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Mean Square Consensus of Multi-agent Systems with Multiplicative Noises and Time Delays under Directed Fixed Topologies

Lei Song, Dan Huang*, Sing Kiong Nguang, and Shan Fu

Abstract: This paper investigates the mean square consensus problem of multi-agent systems impacted by the combined uncertainty of multiplicative noises and time delays. Considering general network under directed fixed topologies, we propose consensus protocol that take into account both the multiplicative noises and time delays. Using tools from stochastic differential delay equation (SDDE), martingale theory and stochastic inequality, we establish sufficient conditions and obtain the explicit consensus gain and delay upper bounds under which the proposed protocol leads to mean square consensus. In addition, we compare the impact of multiplicative and additive noise and reach the conclusion that multiplicative noises have the property of stabilizing effect. Simulations demonstrate the theoretical results.

Keywords: Mean square consensus, multi-agent systems, multiplicative noises, time delays.

1. INTRODUCTION

Consensus problems have a long history in distributed computing [1], management science [2] and statistical physics [3]. In the control systems society, the research effort can be traced back to the work [4], where an asynchronous agreement problem was studied for distributed decision-making problems. Due to widespread applications, there has been a recent surge of interests in consensus problems and the like, see the recent survey [5] and extensive reference therein.

Naturally, consensus problems arise in the system with distributed structure when different coupling parts are seeking an agreement on some certain quantity. Therefore, in multi-agent systems, the flow of information and control among agents over a network plays a crucial role in determining consensus. This point is reflected by some foundational work in different sides, e.g., [6] and [7] focus on consensus protocol based on given strategy of information interaction, [8] and [9] study the cases of switching and dynamical changing network topology, respectively. Nevertheless, in practical applications, many uncertainties will unavoidably make some impact on the information acquisition and transmission that should be considered in protocol design. Except the factor of network topology mentioned above, the noise and delay effects are equally important and should be taken into consideration simultaneously when studying consensus problems.

Specifically, noises often come from measurement errors or communication disturbance during the interaction process between any two agents, which can lead to the divergence of consensus value. Significant earlier studies about this topic are the case of consensus with additive noise. In the discrete-time setting, Huang and Manton develop a stochastic approximation type of algorithm to tackle noises in [10], and further investigate the cases with fixed or randomly varying topology in [11] and Markovian or arbitrary switching topology in [12]. In the continuous-time setting, Li and Zhang propose convergence and robustness conditions of the control weights and obtain necessary and sufficient conditions for mean square average consensus under a fixed topology in [13], and then apply the algorithm to networks with quantized data and packet losses in [14]. Other works are mainly motivated by those above, e.g., [15] propose a new algorithm to reach an (ε,δ) consensus based on the conclusion of [10]. However researchers have paid more attention on the case of consensus with relative-state-dependent noises or linearized multiplicative noises more recently, where the noise intensities are assumed as a function of relative states. Physically, the typical examples are the logarithmic quantization model in the stochastic framework and the distributed averaging system with Gaussian fading communication channels [16]. In [16] and [17], Li *et al.* develop several small gain consensus gain theorems to give sufficient conditions to ensure mean square and almost sure

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Lei Song, Dan Huang, and Shan Fu are with the School of Aeronautics and Astronautics, Shanghai Jiao Tong University, Shanghai 200240, China (e-mails: {songlei_24, huangdan, sfu}@sjtu.edu.cn). Sing Kiong Nguang is with the Department of Electrical and Computer Engineering, The University of Auckland, Auckland 1142, New Zealand (e-mail: sk.nguang@auckland.ac.nz).
* Corresponding author.

consensus of high-dimensional first-order agents with relative-state-dependent noises and matrix-valued intensity function. In [18], Ni and Li give a constant consensus gain that can resolve the consensus problem that the measurement noises are proportional to the relative states. In [19], Long *et al.* deal with the case in the discrete-time settings and give sufficient conditions for mean square and almost sure consensus under fixed, dynamically switching and random switching topologies. Some practical applications of the noisy setting can be found recently, e.g., consensus of leader-following multi-agent systems [20] and cooperative simultaneous attack of multi missiles [21].

On the other hand, all the aforementioned references do not consider the co-existence of delays and noises while time delays almost surely appear in multi-agent systems due to the communication or measurement limit. Two types of time delays, have been considered in the literature, i.e., communication delay and input delay [5], which model the effect of communication and measurement respectively. There have been only a few researchers focusing on the topic both with noises and time delays which poses significant challenges to the analysis of consensus. In the study of discrete-time setting [22], Liu, Xie, and Zhang introduce an auxiliary system for the first time to overcome the the challenges induced by both the transmission delays and noises. Meanwhile, in the study of continuous-time setting [23], Liu *et al.* develop intensely beneficial method to overcome the challenges induced by the co-existence of additive delays and noises. By introducing a Gronwall-Bellman-Halanay type inequality firstly, the tools of stochastic differential delay equation (SDDE) are taken to establish the sufficient conditions of mean square average consensus. Based on this, further in [24], Liu *et al.* derive the conditions of almost sure average consensus and pth moment average consensus. In [25], by adopting the method proposed in [23], Djaidja and Wu study the leader-following multi-agent systems with additive delays and noises. Moreover, in [26], Sun *et al.* propose the sufficient and necessary condition of mean square average consensus under additive delays and noises, while the proof based on the second-order Taylor expansion is much more conservative. None of the above mentioned work, however, has investigated stochastic consensus problems of networks with relative-state-dependent noises or multiplicative noises and delays. To our best knowledge, there is still lack of good results in this case.

In this paper, we aim to investigate the mean square consensus problem of multi-agent systems with multiplicative noises and time delays under directed fixed topologies. Following the known results in two aspects of research, i.e., the impact of multiplicative noises and the coexistence of additive noises and time delays, we take the tools of SDDE, martingale theory and stochastic inequality to provide sufficient conditions under which the proposed consensus protocol leads to mean square consensus. Explicit consensus gain and delay upper bounds for guaranteeing consensus are obtained, which is our main contribution. In addition, we compare the impact of multiplicative and additive noises and reach the conclusion that multiplicative noises have the property of stabilizing effect.

