{\mathrm{T}}P & -\gamma I_{2(n-m)} \end{bmatrix} \begin{bmatrix} E(t) \\ \bar{E} \end{bmatrix} \\ &+ E^{\mathrm{T}}(t)[2PB_2]\bar{F}(t) \\ &+ \gamma\tau^2\left\{\begin{bmatrix} E(t) \\ \bar{E} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \Phi^{\mathrm{T}} \\ -B_1^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \Phi & -B_1 \end{bmatrix} \begin{bmatrix} E(t) \\ \bar{E} \end{bmatrix}\right. \\ &+ 2E^{\mathrm{T}}(t)\Phi^{\mathrm{T}}B_2\bar{F}(t) + 2\bar{E}^{\mathrm{T}}(-B_1)^{\mathrm{T}}B_2\bar{F}(t) \\ &\left. + \bar{F}(t)^{\mathrm{T}}B_2^{\mathrm{T}}B_2\bar{F}(t)\right\}. \end{aligned} \tag{15}$$

We can easily know from the definition of the matrix $B_2^T B_2 = I_{n-m}$. And by Lemma 3 and (10), we have

$$\begin{aligned}
&2E^{\mathrm{T}}(t)[\Phi^{\mathrm{T}}B_2]\bar{F}(t)\\
&\quad\leq \frac{1}{k_1}E^{\mathrm{T}}(t)(\Phi^{\mathrm{T}}B_2)(\Phi^{\mathrm{T}}B_2)^{\mathrm{T}}E(t)+k_1\bar{F}^{\mathrm{T}}(t)\bar{F}(t)\\
&\quad\leq \frac{1}{k_1}E^{\mathrm{T}}(t)(\Phi^{\mathrm{T}}B_2)(\Phi^{\mathrm{T}}B_2)^{\mathrm{T}}E(t)\\
&\qquad +k_1(\rho_1+\rho_2)^2E^{\mathrm{T}}(t)E(t)\\
&\quad= E^{\mathrm{T}}(t)[\frac{1}{k_1}(\Phi^{\mathrm{T}}B_2)(\Phi^{\mathrm{T}}B_2)^{\mathrm{T}}+k_1(\rho_1+\rho_2)^2]E(t)\\
&\quad= E^{\mathrm{T}}(t)[\frac{1}{k_1}\Phi^{\mathrm{T}}B_2B_2^{\mathrm{T}}\Phi+k_1(\rho_1+\rho_2)^2]E(t), \qquad (16)\\
&2\bar{E}^{\mathrm{T}}(t)[(-B_1)^{\mathrm{T}}B_2]\bar{F}(t)\\
&\quad\leq \bar{E}^{\mathrm{T}}(t)[\frac{1}{k_2}(B_1^{\mathrm{T}}B_2)(B_1^{\mathrm{T}}B_2)^{\mathrm{T}}]\bar{E}(t)+k_2\bar{F}^{\mathrm{T}}(t)\bar{F}(t)\\
&\quad\leq \bar{E}^{\mathrm{T}}(t)[\frac{1}{k_2}(B_1^{\mathrm{T}}B_2B_2^{\mathrm{T}}B_1]\bar{E}(t)\\
&\qquad +k_2(\rho_1+\rho_2)^2E^{\mathrm{T}}(t)E(t), \qquad (17)\\
&\bar{F}(t)^{\mathrm{T}}B_2^{\mathrm{T}}B_2\bar{F}(t)\\
&\quad=\bar{F}(t)^{\mathrm{T}}\bar{F}(t)\leqslant(\rho_1+\rho_2)^2E^{\mathrm{T}}(t)E(t), \qquad (18)\\
&2E^{\mathrm{T}}(t)PB_2\bar{F}(t)\\
&\quad\leq E^{\mathrm{T}}(t)[\frac{1}{k}(PB_2B_2^{\mathrm{T}}P)]E(t)+k\bar{F}(t)^{\mathrm{T}}\bar{F}(t)\\
&\quad\leq E^{\mathrm{T}}(t)[\frac{1}{k}(PB_2B_2^{\mathrm{T}}P)+k(\rho_1+\rho_2)^2]E(t), \qquad (19)
\end{aligned}$$

