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Stability of Nonlinear Systems with Variable-time Impulses: B-equivalence Method

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Abstract: This paper addresses the stability problem of nonlinear systems with variable-time impulses. By B-equivalence method, we shall show that under the well-selected conditions each solution of the considered systems will intersect each surface of discontinuity exactly once, and that the considered systems can be reduced to the fixed-time impulsive ones, which can be regarded as the comparison systems of the considered variable-time impulsive systems. Based on the stability theory of fixed-time impulsive systems, we propose a set of stability criteria for the variable-time impulsive systems. The theoretical results are illustrated by impulsive stabilization of Chua circuit.

Keywords: B-equivalence, Chua circuit, impulsive control systems, variable-time impulse.

1. INTRODUCTION

Generally, an impulsive system is a discontinuous dynamical system comprised of three parts: a continuous-time subsystem (namely, differential system or difference system), a discrete-time subsystem (namely, a state jump operator), and a switching rule which determines the impulse moments [1, 2]. According to different switching rule, impulsive systems can be divided commonly into three types: systems with fixed-time impulses; systems with variable-time impulses, and more generally, systems with event-dependent impulses. The impulse moments in fixed-time impulsive system are prescribed, while in a variable-time impulsive system they are not prescribed, and not known until one starts to look for a certain solution. Generally, variable-time impulses arise naturally in control, biological and physiological systems including nonlinear control systems, artificial neural networks, and variable moments of impulses are often of state dependence, and therefore, different solutions of variable-time impulsive system have different moments of impulses. Obviously, the variable-time impulsive systems are of much more importance in modeling and control, and of much more analytic difficulties than the fixed-time impulsive systems.

In recent years, much attention has been focused on the impulsive systems and impulsive control because of their immense application prospective. For example, the state of electronic networks is often subject to instantaneous and experience abrupt change at certain instants which may be caused by switching phenomenon or other sudden noise [3, 4]. Because of the complex and analytical difficulties, most existing publications about impulsive control systems only focused on the case of fixed-time impulses; the readers are referred to the references [1–11]. In the existing publications [1, 2, 12–15]for variable-time impulsive systems, comparison system method is often applied to analyze the system stability. In [2] and [12–15], several types of comparison systems for variable-time impulsive systems have been formulated. However, all these comparison systems are also variable-time impulsive systems but with one dimension, and it is usually difficult to verify or derive the system conditions and system parameters of the corresponding comparison systems.

Recently, Akhmet and his colleague in [16–22] proposed a powerful analytical tool for discontinuous systems, namely, B-equivalence, by means of which one can reduce a variable-time impulsive system to a fixed-time impulsive system. The reduced fixed-time impulsive system is expected to be the comparison system with impulse moments θ_k of the variable-time impulsive system with the impulse moments being governed by $t = \theta_k + \tau_k(x(t))$.

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It is also worth noting that the jump operator in comparison system might be very complex map and difficult to be applied to stability analysis. Sayli and Yilmaz in [23] tried to investigate the asymptotic stability of variable-time bidirectional associative memory (BAM) neural networks by means of B-equivalence. The main difficulty in use of B-equivalence method is to formulate or estimate the relationship between the original jump operator in variable-time impulsive system and new jump operator in corresponding fixed-time impulsive system. Sayli and Yilmaz in [23] just simply assumed that new jump operator is linear with respect to system states (although it may be feasible theoretically). Unfortunately, it is very difficult to determine the coefficient in the assumption (A5) in [23]. Therefore, to the best of our knowledge there are very few (if any) publications in literature where the reduction principle based on B-equivalence can be effectively applied to stability analysis on variable-time impulsive systems.

In the present paper, we shall formulate a throughout theoretical framework of reduction and comparison principle by B-equivalence for variable-time impulsive control systems. More specifically, we shall restate the sufficient conditions that ensure every solution of system intersects each surface of discontinuity exactly once, analyze and formulate the relationship between original jump operator and corresponding new one, theoretically prove that the stability properties of corresponding comparison system imply the same stability properties of the considered variable-time impulsive control systems, and finally establish a set of stability criteria by use of comparison system with fixed-time impulses.

This paper is organized as follows: In the next section, the considered systems are formulated and some preliminaries are presented. In Section 3, we state the conditions of absence of beating, and then we introduce the corresponding B-equivalent system in Section 4. A number of criteria for global exponential stability/stabilization of variable-time impulsive systems are established in Section 5. Chua circuit as the numerical example is presented to illustrate the theoretical results in Section 6, with conclusions drawn in Section 6.