The following notations will be used throughout this paper: I_n denotes the $n \times n$dimensional identity matrix. $\mathbf{1}_n$ denotes the n dimensional one vector . For a vector or matrix A, A^T denotes its transpose, if A is a vector then $|A|$ denotes its modulus, if A is a matrix then $tr(A)$ denotes its trace, $\|A\|$ denotes its operator norm, and $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ are its maximum and minimum eigenvalues, respectively. For a given random variable X $\mathrm{E}[\mathrm{X}]$ denotes its mathematical expectation.

2. PROBLEM FORMULATION

We consider the consensus seeking problem of multi-agent systems with multiplicative noises and time delays. In our case, the dynamics of each agent is the continuous-time first-order integrator

$$\dot{x}_i = u_i,\ i = 1, \ldots, n, \tag{1}$$

where x_i, u_i are the state and control input of agent i. u_i is designed based on the information of itself and its local information of neighbors under a communication network.

Denote the network topology as a weighted directed graph (digraph) $\mathscr{G} = (\mathscr{V}, \mathscr{E}, \mathscr{A})$ composed of set of nodes $\mathscr{V} = (v_1, \ldots, v_n)$, set of edges $\mathscr{E} \subset \mathscr{V} \times \mathscr{V}$, and a weighted adjacency matrix $\mathscr{A} = [a_{ij}]_{n \times n}$ with nonnegative adjacency elements a_{ij}. An edge of $\mathscr{G}$ is denoted by $e_{ij} = (v_i, v_j) \in \mathscr{E}$. The adjacency elements associated with the edges of the graph are positive, i.e., $e_{ij} \in \mathscr{E} \Leftrightarrow a_{ij} > 0$. It is assumed that $a_{ii} = 0$ for $i = 1, \ldots, n$. The set of neighbors of node v_i is denoted by $\mathscr{N}_i = \{v_j = \mathscr{V} : (v_j, v_i) \in \mathscr{E}\}$. The graph Laplacian L of the network is defined by $L = D - \mathscr{A}$, where $D = diag(d_1, \ldots, d_n)$ is the in-degree matrix of $\mathscr{G}$ with elements $d_i = \sum_{j \neq i} a_{ij}$. A digraph is strongly connected if there is a directed path connecting any two arbitrary nodes in the graph.

To tackle the multiplicative noises and time-delays, we firstly model the information received by agent from its neighbors by

$$\begin{aligned} y_{ji} =& x_j(t - \tau_{ij}(t)) \\ &+ \sigma_{ji} \left| x_j(t - \tau_{ij}(t)) - x_i(t - \tau_{ij}(t)) \right| \xi_{ji}(t),\ j \in \mathscr{N}_i, \end{aligned} \tag{2}$$

where the noise processes $\{\xi_{ji}(t) : i, j = 1, \ldots, n\}$ are assumed as independent standard white noises, that is,

$$\int_0^t \xi_{ji}(s)ds = w_{ji}(t), t \geq 0,$$

where $\{w_{ji}(t) : i, j = 1, \ldots, n\}$ are independent Brownian motions, and $\sigma_{ji} \geq 0$ represent the noise intensity. The time-varying delays $\tau_{ij}(t)$ lie in $[0,\tau]$ for some $\tau > 0$ and assumed to be continuous in t. Thus an admissible consensus protocol should be designed as such a group of controls $\{u_i, i = 1,\dots,n\}$ under this mode of information exchange, to make all the states of all agents asymptotically approach a common value in some sense . For the ith agent we consider the following consensus controller with its own measurement time-delays:

$$u_i = \alpha \sum_{j\in\mathscr{N}_i} a_{ij}[y_{ji} - x_i(t-\tau_{ij}(t))], \tag{3}$$

where $\alpha > 0$ is the consensus gain which will be determined. In this paper, we focus on the case where the time delays are uniform for all $i, j = 1,\dots,n$, i.e.,

$$\tau_{ij}(t) = \tau(t). \tag{4}$$

By (1)-(4), we have

$$\begin{aligned}\dot{x}_i = &\alpha \sum_{j\in\mathscr{N}_i} a_{ij}\left(x_j(t-\tau(t)) - x_i(t-\tau(t))\right)\\ &+\alpha \sum_{j\in\mathscr{N}_i} a_{ij}\sigma_{ji}\left|x_j(t-\tau(t)) - x_i(t-\tau(t))\right|\xi_{ji}(t),\\ &\qquad\qquad\qquad\qquad\qquad i = 1,\dots,n.\end{aligned} \tag{5}$$

To get the compact form of the entire system, define two matrix

$$\Theta = diag(\Theta_1,\dots,\Theta_n)$$

with

$$\Theta_i = [a_{i1}\sigma_{1i}, a_{i2}\sigma_{2i},\dots,a_{in}\sigma_{ni}]$$

and

$$y(t-\tau(t)) = diag(y_1(t-\tau(t)),\dots,y_n(t-\tau(t)))$$

with

$$\begin{aligned}y_i(t-\tau(t)) =&diag(|x_1(t-\tau(t)) - x_i(t-\tau(t))|,\\ &\dots,|x_n(t-\tau(t)) - x_i(t-\tau(t))|).\end{aligned}$$

The collective dynamics of system can be written in the form of the SDDE

$$dx(t) = -\alpha Lx(t-\tau(t))dt + \alpha\Theta y(t-\tau(t))dW(t), \tag{6}$$

where $\{W(t), t > 0\}$ is the $\mathbb{R}^{n^2}$ -valued standard Brownian motion defined on a probability space $(\Omega,\mathscr{F},P)$

We introduce the notion of mean square consensus for the multi-agent system (1) under consensus protocol (3) in an uncertain environment.