then substituting (16)-(19) back into (15) yields

$$\begin{aligned}
\dot{V}(t)\leq&\begin{bmatrix}E(t)\\ \bar{E}\end{bmatrix}^{\mathrm{T}}\left\{\begin{bmatrix}\Omega & -PB_1\\ -B_1^{\mathrm{T}}P & -\gamma I_{2(n-m)}\end{bmatrix}\right.\\
&+\gamma\tau^2\left(\begin{bmatrix}\Phi^{\mathrm{T}}\\ -B_1^{\mathrm{T}}\end{bmatrix}\begin{bmatrix}\Phi & -B_1\end{bmatrix}\right.\\
&\left.\left.+\begin{bmatrix}\Omega_2+\Omega_1 & 0\\ 0 & \frac{1}{k_2}B_1^{\mathrm{T}}B_2B_2^{\mathrm{T}}B_1\end{bmatrix}\right)\right\}\begin{bmatrix}E(t)\\ \bar{E}\end{bmatrix}\\
=&\begin{bmatrix}E(t)\\ \bar{E}\end{bmatrix}^{\mathrm{T}}(E_2+\gamma\tau^2F_2)\begin{bmatrix}E(t)\\ \bar{E}\end{bmatrix}.
\end{aligned}$$

In view of (11), we know $\dot{V}(t)<0$, which implies that $\lim_{t\to\infty}E(t)=0$, that is, the containment control of system (1) under protocol (2) is achieved. The proof is completed.

Remark 3: It is clear that $F_2\geq 0$ and $\gamma\tau^2>0$. So (11) maybe hold only when the $E_2<0$ holds. And (11) can hold when τ is sufficiently close to zero if $E_2<0$ holds.

Remark 4: By the Schur complement theorem, the condition that $E_2<0$ holds if and only if $\Omega<0$ holds. The condition that $\Omega<0$ holds only when $P\Phi^{\mathrm{T}}+\Phi P<0$ holds. So it is worth pointing out that a necessary condition for (11) is that Assumption 1 holds. If Assumption 1 does not hold, L_1must have at least one zero eigenvalue and yields a contradiction.

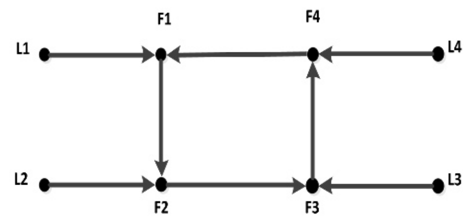

Fig. 1. Fixed directed network topology.

4. NUMERICAL SIMULATIONS

In this section, we present some simulation results to validate the previous theoretical results. We consider a group of agents with 4 leaders and 4 followers. The fixed communication delay is τ. For simplicity, we suppose that all the edge weights are 1 in the following example. Consider a network consisting of four followers and four leaders, whose dynamics are given by (1), the nonlinear dynamics of each agent is described by $f(t,x_i(t),v_i(t)) = \sin(t)x_i(t)+0.5\cos(t)v_i(t)$, in view of Assumption 2, we have $\rho_1=1$ and $\rho_2=0.5$.

For illustration, let the communication graph G be given as in Fig. 1. It can be seen that for each follower, there exists at least one leader that has a directed path to that follower. Thus, Assumption 1 holds in this case. For the graph G, the matrix L_1 is given as

$$L_1=\begin{bmatrix}2&0&0&-1\\-1&2&0&0\\0&-1&2&0\\0&0&-1&2\end{bmatrix},$$

whose eigenvalues are 3, 1, $2\pm i$. From Lemma 4, we obtain that feedback gains α and β in protocol (2) should satisfy $\frac{\beta^2}{\alpha}>0.1$. Here we take $\alpha=10$, $\beta=8$. Using Matlab's LMI Toolbox to solve LMI (11), we obtain a set of feasible solutions as follows:

$$\begin{aligned}
&k=0.1725,\ \ \gamma=10.6824,\ \ \tau=0.0109,\\
&k_1=8,\ \ k_2=0.75,\ \ P=\begin{bmatrix}P_1 & P_2\end{bmatrix},
\end{aligned}$$

where

$$P_1=\begin{bmatrix}2.3605&0.0802&0.0111&0.0284\\0.0802&0.0993&0.0140&0.0299\\0.0111&0.0140&2.3605&0.0802\\0.0284&0.0299&0.0802&0.0933\\0.0033&0.0066&0.0111&0.0140\\0.0066&0.0155&0.0284&0.0299\\0.0111&0.0284&0.0033&0.0066\\0.0140&0.0299&0.0066&0.0155\end{bmatrix},$$

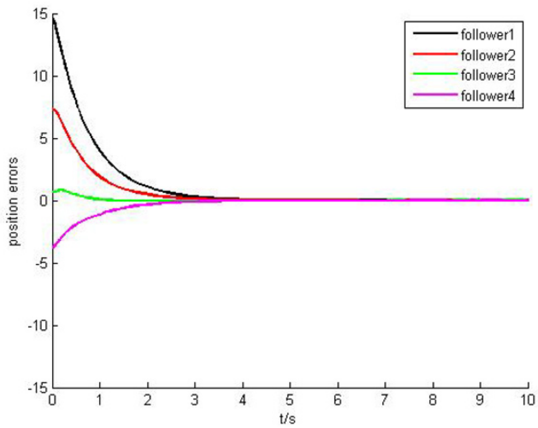

Fig. 2. The errors of position trajectories.