2. PROBLEM STATEMENT AND PRELIMINARIES

Notations: Throughout this paper, we denote by P^T the transpose of matrix, P by $P>0\,(<0)$ the symmetrical and positive (negative) definite matrix P and by $\|*\|$ the Euclidian norm of a square matrix or a vector. Let $\mathbb{R}_+=[0,+\infty)$, $Z_+=\{1\ 2\ 3\ \cdots\}$, and we denote $\Gamma_i=\{(t,x(t))\in R_+\times G: t=\theta_i+\tau_i(x(t)),\ t\in R_+,\ x\in G,\ G\subset R^n\}$. the ith surface of discontinuity.

In this paper, we will consider the following systems:

$$\begin{cases}\dot{x}=f(t,x), & t>0, t\neq\theta_i+\tau_i(x),\\ \Delta x|_{t=\theta_i+\tau_i(x)}=J_i(x), & i\in Z_+,\end{cases}\tag{1}$$

where $x\in G\subseteq\mathbb{R}^n$, $f(t,x)$ is continuous on $\mathbb{R}_+\times G$ with $f(t,\ 0)=0$, and moreover, satisfies the Lipschitzian condition with respect to x, i.e., for all $t\in\mathbb{R}_+$, there exists a positive number l_f such that $\|f(t,x)-f(t,y)\|\leq l_f\|x-y\|$, for all $x,y\in G$, $t\in\mathbb{R}_+$. $J_i(x): G\to G$, $\tau_i(x): G\to\mathbb{R}$ are continuous functions, for all $i\in\mathbb{Z}_+$, satisfying that $J_i(0)=0$, $\tau_i(0)=0$, and there exist positive numbers l_J and l_τ such that $\|J_i(x)-J_i(y)\|\leq l_J\|x-y\|$, and $\|\tau_i(x)-\tau_i(y)\|\leq l_\tau\|x-y\|$, for all $i\in\mathbb{Z}_+$, $x,y\in G$. Furthermore, we assume that there exists a positive number ν such that $0\leq\tau_i(x)<\nu$, for all $i\in\mathbb{Z}_+$ and $x\in G$. For analytical simplification, we assume that each solution $x(t)$ of (1) is left-continuous, i.e., $\lim_{t\to\xi-0}x(t)=x(\xi)$. We denote $\Delta x(\xi)=x(\xi+0)-x(\xi)$, and assume that the sequence $\{\theta_i\}_{i=1}^{\infty}$ satisfies that , and $\lim_{i\to\infty}\theta_i=\infty$.

The following definitions and lemmas are necessary in the sequel.

Definition 1 [1]**:** Let $V: R_+\times R^n\to R_+$, then V is said to belong to class Σ if

a) V is continuous in $(\tau_{i-1},\ \tau_i]\times R^n$ and for each $x\in R^n$, $i=1,2,\cdots$, $\lim_{(t,y)\to(\tau_i^+,x)}V(t,y)=V(\tau_i^+,x)$ exists.

b) V is locally Lipschitzian in s.

From this definition, we can see that V associated with impulsive system (1) is the analog of Lyapunov function for stability analysis of ODE. Because these Lyapunov-like functions are generally discontinuous, a generalized derivative should be defined, which is known as the right and upper Dini's derivative.

Definition 2 [1]**:** For $(t,x)\in(\tau_{i-1},\ \tau_i]\times R^n$, the right and upper Dini's derivative of $V\in\Sigma$ with respect to time variable is defined as

$$D^+V(t,x)\equiv\lim_{h\to0^+}\sup\frac{1}{h}\{V[t+h,x+hf(t,x)]-V(t,x)\}.$$

Definition 3 [1]**:** The origin of system (1) is said to be globally exponentially stable if there exist some constants $\alpha>0$ and $M>0$ such that $\|x(t,t_0,\ x(t_0))\|\leq Me^{-\alpha(t-t_0)}$, for any $t\geq t_0$.

3. CONDITIONS FOR ABSENCE OF BEATING

In this section, by means of the B-equivalent method, we shall make such assumptions that each solution of (1) intersects each surface of discontinuity exactly once, and then try to reduce variable-time impulsive system (1) to fixed-time impulsive systems as its comparison system. For this purpose, we make the following assumptions.

(A1) There exist positive numbers $\underline{\theta}$ and $\bar{\theta}$, such that $\underline{\theta}+\nu<\theta_i-\theta_{i-1}<\bar{\theta}-\nu$, for all $i\in\mathbb{Z}_+$.