Defination 1 (mean square consensus [10])**:** The multi-agent system is said to reach mean square consensus if $E|x_i(t)|^2 < \infty$ for all $t \ge 0$, $i = 1,\dots,n$ and there exists a random variable x^* such that $\lim_{t\to\infty} E|x_i(t) - x^*|^2 = 0$ for all $i = 1,\dots,n$.

As mentioned, Liu, Xie, and Zhang [22], Liu *et al.* [23, 24] and related work have investigated the stochastic consensus problem with noises and time delays but without multiplicative noises involved, either in discrete or continuous setting, while the works referring to the topic of multiplicative noises [16–20] do not consider the effect of time delays. We aim to develop a consensus algorithm and corresponding conditions to tackle the consensus problem of multi-agent systems with multiplicative noises and time delays.

3. MAIN RESULTS

In this paper, we deal with the case that the network topology is directed fixed. To achieve our perspective, we make the following assumptions.

Assumption 1: The fixed digraph $\mathscr{G}$ contains a spanning tree.

Assumption 2: The interaction mode between agents is cooperative rather than competitive for all $i, j = 1,\dots,n$, i.e. $a > 0$.

Note that $-L$ is not Hurwitz and may be interpreted as the generator of a continuous time Markov chain. Therefore, we could make state transformation and system decomposition for the simplification of analysis by relevant method. According to some known results in [11] and [18], there exists a nonsingular matrix $\phi = [\mathbf{1}_n, \varphi]$ to make

$$\phi^{-1}(-L)\phi = \begin{pmatrix} 0 & 0\\ 0 & -\bar{L}\end{pmatrix}, \tag{7}$$

where $-\bar{L}\in\mathbb{R}^{(n-1)\times(n-1)}$ is Hurwitz and φ is a $n\times(n-1)$ matrix. Denote $\phi^{-1} = [\ \pi \quad \psi\ ]^T$, where π is the unique invariant probability measure of the Markov chain with the generator $-L$ and ψ is a $(n-1)\times n$ matrix. Obviously, there exist a positive definite matrix P such that

$$P(-\bar{L}) + (-\bar{L}^T)P = -I_{n-1}. \tag{8}$$

Further, let $\tilde{x}(t) = \phi^{-1}x(t) \equiv [\tilde{x}_1(t), \bar{x}(t)^T]^T$, where $\tilde{x}_1(t)\in\mathbb{R}$ and $\bar{x}(t)\in\mathbb{R}^{n-1}$, and we get

$$x(t) = \phi\tilde{x}(t) = \tilde{x}_1(t)\mathbf{1}_n + \varphi\bar{x}(t). \tag{9}$$

For each agent, $x_i(t) = \tilde{x}_1(t) + \varphi_i\bar{x}(t)$, where φ_i is the ith row of φ. Clearly,

$$|x_i(t-\tau(t)) - x_j(t-\tau(t))| = |(\varphi_i - \varphi_j)\bar{x}(t-\tau(t))|. \tag{10}$$

By those above, we finally get the equivalent system of (6), i.e.,

$$d\tilde{x}_1(t) = \alpha\pi\Theta y(t-\tau(t))dW(t), \tag{11}$$

$$d\bar{x}(t) = -\alpha\bar{L}\bar{x}(t-\tau(t))dt + \alpha\psi\Theta y(t-\tau(t))dW(t). \tag{12}$$

Before starting the main result of this paper, we give the following Lemmas.

Lemma 1 [27]**:** Let B, C and D be real matrices of appropriate dimensional with $\|D\| \leq 1$. Then, we have, for any scalar $\zeta > 0$

$$BDC + C^T D^T B^T \leq \zeta BB^T + \zeta^{-1} C^T C. \tag{13}$$

Lemma 2: Let t_0 and r be non-negative constants. Let $m : [t_0 - r, \infty) \mapsto \mathbb{R}^+$ be continuous and satisfy

$$\begin{aligned} D^+ m(t) :&= \limsup_{h \to 0^+} \frac{m(t+h) - m(t)}{h} \\ &\leq -\mu m(t) + \lambda \sup_{-r \leq s \leq 0} m(t+s) \end{aligned}$$

on $[t_0, \infty)$, where μ and λ are constants satisfying $\mu > \lambda > 0$. Then,

$$m(t) \leq m_0 \exp\left[-\rho(t - t_0)\right]$$

hold on $[t_0, \infty)$, where $\rho > 0$ is the root of $-\rho = -\mu + \lambda e^{\rho r}$ and $m_0 = \sup_{-\tau \leq s \leq 0} m(t_0 + s)$.

Proof: The proof is similar to Lemma 5.1 [23] by setting the parameters as $\gamma(t) = 0$ and $c(t) = 1$. Hence it is omitted. □

Lemma 3: Applying the protocol to the system (1) and (2), if Assumption 1 and Assumption 2 hold, then $\int_\tau^t 2\alpha \bar{x}^T(t) P \psi \Theta y(t - \tau(t)) dW(t)$ is a martingale and

$$E \int_\tau^t 2\alpha \bar{x}^T(t) P \psi \Theta y(t - \tau(t)) dW(t) = 0. \tag{14}$$

Proof: Firstly, we aim to obtain the estimation of the solution of the SDDE (12). Denote $f(\bar{x}(t), \bar{x}(t - \tau(t)), t) = -\alpha \bar{L} \bar{x}(t - \tau(t))$ and $g(\bar{x}(t), \bar{x}(t - \tau(t)), t) = \alpha \psi \Theta y(t - \tau(t))$. Obviously, there exist a positive constant k_1 such that

$$\begin{aligned} &|f(\bar{x}(t_1), \bar{x}(t_1 - \tau(t_1)), t_1) - f(\bar{x}(t_2), \bar{x}(t_2 - \tau(t_2)), t_2)|^2 \\ &\qquad \leq k_1 |\bar{x}(t_1 - \tau(t_1)) - \bar{x}(t_2 - \tau(t_2))|^2. \end{aligned} \tag{15}$$