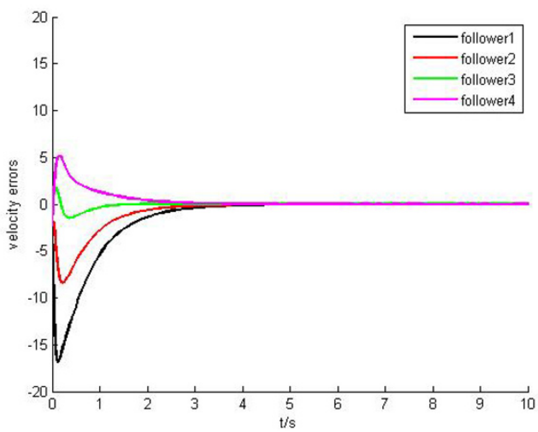

Fig. 3. The errors of velocity trajectories.

$$P_2 = \begin{bmatrix} 0.0033 & 0.0066 & 0.0111 & 0.0140 \\ 0.0066 & 0.0155 & 0.0284 & 0.0299 \\ 0.0111 & 0.0284 & 0.0033 & 0.0066 \\ 0.0140 & 0.0299 & 0.0066 & 0.0155 \\ 2.3605 & 0.0802 & 0.0111 & 0.0284 \\ 0.0802 & 0.0993 & 0.0140 & 0.0299 \\ 0.0111 & 0.0140 & 2.3605 & 0.0802 \\ 0.0284 & 0.0299 & 0.0802 & 0.0993 \end{bmatrix}.$$

Figs. 2 and 3 describe the position and velocity state errors trajectories of all followers in (1) under the designed protocol (2), respectively. It is easy to see that the containment errors asymptotically converge to the zero. That implies that the system achieved containment control.

Figs. 4 and 5 describe the velocity and position state trajectories of the agents under the designed protocol (2) respectively. It is easy to see that the states of the four followers asymptotically converge to the convex hull spanned by those of the leaders. From which it can be observed that the containment control problem is indeed solved.

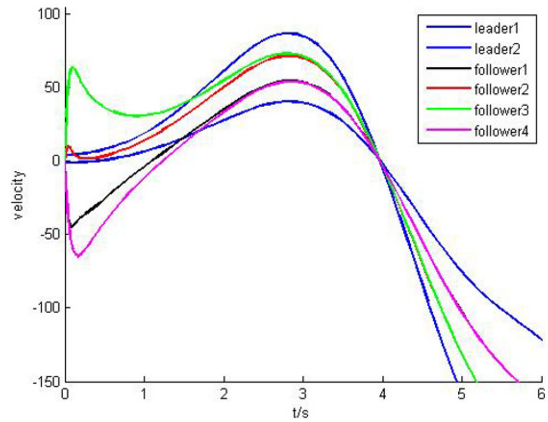

Fig. 4. The velocity trajectories of agents.

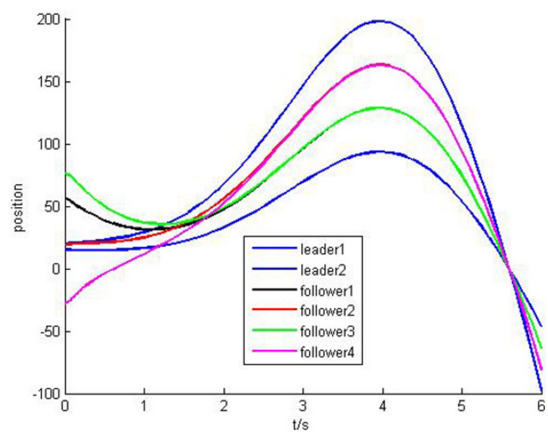

Fig. 5. The position trajectories of agents.

5. CONCLUSION

By using algebraic graph theory, matrix theory, and stability theory, the containment problems of multi-agent systems with nonlinear dynamics and fixed communication time-delays are investigated. This paper studied the distributed containment control problem of mobile autonomous agents under fixed directed network topologies. Some sufficient conditions on the directed network topology, feedback gains, and time-delays to guarantee distributed containment control are presented. During our past research work, we found it is very difficult to get the upper bound of time-delays for nonlinear multi-agent systems. The problem for linear multi-agent systems had been investigated and had better results in [24, 29]. Some works had been done about the feedback gains selecting for multi-agent systems with nonlinear dynamics [31]. Containment control of second-order multi-agent systems with nonlinear dynamics and varying time-delays under fixed or switching topologies is also very interesting to us;

this case will be investigated in our future research work.

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