This assumption implies that $\underline{\theta} < [\theta_{i+1} + \tau_{i+1}(x)] - [\theta_i + \tau_i(x)] < \bar{\theta}$, and therefore no "beating phenomenon" will occur.

(A2) $\tau_i(x + J_i(x)) \le \tau_i(x)$, for all $i \in \mathbb{Z}^+$, $x \in G$.

(A3) $l_\tau \cdot M_f < 1$, where $M_f = \sup_{(t,x)\in R_+\times G} \|f(t,x)\| < +\infty$.

From (A1)-(A3), we have following observations.

Observation 1: Assume that (A1) holds, then each solution of (1), which intersects surface Γ_i and Γ_k, $(i < k-1)$, must intersects all surfaces Γ_j, $(i < j < k)$, between Γ_i and Γ_k.

Proof: Suppose $x(t)$ be a solution of (1), which intersects Γ_i and Γ_k. That is, there exist ξ_i and ξ_k $(\xi_i < \xi_k)$, such that

$$\xi_i = \theta_i + \tau_i(x(\xi_i)), \quad \xi_k = \theta_k + \tau_k(x(\xi_k)).$$

Define a function $\varphi(t) = t - \theta_j - \tau_j(x(t))$, $(i < j < k)$. Then $\varphi(t)$ is continuous with respect to t because of the continuity of τ_j.

Note that

$$\begin{aligned}\varphi(\xi_i) &= \xi_i - \theta_j - \tau_j(x(\xi_i)) = \theta_i + \tau_i(x(\xi_i)) \\ &\quad - \theta_j - \tau_j(x(\xi_i)) \\ &= -[(\theta_j - \theta_{j-1} + \tau_j(x(\xi_i)) - \tau_{j-1}(x(\xi_i))) + \cdots \\ &\quad + (\theta_{i+1} - \theta_i + \tau_{i+1}(x(\xi_i)) - \tau_i(x(\xi_i)))] \\ &< -(j-i-1)\underline{\theta} \le 0.\end{aligned}$$

Therefore, there exists a positive number ξ_j, $(\xi_i < \xi_j < \xi_k)$, such that $\varphi(\xi_j) = 0$, i.e., $\xi_j = \theta_j + \tau_j(x(\xi_j))$. Thus, $x(t)$ intersects each impulse surface Γ_j, $(i < j < k)$. The observation holds. □

Observation 2: Assume that (A1) holds, and $x(t)$: $\mathbb{Z}_+ \to G$ is a solution of (1). Then $x(t)$ intersects every surface Γ_i, $i \in \mathbb{Z}_+$.

Proof: Assume on the contrary that $x(t)$ does not intersect Γ_j for some $j \in \mathbb{Z}_+$. Observation 1 implies, the solution $x(t)$ does not intersect all the surfaces Γ_i for all $i < j$ or $i > j$. Introduce a new function $r(t) = t - [\theta_j + \tau_j(x(t))]$. Note that $t - (\theta_j + \nu) < r(t) < t - (\theta_j - \nu)$. One observes that $r(\theta_j - \nu) < 0 < r(\theta_j + \nu)$. Therefore, by the continuity of $r(t)$, there exists $\xi \in (\theta_j - \nu,\ \theta_j + \nu)$ such that $r(\xi) = 0$, i.e., $\xi = \theta_j + \tau_j(x(\xi))$. The observation holds. □

Observation 3: Assume that (A2) and (A3) hold. Every solution of (1) intersects the surface Γ_i at most once.

Proof: Assume on the contrary that there is a solution $x(t)$ intersects the surface Γ_j at $(s,\ x(s))$ and $(s_1, x(s_1))$, where we assume that $s < s_1$ and there exists no discontinuity point of $x(t)$ between s and s_1. Then $s = \theta_j + \tau_j(x(s))$ and $s_1 = \theta_j + \tau_j(x(s_1))$, and

$$\begin{aligned}0 &< s_1 - s \\ &= \tau_j(x(s_1)) - \tau_j(x(s)) \\ &\le \tau_j(x(s_1)) - \tau_j(x(s) + J_j(x(s))) \\ &\le l_\tau \|x(s_1) - [x(s) + J_j(x(s))]\| \\ &= l_\tau \left\| \int_s^{s_1} f(u, x(u))\, du \right\| \\ &\le l_\tau M_f(s_1 - s) < s_1 - s.\end{aligned}$$

This contradicts (A3). The observation holds. □

Based on the above observations, the following result is immediate.