Notice that

$$\begin{aligned} &\Theta\left(y(t - \tau(t))\right)(\Theta y(t - \tau(t)))^T \\ &= diag\left(\sum_{j=1}^n \left(a_{j1}\sigma_{1j} |x_j(t) - x_i(t)|\right)^2, \right. \\ &\qquad \left. \ldots, \sum_{j=1}^n \left(a_{jn}\sigma_{nj} |x_j(t) - x_n(t)|\right)^2 \right) \end{aligned}$$

which is a diagonal matrix. Then, for $g(\bar{x}(t), \bar{x}(t - \tau(t)), t)$, we have

$$\begin{aligned} &|g(\bar{x}(t_1), \bar{x}(t_1 - \tau(t_1)), t_1) - g(\bar{x}(t_2), \bar{x}(t_2 - \tau(t_2)), t_2)|^2 \\ &= |\alpha \psi \Theta\left(y(t_1 - \tau(t_1)) - y(t_2 - \tau(t_2))\right)|^2 \\ &\leq \alpha^2 tr\left(\psi^T \psi\right) tr[\Theta\left(y(t_1 - \tau(t_1)) - y(t_2 - \tau(t_2))\right) \times \\ &\quad (\Theta y(t_1 - \tau(t_1)) - y(t_2 - \tau(t_2)))^T \\ &\leq \alpha^2 tr\left(\psi^T \psi\right) \sum_{i=1}^n \sum_{j \in \mathcal{N}_i} (\sigma_{ji} |(\varphi_i - \varphi_j) \times \\ &\quad (\bar{x}(t_1 - \tau(t_1)) - \bar{x}(t_2 - \tau(t_2)))|)^2 \\ &\leq \left(\alpha^2 tr\left(\psi^T \psi\right) \sum_{i=1}^n \sum_{j \in \mathcal{N}_i} (\sigma_{ji} |(\varphi_i - \varphi_j)|)^2\right) \times \\ &\quad |\bar{x}(t_1 - \tau(t_1)) - \bar{x}(t_2 - \tau(t_2))|^2 \\ &= \alpha^2 c_1 |\bar{x}(t_1 - \tau(t_1)) - \bar{x}(t_2 - \tau(t_2))|^2 \\ &\leq k_1 |\bar{x}(t_1 - \tau(t_1)) - \bar{x}(t_2 - \tau(t_2))|^2, \end{aligned} \tag{16}$$

where

$$c_1 = tr\left(\psi^T \psi\right) \sum_{i=1}^n \sum_{j \in \mathcal{N}_i} (\sigma_{ji} |(\varphi_i - \varphi_j)|)^2. \tag{17}$$

By (15) and (16), the Lipschitz condition is satisfied. Then,

$$\begin{aligned} \left|\bar{x}^T(t) f(\bar{x}(t), \bar{x}(t - \tau(t)), t)\right| &= \left|-\alpha \bar{x}^T(t) \bar{L} \bar{x}(t - \tau(t))\right| \\ &\leq \frac{\alpha}{2}\left(\left|\bar{x}^T(t)\right|^2 + \|\bar{L}\| \left|\bar{x}(t - \tau(t))\right|^2\right) \\ &\leq k_2\left(1 + \left|\bar{x}^T(t)\right|^2 + \left|\bar{x}(t - \tau(t))\right|^2\right), \end{aligned}$$

where k_2 is a positive constant.

By Theorem 7.14 in [28], the solution $\bar{x}(t)$ obeys , for any $p \geq 2$

$$\begin{aligned} &E\left(\sup_{-\tau \leq t \leq T} |\bar{x}(t)|^p\right) \\ &\leq 2^{\frac{1}{2}(p+2)}\left(1 + E|\bar{x}(0)|^p\right) \mathrm{e}^{2K_1 p(10p+1)T} \equiv c_{T,p}, \end{aligned} \tag{18}$$

where $c_{T,p}$ is a positive constant and then we have

$$E\left(\sup_{0 \leq t \leq T} |\bar{x}(t - \tau(t))|^p\right) \leq c_{T,p}. \tag{19}$$

By (18) and (19), we have

$$\begin{aligned} &E \int_\tau^t \left|2\alpha \bar{x}^T(t) P \psi \Theta y(t - \tau(t))\right|^2 ds \\ &\leq 2E \int_\tau^t \left(\left|\alpha \bar{x}^T(t) P\right|^4 + \left|\psi \Theta y(t - \tau(t))\right|^4\right) ds \\ &\leq 2(\alpha \|P\|)^4 \int_\tau^t E|\bar{x}(s)|^4 ds + K_1^2 \int_\tau^t E|x(t - \tau(t))|^4 ds < \infty. \end{aligned}$$

Therefore $\int_\tau^t 2\alpha \bar{x}^T(t) P \psi \Theta y(t - \tau(t)) dW(t)$ is a martingale. By the property of Itô's integral, we have (14). □

Theorem 1: If Assumption 1 and Assumption 2 hold, then the consensus protocol (3) leads to mean square consensus for the agents in (1), if the consensus gain satisfies

$$\alpha < \frac{\lambda_{\min}(P)}{(2 + \varepsilon) c_2 \lambda_{\max}(P)}, \tag{20}$$

and for a given α, the time-delay satisfies

$$\tau < \frac{-c_\tau + \sqrt{c_\tau^2 - 4c_{\tau^2} c_{\tau^0}}}{2c_{\tau^2}}, \tag{21}$$

where the included parameters are determined as follows:

$$c_2=\sum_{i=1}^{n}(\psi^T P\psi)_{ii}\sum_{i\in\mathscr{N}_i}(\sigma_{ji}|\varphi_i-\varphi_j|)^2, \tag{22}$$

$$c_{\tau^2}=\frac{2\varepsilon^{-1}\alpha^2\left\|P^2\right\|\left\|\bar{L}^2\right\|^2}{c_1\lambda_{\min}(P)}+\frac{4c_1\alpha^4\left\|\bar{L}^2\right\|}{\lambda_{\min}(P)}, \tag{23}$$

$$c_{\tau}=\frac{2\varepsilon^{-1}\alpha^2 c_1\left\|P^2\right\|\left\|\bar{L}^2\right\|}{c_2\lambda_{\min}(P)}+\frac{4c_1c_2\alpha^4}{\lambda_{\min}(P)}, \tag{24}$$

$$c_{\tau^0}=-\frac{\alpha}{\lambda_{\max}(P)}+\frac{(2+\varepsilon)c_1\alpha^2}{\lambda_{\min}(P)}, \tag{25}$$

and ε is a positive constant and arbitrarily selected.