Theorem 1: Assume that (A1)-(A3) hold, then every solution $x(t) : R_+ \to G$ of (1) intersects each the surface Γ_i, $i \in Z_+$, exactly once.

4. REDUCTION TO FIXED-TIME IMPULSIVE SYSTEM

In this section, we shall present a fixed-time impulsive system which can be regarded as the comparison system of the considered variable-time impulsive system (1), and then discuss the relationship between both systems.

Let $x^0(t) = x(t, \theta_i, x)$ be a solution of the first equation of model (1) in time interval $[\theta_i,\ \xi_i]$, where we denote by ξ_i the meeting moment of the solution with the surface of discontinuity so that $\xi_i = \theta_i + \tau_i\left(x^0(\xi_i)\right)$. Note that $\xi_i \ge \theta_i$ because of $0 \le \tau_i(x) < \nu$, for all $i \in \mathbb{Z}_+$, $x \in G$. Let also $x^1(t)$ be a solution of (1a) in time interval $(\theta_i,\ \xi_i]$ such that,

$$x_1(\xi_i) = x^0(\xi_i+) = x^0(\xi_i) + J_i\left(x^0(\xi_i)\right).$$

Define the following map:

$$\begin{aligned}W_i(x) &= x^1(\theta_i) - x \\ &= x^1(\xi_i) + \int_{\xi_i}^{\theta_i} f\left(s,\ x^1(s)\right) ds - x \\ &= x^0(\xi_i) + J_i\left(x^0(\xi_i)\right) + \int_{\xi_i}^{\theta_i} f(s,\ x_1(s)) ds - x \\ &= \int_{\theta_i}^{\xi_i} f\left(s,\ x^0(s)\right)\, ds \\ &\quad + J_i\left(x + \int_{\theta_i}^{\xi_i} f\left(s,\ x^0(s)\right)\, ds\right) \\ &\quad + \int_{\xi_i}^{\theta_i} f\left(s,\ x^1(s)\right) ds. \end{aligned} \tag{2}$$

From the definition of $W_i(x)$ together with Fig. 1, we have the following observations without proof:

(i) $x^0(t) = x(t, \theta_i, x)$ can be extended as the solution of (1) in R_+;

(ii) $x^1(t) = x\left(t,\ \xi_i,\ x^0(\xi_i+)\right)$ can be extended as the solution of the following fixed-time impulsive system in R_+:

$$\begin{cases}\dot{x}(t) = f(t,\ x(t))\, t \ne \theta_i, \\ \Delta x|_{t=\theta_i} = W_i(x).\end{cases} \tag{3}$$

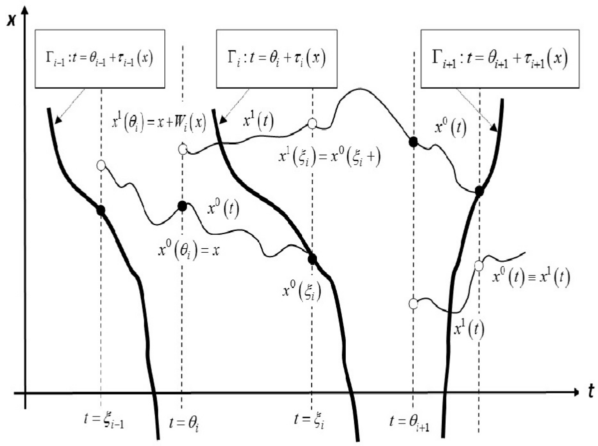

Fig. 1. Construction Principle of the map $W_i(x)$.

(iii) For all $i \in Z_+$, on time interval $(\xi_{i-1},\ \theta_i]$, $x^0(t) \equiv x^1(t)$, and $x^1(\theta_i+) \equiv x^0(\theta_i) + W_i(x^0(\theta_i))$, $x^1(\xi_i) = x^0(\xi_i+) = x^0(\xi_i) + J_i(x^0(\xi_i))$.