Proof: Let $V(t)=\bar{x}^T(t)P\bar{x}(t)$. By applying Itô's formula, we have

$$\begin{aligned}dV(t)=&\alpha\bar{x}^T(t)(-\bar{L}^TP-P\bar{L})\bar{x}(t)dt\\&+2\alpha\bar{x}^T(t)P\bar{L}[\bar{x}(t)-\bar{x}(t-\tau(t))]dt\\&+\alpha^2Tr[\psi^TP\psi(\Theta y(t-\tau(t)))(\Theta y(t-\tau(t)))^T]dt\\&+2\alpha\bar{x}^T(t)P\psi\Theta y(t-\tau(t))dW(t)\end{aligned} \tag{26}$$

By (8), (24) can be simplified as

$$\begin{aligned}dV(t)=&-\alpha\bar{x}^T(t)\bar{x}(t)dt\\&+2\alpha\bar{x}^T(t)P\bar{L}[\bar{x}(t)-\bar{x}(t-\tau(t))]dt\\&+\alpha^2Tr[\psi^TP\psi(\Theta y(t-\tau(t)))(\Theta y(t-\tau(t)))^T]dt\\&+2\alpha\bar{x}^T(t)P\psi\Theta y(t-\tau(t))dW(t).\end{aligned} \tag{27}$$

Simple calculation shows that

$$\begin{aligned}&Tr\left[\psi^TP\psi(\Theta y(t-\tau(t)))(\Theta y(t-\tau(t)))^T\right]\\&=\sum_{i=1}^{n}[\psi^TP\psi]_{ii}\sum_{i\in\mathscr{N}_i}(\sigma_{ji}|(\varphi_i-\varphi_j)\bar{x}(t-\tau(t))|)^2\\&\le\left(\sum_{i=1}^{n}(\psi^TP\psi)_{ii}\sum_{i\in\mathscr{N}_i}(\sigma_{ji}|\varphi_i-\varphi_j|)^2\right)|\bar{x}(t-\tau(t))|^2\\&\equiv c_2|\bar{x}(t-\tau(t))|^2.\end{aligned} \tag{28}$$

We have $\frac{V(t)}{\lambda_{\max}(P)}\le\bar{x}^T(t)\bar{x}(t)$. Therefore,

$$\begin{aligned}dV(t)\le&\\&-\frac{\alpha}{\lambda_{\max}(P)}V(t)dt+2\alpha\bar{x}^T(t)P\bar{L}[\bar{x}(t)-\bar{x}(t-\tau(t))]dt\\&+c_2\alpha^2|\bar{x}(t-\tau(t))|^2dt+2\alpha\bar{x}^T(t)P\psi\Theta y(t-\tau(t))dW(t),\end{aligned} \tag{29}$$

By using Lemma 1, for any $\zeta>0$, we have

$$\begin{aligned}&2\bar{x}^T(t)P\bar{L}[\bar{x}(t)-\bar{x}(t-\tau(t))]\\&\le\zeta\bar{x}^T(t)P^2\bar{x}(t)+\zeta^{-1}[\bar{L}(\bar{x}(t)-\bar{x}(t-\tau(t)))]^2,\end{aligned} \tag{30}$$

and by discrete Hölder inequality,

$$\begin{aligned}|\bar{x}(t-\tau(t))|^2&=|\bar{x}(t)-(\bar{x}(t)-\bar{x}(t-\tau(t)))|^2\\&\le2|\bar{x}(t)|^2+2|\bar{x}(t)-\bar{x}(t-\tau(t))|^2.\end{aligned} \tag{31}$$

Inserting (30) and (31) in (29) gives

$$\begin{aligned}dV(t)\le&-\frac{\alpha}{\lambda_{\max}(P)}V(t)dt+\alpha\zeta\bar{x}^T(t)P^2\bar{x}(t)dt\\&+\alpha\zeta^{-1}[\bar{L}(\bar{x}(t)-\bar{x}(t-\tau(t)))]^2dt+2c_2\alpha^2|\bar{x}(t)|^2dt\\&+2c_2\alpha|\bar{x}(t)-\bar{x}(t-\tau(t))|^2dt\\&+2\alpha\bar{x}^T(t)P\psi\Theta y(t-\tau(t))dW(t).\end{aligned} \tag{32}$$

Now, by integrating (32) from τ to t, and taking the mathematical expectation for both sides and using Lemma 3 yields

$$\begin{aligned}&E[V(t)]-E[V(\tau)]\\&\le-\frac{\alpha}{\lambda_{\max}(P)}\int_{\tau}^{t}E[V(s)]ds\\&+\alpha\zeta\int_{\tau}^{t}E\left[\bar{x}^T(s)P^2\bar{x}(s)\right]ds+2c_2\alpha^2\int_{\tau}^{t}E|\bar{x}(s)|^2ds\\&+\alpha\zeta^{-1}\int_{\tau}^{t}E[\bar{L}(\bar{x}(s)-\bar{x}(s-\tau(s)))]^2ds\\&+2c_2\alpha^2\int_{\tau}^{t}E|\bar{x}(s)-\bar{x}(s-\tau(s))|^2ds.\end{aligned} \tag{33}$$