(iv) For all $i \in Z_+$, on time interval $(\theta_i,\ \xi_i]$,

$$\begin{aligned}
&x^1(t) - x^0(t) \\
&= x + W_i(x) + \int_{\theta_i}^{t} f(u, x^1(u))du - x \\
&\quad - \int_{\theta_i}^{t} f(u, x^0(u))du \\
&= W_i(x) + \int_{\theta_i}^{t} \left(f(u, x^1(u)) - f(u, x^0(u))\right)du. \qquad (4)
\end{aligned}$$

5. STABILITY ANALYSIS ON VARIABLE-TIME IMPULSIVE SYSTEMS

From (2), we have

$$\begin{aligned}
\|W_i(x)\| &= \Bigg\| \int_{\theta_i}^{\xi_i} f(s,\ x^0(s))\ ds \\
&\quad + J_i\left(x + \int_{\theta_i}^{\xi_i} f(s,\ x^0(s))\ ds\right) \\
&\quad + \int_{\xi_i}^{\theta_i} f(s,\ x^1(s))ds \Bigg\| \\
&\le 2M_f(\xi_i - \theta_i) + l_J\|x\| + l_J M_f(\xi_i - \theta_i) \\
&= [2 + l_J] M_f\ \tau_i(x^0(\xi_i)) + l_J\|x\|.
\end{aligned}$$

Note that

$$\begin{aligned}
\tau_i(x^0(\xi_i)) &\le l_\tau \|x^0(\xi_i)\| \\
&= l_\tau \left\| x + \int_{\theta_i}^{\xi_i} f(s,\ x^0(s))\ ds \right\| \\
&\le l_\tau\|x\| + l_\tau \left\| \int_{\theta_i}^{\xi_i} f(s,\ x^0(s))\ ds \right\| \\
&\le l_\tau\|x\| + l_\tau M_f(\xi_i - \theta_i) \\
&= l_\tau\|x\| + l_\tau M_f\ \tau_i(x^0(\xi_i)),
\end{aligned}$$

which implies that

$$\tau_i(x^0(\xi_i)) \le (1 - l_\tau M_f)^{-1} l_\tau \|x\|. \qquad (5)$$

Therefore,

$$\|W_i(x)\| \le \left[(2 + l_J) M_f\ (1 - l_\tau M_f)^{-1} l_\tau + l_J\right] \|x\|. \qquad (6)$$

Moreover, from (4) and using Gronwall-Bellman Inequality, we have, for all $t \in (\theta_i,\ \xi_i]$,

$$\begin{aligned}
&\|x^1(t) - x^0(t)\| \\
&\le \|W_i(x)\| + \int_{\theta_i}^{t} \|f(u, x^1(u)) - f(u, x^0(u))\|\ du \\
&\le \|W_i(x)\| + l_f \int_{\theta_i}^{t} \|x^1(u) - x^0(u)\|\ du \\
&\le \|W_i(x)\| \exp\{l_f(t - \theta_i)\} \\
&\le \left[(2 + l_J) M_f\ (1 - l_\tau M_f)^{-1} l_\tau + l_J\right] e^{l_f v}\ \|x\|.
\end{aligned}$$

That is, for all $t \in (\theta_i,\ \xi_i]$,

$$\begin{aligned}
&\|x^1(t) - x^0(t)\| \\
&\le \left[(2 + l_J) M_f\ (1 - l_\tau M_f)^{-1} l_\tau + l_J\right] e^{l_f v}\ \|x\|. \qquad (7)
\end{aligned}$$

Based on the above discussion, the following theorem is immediate.

Theorem 2: The variable-time impulsive system (1) has the same stability property with the fixed-time impulsive system (3).

Remark 1: Theorem 2 implies that system (3) can be regarded as the comparison system of system (1) in the context of stability analysis. Therefore, we shall study the stability of (1) by considering the stability of (3).

Theorem 3: Suppose that there exists $V \in \Sigma$ such that

$$\alpha\|x(t)\|^p \le V(t, x(t)) \le \beta\|x(t)\|^p, \qquad (8)$$

and

$$\begin{cases} D^+V(t, x(t)) \le bV(t, x(t)), t \ne \theta_k,\ \ k \in Z_+, \\ V(t, x(t) + W_i(x(t))) \le d_k V(t, x(t)),\ \ t = \theta_k, \end{cases} \qquad (9)$$

where $\alpha > 0$, $\beta > 0$, $p > 0$, $b > 0$, and $d_k > 0$. If there exists a positive number r,such that

$$\ln d_k + b(\bar{\theta} - v) \le -r. \qquad (10)$$

Then the origin of system (3) is globally exponentially stable, and therefore, the origin of system (1) is globally exponentially stable.

Proof: To prove this theorem, we first show the following claim holds:

For any $t \in (\theta_k, \ \theta_{k+1}], \ k \in Z_+$, we have

$$V(t, \ x(t)) \le V_0 \exp\left\{bt + \sum_{i=1}^{k} \ln d_i\right\}, \tag{11}$$

where $V_0 = V(0, \ x(0))$. We now prove the claim (11) by means of mathematical induction.