Rewriting (33) in differential form gives

$$\begin{aligned}D^+E[V(t)]&=\limsup_{h\to0}\frac{E[V(t+h)]-E[V(t)]}{h}\\&\le-\frac{\alpha}{\lambda_{\max}(P)}E[V(t)]+\alpha\zeta E\left[\bar{x}^T(t)P^2\bar{x}(t)\right]\\&+\alpha\zeta^{-1}E[\bar{L}(\bar{x}(t)-\bar{x}(t-\tau(t)))]^2+2c_2\alpha^2E|\bar{x}(t)|^2\\&+2c_2\alpha^2E|\bar{x}(t)-\bar{x}(t-\tau(t))|^2.\end{aligned} \tag{34}$$

We aim to obtain an estimate of the right-hand side of (34). Note that

$$E\left[\bar{x}^T(t)P^2\bar{x}(t)\right]\le\frac{\left\|P^2\right\|}{\lambda_{\min}(P)}E[V(t)], \tag{35}$$

and

$$E|\bar{x}(t)|^2\le\frac{E[V(t)]}{\lambda_{\min}(P)}. \tag{36}$$

On the other hand, from (12) we have

$$\begin{aligned}\bar{x}(t)-\bar{x}(t-\tau(t))=&-\alpha\int_{t-\tau(t)}^{t}\bar{L}\bar{x}(s-\tau(s))ds\\&+\alpha\int_{t-\tau(t)}^{t}\psi\Theta y(s-\tau(s))dW(s).\end{aligned}$$

This implies

$$\begin{aligned}E|\bar{x}(t)-\bar{x}(t-\tau(t))|^2=&E\left|-\alpha\int_{t-\tau(t)}^{t}\bar{L}\bar{x}(s-\tau(s))ds+\right.\\&\left.\alpha\int_{t-\tau(t)}^{t}\psi\Theta y(s-\tau(s))dW(s)\right|^2.\end{aligned}$$

Further,

$$\begin{aligned}
&E|\bar{x}(t)-\bar{x}(t-\tau(t))|^2\\
\leq&2E\left|-\alpha\int_{t-\tau(t)}^{t}\bar{L}\bar{x}(s-\tau(s))ds\right|^2\\
&+2E\left|\alpha\int_{t-\tau(t)}^{t}\psi\Theta y(s-\tau(s))dW(s)\right|^2\\
\leq&2\alpha^2\tau\int_{t-\tau(t)}^{t}E|\bar{L}\bar{x}(s-\tau(s))|^2ds\\
&+2\alpha^2E\int_{t-\tau(t)}^{t}|\psi\Theta y(s-\tau(s))|^2ds\\
\leq&2\alpha^2\tau\left\|\bar{L}^2\right\|\int_{t-\tau}^{t}E|\bar{x}(s-\tau(s))|^2ds\\
&+2\alpha^2c_1\int_{t-\tau}^{t}E|\bar{x}(s-\tau(s))|^2ds\\
=&\left(2\alpha^2\tau\left\|\bar{L}^2\right\|+2\alpha^2c_1\right)\int_{t-\tau}^{t}E|\bar{x}(s-\tau(s))|^2ds\\
\leq&\frac{2\alpha^2\tau\left\|\bar{L}^2\right\|+2\alpha^2c_1}{\lambda_{\min}(P)}\int_{t-\tau}^{t}E\left[V(s-\tau(s))\right]ds\\
\leq&\frac{2\alpha^2\tau^2\left\|\bar{L}^2\right\|+2\alpha^2c_1\tau}{\lambda_{\min}(P)}\underset{-2\tau\leq s\leq 0}{Sup}E\left[V(t+s)\right],
\end{aligned}\tag{37}$$

while the first inequality above is a direct application of discrete Hölder inequality, the second inequality has used Hölder inequality and Itô's formula and the third inequality has used the result of (16). Similarly, we have

$$\begin{aligned}
&E|\bar{L}\left(\bar{x}(t)-\bar{x}(t-\tau(t))\right)|^2\\
\leq&\frac{2\alpha^2\tau^2\left\|\bar{L}^2\right\|^2+2\alpha^2c_1\tau\left\|\bar{L}^2\right\|}{\lambda_{\min}(P)}\underset{-2\tau\leq t\leq 0}{Sup}E\left[V(t+s)\right].
\end{aligned}\tag{38}$$

Then we obtain

$$\begin{aligned}
&D^+E\left[V(t)\right]\\
&\leq\left(-\frac{\alpha}{\lambda_{\max}(P)}+\frac{\alpha\zeta\left\|P^2\right\|}{\lambda_{\min}(P)}+\frac{2c_2\alpha^2}{\lambda_{\min}(P)}\right)E\left[V(t)\right]\\
&+\left(\alpha\zeta^{-1}\left\|\bar{L}^2\right\|+2c_2\alpha^2\right)\times\frac{2\alpha^2\tau^2\left\|\bar{L}^2\right\|+2\alpha^2c_1\tau}{\lambda_{\min}(P)}\\
&\times\underset{-2\tau\leq t\leq 0}{Sup}E\left[V(t+s)\right].
\end{aligned}\tag{39}$$

Substitute another arbitrary constant $\varepsilon=(\alpha c_2)^{-1}\left\|P^2\right\|\zeta$ into (39) , it can be simplified as

$$\begin{aligned}
D^+E\left[V(t)\right]\leq&\left(-\frac{\alpha}{\lambda_{\max}(P)}+\frac{(2+\varepsilon)c_2\alpha^2}{\lambda_{\min}(P)}\right)E\left[V(t)\right]\\
+&\left(\frac{\varepsilon^{-1}\left\|P^2\right\|\left\|\bar{L}^2\right\|}{\alpha c_2}+2c_2\alpha^2\right)\times\frac{2\alpha^2\tau^2\left\|\bar{L}^2\right\|+2\alpha^2c_1\tau}{\lambda_{\min}(P)}\\
&\times\underset{-2\tau\leq t\leq 0}{Sup}E\left[V(t+s)\right].
\end{aligned}\tag{40}$$