(i) When $k = 0$, i.e., $t \in (0, \ \theta_1]$, it follows from (9) that

$$V(t, \ x(t)) \le V_0 \exp\{bt\},$$

and

$$\begin{aligned} V(\theta_1, \ x(\theta_1) + W_1(x(\theta_1))) &\le d_1 V(\theta_1, \ x(\theta_1)) \\ &\le d_1 V_0 \exp\{b\theta_1\}. \end{aligned}$$

(ii) When $k = 1$, i.e., $t \in (\theta_1, \ \theta_2]$, it follows from (9) that

$$\begin{aligned} V(t, \ x(t)) &\le V(\theta_1, \ x(\theta_1+)) \exp\{b(t-\theta_1)\} \\ &\le d_1 V_0 \exp\{bt\} = V_0 \exp\{bt + \ln d_1\}, \end{aligned}$$

and

$$\begin{aligned} V(\theta_2, \ x(\theta_2) + W_2(x(\theta_2))) &\le d_2 V(\theta_2, \ x(\theta_2)) \\ &\le d_1 d_2 V_0 \exp\{b\theta_2\}. \end{aligned}$$

Therefore, the claim (11) is true when $k = 1$, i.e., $t \in (\theta_1, \ \theta_2]$.

(iii) Suppose that the claim holds when $k = s, \ s > 1$. That is,

$$V(t, \ x(t)) \le V_0 \exp\left\{bt + \sum_{i=1}^{s} \ln d_i\right\}, \quad t \in (\theta_s, \ \theta_{s+1}].$$

(iv) When $k = s+1$, i.e., $t \in (\theta_{s+1}, \ \theta_{s+2}]$, we have

$$\begin{aligned} &V(t, \ x(t)) \\ &\le V(\theta_{s+1}, \ x(\theta_{s+1}) + W_{s+1}(x(\theta_{s+1}))) \exp\{b(t-\theta_{s+1})\} \\ &\le d_{s+1} V_0 \exp\left\{bt + \sum_{i=1}^{s} \ln d_i\right\} \\ &= V_0 \exp\left\{bt + \sum_{i=1}^{s+1} \ln d_i\right\}. \end{aligned}$$

This implies that the claim (11) holds for $k = s+1$, i.e., $t \in (\theta_{s+1}, \ \theta_{s+2}]$, and therefore, it holds for all $t \in (\theta_k, \ \theta_{k+1}]$, $k \in Z_+$.

Note that, for $t \in (\theta_k, \ \theta_{k+1}]$, $k \in Z_+$ we have

$$\begin{aligned} t \le \theta_{k+1} &= \theta_{k+1} - \theta_k + \theta_k - \theta_{k-1} + \cdots + \theta_1 - \theta_0 \\ &\le \sum_{i=1}^{k+1} (\theta_i - \theta_{i-1}) \le (k+1)\left(\bar{\theta} - v\right). \end{aligned}$$

Therefore, $k \ge \frac{t}{\bar{\theta}-v} - 1$, and

$$\begin{aligned} &V(t, \ x(t)) \\ &\le V_0 \exp\left\{bt + \sum_{i=1}^{k} \ln d_i\right\} \\ &\le V_0 \exp\left\{b \sum_{i=1}^{k+1} (\theta_i - \theta_{i-1}) + \sum_{i=1}^{k} \ln d_i\right\} \\ &= V_0 \exp\left\{b\left(\bar{\theta} - v\right)\right\} \exp\left\{\sum_{i=1}^{k} [\ln d_i + b(\theta_i - \theta_{i-1})]\right\} \\ &\le V_0 \exp\left\{b\left(\bar{\theta} - v\right)\right\} \exp\{-k\gamma\} \\ &< V_0 \exp\left\{b\left(\bar{\theta} - v\right) + \gamma\right\} \exp\left\{-\frac{\gamma}{\bar{\theta} - v} t\right\} \end{aligned}$$

The last inequality together with (8) implies that the origin of system (3) is globally exponentially stable. □

Remark 2: Theorem 3 present a set of sufficient conditions for the exponential stability of fixed-time impulsive system (3) in terms of Lyapunov function. Based on Theorem 3 and by selecting some suitable Lyapunov functions, we can derive some alternative stability criteria which are easy to be verified for variable-time impulsive systems (1).

6. STABILIZING CHUA CIRCUIT

In this section, we take the Chua circuit as an example to illustrate the effectiveness of the theoretical results mentioned in the previous section. The mathematical model of Chua circuit is the following equations:

$$\begin{cases} \dot{x}_1 = -\alpha\left[x_1 - x_2 + g_1(x_1)\right], \\ \dot{x}_2 = x_1 - x_2 + x_3, \\ \dot{x}_3 = -\beta x_2. \end{cases} \tag{12}$$

The Chua circuit (12) exhibits the chaotic behavior with the attractor

$$G = \left\{(x_1, x_2, x_3) \in R^3 \mid \ |x_i| \le 2, i = 1, 2, 3\right\},$$

when $\alpha = 9.2156$, $\beta = 15.9946$,

$$g_1(x_1) = bx_1 + \frac{1}{2}(a-b)\left[|x_1+1| - |x_1-1|\right],$$

and $a = -1.24905$, $b = -0.75735$.

This system can be rewritten as the compact form:

$$\dot{x}(t) = Ax(t) + g(x(t)) \tag{13}$$

with $x(t) = [x_1(t), \ x_2(t), \ x_3(t)]^T \in R^3$, and

$$A = \begin{bmatrix} -\alpha(1+b) & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{bmatrix},$$

$$g(x(t)) = \left[\frac{1}{2}(a-b)(|x_1+1| - |x_1-1|), 0, 0\right]^T.$$

To stabilize this system, we take the impulsive controller

$$\Delta x|_{t=\theta_k+\tau_k(x(t))} = B_k x(t), \tag{14}$$

where $\tau_k(x(t)) = \gamma_k |x_1|$, $\gamma_k > 0$, B_k, $k \in Z_+$ are constant scalars.

Therefore, the controlled system is of the form

$$\begin{cases} \dot{x}(t) = Ax(t) + g(x(t)), & t \neq \theta_k + \gamma_k |x_1|, \\ \Delta x|_{t=\theta_k+\gamma_k|x_1|} = B_k x(t), & k \in Z_+. \end{cases} \tag{15}$$

Note that $v = 2\max\{\gamma_k\}$, $l_\tau = \max\{\gamma_k\}$, and

$$\begin{aligned} \|Ax(t) + g(x(t))\| &\leq \|A\| \|x(t)\| + \|g(x(t))\| \\ &\leq \|A\| \|x(t)\| + |a-b| \\ &= 18.5\sqrt{12} + 0.4917 \\ &\leq 64.5776 \end{aligned}$$

which implies $M_f = \sup_{(t,x)\in R_+\times G} \|f(t,x)\| \leq 64.5776$. Then, from Theorem 1 the following corollary is immediate.

Corollary 1: If the controlled system (15) satisfies the following conditions:

(i) There exist positive numbers $\underline{\theta}$ and $\bar{\theta}$, such that $\underline{\theta} + 2\max\{\gamma_k\} < \theta_i - \theta_{i-1} < \bar{\theta} - 2\max\{\gamma_k\}$, for all $i \in \mathbb{Z}_+$.

(ii) $|1 + B_i| \leq 1$, for all $i \in \mathbb{Z}^+$.

(iii) $\max\{\gamma_k\} \cdot M_f < 1$, where $M_f = \sup_{(t,x)\in R_+\times G} \|f(t,x)\| \leq 64.5776$.

Then the assumptions (A1)-(A3) hold, and therefore, every solution $x(t) : R_+ \to G$ of (15) intersects each surface $\Gamma_i = \{(t, x(t)) \in R_+ \times G : \quad t \neq \theta_k + \gamma_k |x_1|\}$, $i \in Z_+$, exactly once.

Now, it is time to define the map $W_k(x)$ based on (2) as following:

$$\begin{aligned} W_k(x) &= \int_{\theta_k}^{\xi_k} \left[Ax^0(u) + g\left(x^0(u)\right)\right] du \\ &\quad + B_k x + B_k \int_{\theta_k}^{\xi_k} \left[Ax^0(u) + g\left(x^0(u)\right)\right] du \\ &\quad + \int_{\xi_k}^{\theta_k} \left[Ax^1(u) + g\left(x^1(u)\right)\right] du \\ &= B_k x + (I + B_k) \int_{\theta_k}^{\xi_k} \left[Ax^0(u) + g\left(x^0(u)\right)\right] du \\ &\quad + \int_{\xi_k}^{\theta_k} \left[Ax^1(u) + g\left(x^1(u)\right)\right] du. \end{aligned}$$