Therefore, Lemma 2 can be used in (40), where the $m(t)$ is $E\left[V(t)\right]$. Firstly, we have

$$-\frac{\alpha}{\lambda_{\max}(P)}+\frac{(2+\varepsilon)c_2\alpha^2}{\lambda_{\min}(P)}<0,\tag{41}$$

and

$$\begin{aligned}
&-\frac{\alpha}{\lambda_{\max}(P)}+\frac{(2+\varepsilon)c_2\alpha^2}{\lambda_{\min}(P)}+\left(\frac{\varepsilon^{-1}\left\|P^2\right\|\left\|\bar{L}^2\right\|}{\alpha c_2}+2c_2\alpha^2\right)\\
&\times\frac{2\alpha^2\tau^2\left\|\bar{L}^2\right\|+2\alpha^2c_1\tau}{\lambda_{\min}(P)}<0,
\end{aligned}\tag{42}$$

which is a simple quadratic inequality about the time-delay τ. Substitute parameters (22)-(25) into (41) and (42), we have

$$c_{\tau^0}<0,\tag{43}$$

$$c_{\tau^2}\tau^2+c_\tau\tau+c_{\tau^0}<0.\tag{44}$$

By (43), the condition of the consensus gain is given in (20). By Assumption 2, we have $c_{\tau^2}>0$ and $c_\tau>0$. According to the property of quadratic equation, the condition of the time-delay is given in (21).

Then there exists a positive constant ρ such that $-\rho=c_{\tau^0}+(c_{\tau^2}\tau^2+c_\tau\tau)e^{\rho\tau}$ and $E\left[V(t)\right]$ on $[\tau,\infty)$ satisfies

$$E\left[V(t)\right]\leq\underset{-\tau\leq t\leq\tau}{Sup}E\left[V(s)\right]\exp\left[-\rho(t-t_0)\right],\tag{45}$$

which gives $\lim_{t\to\infty}E\left[V(t)\right]=0$, combing this with (36) yields $\lim_{t\to\infty}E|\bar{x}(t)|^2=0$.

Next, by (11), note that

$$\tilde{x}_1(t)=\tilde{x}_1(0)+\int_0^t\alpha\pi\Theta y(s-\tau(s))dW(s).\tag{46}$$

By (19) and Itô's formula, it follows that

$$\begin{aligned}
&\underset{0\leq t\leq T}{sup}E|\tilde{x}_1(t)|^2\\
&\leq 2\underset{0\leq t\leq T}{sup}\left(E|\tilde{x}_1(0)|^2+E\left|\int_0^t\alpha\pi\Theta y(s-\tau(s))dW(s)\right|^2\right)\\
&\leq 2E|\tilde{x}_1(0)|^2+2\underset{t\geq 0}{sup}\left(\int_0^tE|\alpha\pi\Theta y(s-\tau(s))|^2\right)ds\\
&\leq 2E|\tilde{x}_1(0)|^2+2k_3\int_0^tE\left(\sup_{0\leq t\leq T}|\bar{x}(t-\tau(t))|^p\right)ds\\
&\leq 2E|\tilde{x}_1(0)|^2+k_3c_{T,2}<\infty
\end{aligned}$$

for some positive constant k_3. Due to the martingale convergence theorem, we know that as $t\to\infty$, $\tilde{x}_1(t)$ converges to a random variable with finite second-order moment both in mean square and almost surely. Denote the limit variable by

$$x*=\tilde{x}_1(0)+\int_0^\infty\alpha\pi\Theta y(s-\tau(s))dW(s).$$

Thus,

$$\begin{aligned}
&\lim_{t\to\infty}E|x(t)-x^*\mathbf{1}_n|^2\\
&\leq\lim_{t\to\infty}2\left(E|\tilde{x}_1(t)\mathbf{1}_n-x^*\mathbf{1}_n|^2+E|\varphi\bar{x}(t)|^2\right)=0.
\end{aligned}$$

Therefore, we conclude that the consensus protocol (3) is a mean square consensus protocol. This completes the proof. □

Remark 1: By analyzing the effects of the multiplicative and additive noises, we can see the sufficient conditions under multiplicative noises and delays are relatively weaker, which the consensus gain can be a positive constant, see the results in [23]. The reason is that the noise of one agent can be diminished by the noises of its neighbors by the communication network. Therefore the multiplicative noises have the property of stabilizing effect.

4. SIMULATIONS

In this section, we conduct numerical simulations to demonstrate the effectiveness of our main results to show the consensus protocol (3) can lead to mean square consensus of multi-agent systems with multiplicative noises and time delays under directed fixed topologies. On the other hand, the stabilizing effect of multiplicative noises has been demonstrated by the comparison with the consensus problem under additive noises and delays.

Consider a dynamic network of four agents under directed fixed topology with

$$\mathscr{E} = \{(1,3),(2,1),(3,2),(3,1),(4,3)\}$$

which there is a spanning tree, see Fig. 1. Take the initial states as $x_1(0) = 1$, $x_2(0) = 2$, $x_3(0) = -4$, $x_4(0) = 0$, and the corresponding noise intensities are taken as $\sigma_{31} = \sigma_{12} = \sigma_{23} = \sigma_{13} = \sigma_{34} = 0.5$. Under this topology, the positive-definite matrix satisfying (8) can be deduced as $P = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.2812 & -0.0625 \\ 0 & -0.0625 & 0.25 \end{bmatrix}$, with $\lambda_{\max}(P) = 0.5$ and $\lambda_{\min}(P) = 0.2012$. By setting the arbitrary positive constant $\varepsilon = 1$, we can derive that $\alpha < 0.0634$. Then α can be chosen as $\alpha = 0.05$ to satisfy (20), leading to the result of (21)), i.e., $\tau < 0.2403$. Three cases are simulated for the analysis and comparison, see Figs. 2–4. In Fig. 2, the setting is $\alpha = 0.05$ and $\tau(t) = 0.2\,|\cos(\mathrm{t})|$, and the effectiveness of Theorem 1 is verified for the convergence of states of all the agents. In Fig. 3, the setting is $\alpha = 0.05$ and $\tau(t) = 26\,|\cos(\mathrm{t})|$, the movement of states has been obvious periodic and divergent in the end, which reveals that allowable time-delay is limit under a certain consensus gain. Fig. 4 shows the consensus scenario of states of multi-agent system with additive noises and time-delays under the same parameter setting with Fig. 2. According to the known conclusion [23], the constant consensus gain α, cannot eliminate noise effects while it is beneficial to the analysis of the property of noises. By the comparison between Fig. 2 and Fig. 4, we can see that under the same setting, the states of agents under additive noises cannot