Note that

$$\begin{aligned} &\|x + W_k(x)\| \\ &\leq |1 + B_k| \|x\| + |1 + B_k| \cdot M_f \cdot |\xi_k - \theta_k| + M_f \cdot |\xi_k - \theta_k| \\ &= |1 + B_k| \|x\| + M_f [|1 + B_k| + 1] |\xi_k - \theta_k| \\ &= |1 + B_k| \|x\| + M_f [|1 + B_k| + 1] \gamma_k |x_1| \\ &\leq \{|1 + B_k| + \gamma_k M_f [|1 + B_k| + 1]\} \|x\|. \end{aligned} \tag{16}$$

We then obtain the following B-equivalent system of (15):

$$\begin{cases} \dot{x} = Ax + g(x), & t \neq \theta_k, \\ \Delta x|_{t=\theta_k} = W_k(x), & k \in Z_+. \end{cases} \tag{17}$$

Theorem 4: Suppose that the conditions in Corollary 1 are satisfied. If there exists positive number r such that, for all $k \in Z_+$,

$$\ln d_k + \lambda_1 [\theta_k - \theta_{k-1}] \leq -r,$$

where $\lambda_1 = \lambda_{\max}\left(A + A^T + 2(b-a)I\right)$ and $d_k = [|1 + B_k| + \gamma_k M_f (|1 + B_k| + 1)]^2$. Then, the origin of system (17) is globally exponentially stable, and therefore, the origin of system (15) is globally exponentially stable.

Proof: Consider the following Lyapunov function candidate

$$V(t,x) = x^T(t)x(t).$$

When $t \neq \theta_k$, the Dini derivative of $V(t,x)$ with respect to time t along the solution of (17) can be calculated as

$$\begin{aligned} D^+V(t,x(t)) &= 2x^T(t)[Ax(t) + g(x(t))] \\ &= x^T(t)\left[A + A^T\right]x(t) + 2x^T(t)g(x(t)) \\ &\leq x^T(t)\left[A + A^T + 2(b-a)I\right]x(t) \\ &\leq \lambda_1 V(t, x(t)). \end{aligned}$$

On the other hand, from (16), one observes that

$$\begin{aligned} V(\theta_k, x(\theta_k) + W_k(x(\theta_k))) &= [x + W_k(x)]^T [x + W_k(x)] \\ &\leq d_k \|x(\theta_k)\|^2. \end{aligned}$$

The remaining proof follows Theorem 4, and therefore, we omit it. □

Remark 3: Note that the impulse moments are not fixed and prescribed. Therefore, the existing results for fixed-time impulsive systems such as references [1–10, 24–28] are unavailable to this example. It is easy to see that the coefficient in (A5) of [23] is difficult to be determined in this example, therefore, the results in [23] are also inapplicable to this example.

For numerical simulation, we take $\theta_k = 0.2k$, $\gamma_k = 0.001$, $B_k = -0.9$. Then, by simple computation, we have $\lambda_1 = 17.53318$, $d_k = 0.029253$ and therefore, $r = 0.02513$. By Theorem 4, the origin of system (15) is globally exponentially stable, that is, chaotic Chua circuit (12) can be stabilized at the origin under the variable-time impulsive control (14), as shown in Fig. 2.

7. CONCLUSIONS AND DISCUSSIONS

In this paper, we have presented the theoretical framework for analyzing the stability/stabilization of nonlinear systems with variable-time impulses by means of B-equivalence. However, several issues are still open, for examples, (i) the conditions for absence of beating are conservative; (ii) the estimation on the norm of transformation map $W_i(x)$ is also very conservative, which leads to conservative stability/stabilization conditions; (iii) it is expected to extend the presented method to delayed systems or more general impulsive systems.

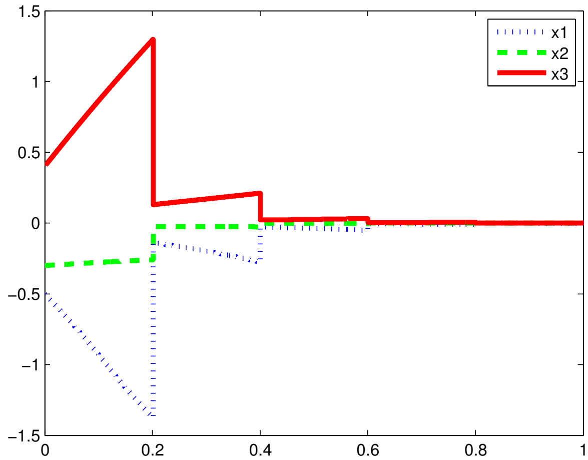

Fig. 2. Time response curve of system (15) where the initial values are $[-0.5, -0.2, 0.4]$.

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