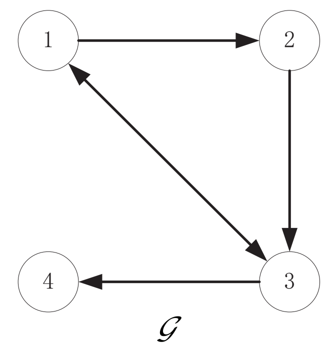

Fig. 1. The digraph of the multi-agent system having a spanning tree.

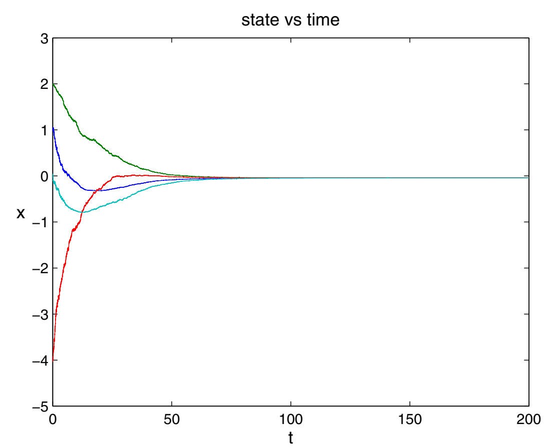

Fig. 2. States of agents with multiplicative noises and time delays ($\alpha = 0.05, \tau(t) = 0.2\,|\cos(\mathrm{t})|$).

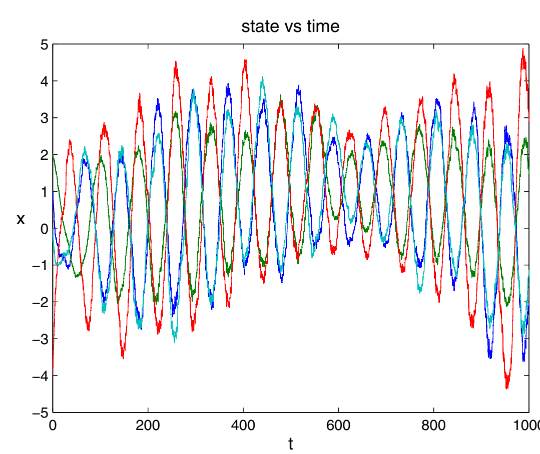

Fig. 3. States of agents with multiplicative noises and time delays ($\alpha = 0.05, \tau(t) = 26\,|\cos(\mathrm{t})|$).

achieve the consensus, which illustrate the stabilizing effect of the multiplicative noises to some extent.

5. CONCLUSIONS

In this paper, the mean square consensus problem of multi-agent systems with multiplicative noises and time

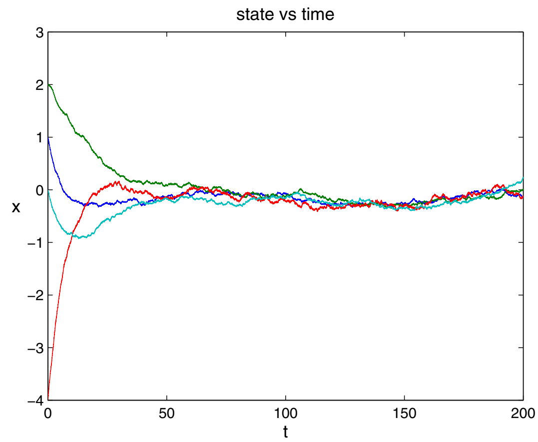

Fig. 4. States of agents with additive noises and time delays ($\alpha = 0.05, \tau(t) = 0.2\,|\cos(t)|$).

delays under directed fixed topologies has been considered. By taking the tools from SDDE, martingale theory and stochastic inequality, together with the property of quadratic inequality, the sufficient conditions composed of the upper bounds of consensus gain and time-delays are obtained for the first time in this case. In addition, by comparing the impact of multiplicative and additive noises, we reach the conclusion that multiplicative noises have the property of stabilizing effect. In future work, the case with multiplicative noises and time delays under switching or stochastic topology is worth investigating.

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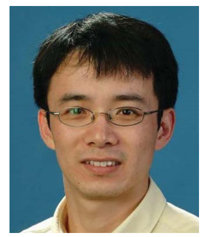
Lei Song is a Ph.D. student with the School of Aeronautics and Astronautics, Shanghai Jiao Tong University, Shanghai, China. His research interests include distributed control theory and multi-agent systems.



Dan Huang received his Ph.D. degree from The University of Auckland, New Zealand, in 2008. Now he is an Associate Professor at Shanghai Jiao Tong University, Shanghai, China. His research interests include distributed control theory and networked control systems.

Sing Kiong Nguang received his Ph.D. degree from The University of Newcastle, Australia in 1995. Now he is a Professor at The University of Auckland, Auckland, New Zealand. His research interests include polynomial control systems and networked control systems.

Shan Fu received his Ph.D. degree from Heriot-Watt University, U.K., in 1995. Now he is a Professor at Shanghai Jiao Tong University, Shanghai, China. His research interests include complex systems and human factors.