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Enhancement on Stability Criteria for Linear Systems with Interval Time-varying Delays

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Abstract: In this paper, the problem of stability for linear systems with interval time-varying delays is investigated. By constructing a suitable augmented Lyapunov-Krasovskii functional and utilizing Wirtinger-based integral inequality, two sufficient conditions for guaranteeing the asymptotic stability of the concerned systems are derived within the framework of linear matrix inequalities (LMIs). The superiority and validity of the proposed criteria are verified by comparing maximum delay bounds under various conditions via two numerical examples.

Keywords: Interval time-varying delay, LMIs, linear systems, Lyapunov method, stability.

1. INTRODUCTION

Time-delay has gained considerable attentions due to one of the constraints in many fields such as physical, industrial and engineering systems such as aircrafts, biological systems, population dynamics, neural networks, networked control systems, and so on. For examples, see [1–3] and references therein. It is well known that time-delay often causes undesirable dynamic behaviors such as oscillation and instability of systems. One major issue in stability analysis of time-delay systems is to develop less conservative delay-dependent stability conditions, which provide upper bounds of time-delay guaranteeing the asymptotic stability. In general, stability analysis for time-delay systems can be classified into two categories. One is delay-dependent analysis which includes the information about the size of time-delay, and the other is delay-independent analysis which do not use the information. Generally, when the size of time-delay is small, the delay-dependent case is less conservative than the delay-independent case. Therefore, in regard to this, a lot of results on delay-dependent stability conditions for time-delay systems have been addressed in the literature [4–8].

Naturally, in the past few years, in order to find improved stability criteria, various techniques such as Jensen's inequality application [9], cross terms [10–13], free-weighting matrices [14], the reciprocally convex approach [15, 16], bounding techniques [17] and delay-range-dependent method [18] were introduced. Sun et al. [19] proposed the triple integral forms of Lyapunov-Krasovskii functional and showed their effectiveness in reducing the conservatism of stability criteria, and after this, many researchers utilize the Lyapunov-Krasovskii functional containing triple integral form in stability analysis for time-delay systems. Above this, the meaningful approaches were proposed in [20–26]. Recently, the Wirtinger-based integral inequality [27] was presented to find more tighter lower bound of integral of single state quadratic term than the lower bound of Jensen's inequality [3], and the inequality was also addressed in [28]. Very recently, based on the result of Wirtinger-based integral inequality, the Wirtinger-based double integral inequality was presented by [29] for the quadratic double integral form. Moreover, in [30], various forms of the Wirtinger-based integral inequality were generalized by the auxiliary function. In [31], the new refined Jensen-based inequalities which are the same with the inequality in Remark 4 [30] were proposed. However, the constructed double integral in Lyapunov-Krasovskii functional is single state quadratic form. Therefore, there is room for further enhancement in stability analysis of time-delay systems. In this regard, we address question "how to construct the Lyapunov-Krasovskii functional?" Notably, it can be remarkable that there are various forms of the functional. In this paper, two types of the functional are proposed to the use of the various cross terms to im-

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prove the stability conditions for time-delay systems.

Motivated by the discussion above, this paper deals with the problem of stability criteria for linear systems with time-varying delays. Here, we consider an interval time-varying delays containing the lower and upper bounds of time-delay. To solve the problem mentioned above, by construction of an augmented Lyapunov-Krasovskii functional and utilization of some mathematical techniques, a stability criterion will be derived in Theorem 1. Based on the result by Theorem 1, a more enhanced stability criterion which utilizes the triple integral functional will be proposed in Theorem 2. In deriving lower bound of double integral terms, Wirtinger-based double integral inequality [29] will be utilized. Through two examples, it will be shown that the stability criteria introduced in Theorems 1 and 2 can improve the feasible region guaranteeing the asymptotic stability for such systems by comparing the results in some recent papers [16, 17, 25, 26, 31–35].

Notation: $\mathbb{R}$, $\mathbb{R}^n$ and $\mathbb{R}^{m\times n}$ denote the sets of real numbers, n-vectors with the l_2-norm $\|\cdot\|$ and $m\times n$ matrices, respectively. $\mathbb{C}_{n,h}=\mathbb{C}([-h,0],\mathbb{R}^n)$ denotes the Banach space of continuous functions mapping the interval $[-h,0]$ into $\mathbb{R}^n$, with the topology of uniform convergence. $\mathbb{S}^n$ and $\mathbb{S}^n_+$ are the sets of symmetric and positive definite $n\times n$ matrices, respectively. I_n, 0_n and $0_{m\cdot n}$ denote $n\times n$ identity matrix, $n\times n$ and $m\times n$ zero matrices, respectively. $X>0$ (<0) represents positive (negative) definite matrix. $X^{\perp}$ denotes a basis for the nullspace of X. $\mathtt{diag}\{\cdots\}$, $\mathtt{sym}\{X\}$ and $\mathtt{col}\{x_1,\ldots,x_n\}$ stand for, respectively, the (block) diagonal matrix, the sum $X+X^T$ with the square matrix X, and the column vector with the vectors $x_1,\ldots,x_2$. The symmetric blocks will be readily denoted by $\star$ when necessary. $X_{[f(t)]}$ means that its elements include the scalar value of $f(t)$ affinely.

2. PRELIMINARIES

Consider the following linear systems with time-varying delays

$$\begin{aligned}&\dot{x}(t)=Ax(t)+A_dx(t-h(t)),\ \ \forall\, t>0,\\ &x(s)=\phi(s),\ \ \forall s\in[-h_U,0],\ \ h_U>0,\end{aligned}\tag{1}$$

where $x(t)\in\mathbb{R}^n$ is the state vector, $A\in\mathbb{R}^{n\times n}$ and $A_d\in\mathbb{R}^{n\times n}$ are known constant matrices, $\phi(s)\in\mathbb{C}_{n,h_U}$ is a given continuous vector valued initial function, and $h(t)$ is a time-varying delay satisfying

$$0\le h_L\le h(t)\le h_U,\ \ \dot{h}(t)\le h_D,\ \ \forall t>0,\tag{2}$$

in which h_L and h_U are known positive scalars and h_D is any constant one.

The aim of this paper is to investigate the stability analysis for systems (1). Moreover, to derive a main result, the following lemmas are used.

Lemma 1: For any matrix $R\in\mathbb{S}^n_+$, given scalars a and b satisfying $a<b$, the following inequalities hold for all continuously differentiable function x in $[a,b]\to\mathbb{R}^n$:

i) Wirtinger-based inequality [27]

$$\begin{aligned}&(b-a)\int_a^b x^T(s)Rx(s)ds\\ &\ge\left(\int_a^b x(s)ds\right)^T R\left(\int_a^b x(s)ds\right)\\ &+3\mathscr{J}_1^T(x)R\mathscr{J}_1(x),\end{aligned}$$

ii) Wirtinger-based double inequality [29]

$$\begin{aligned}&\frac{(b-a)^2}{2}\int_a^b\int_s^b x^T(u)Rx(u)duds\\ &\ge\left(\int_a^b\int_s^b x(u)duds\right)^T R\left(\int_a^b\int_s^b x(u)duds\right)\\ &+2\mathscr{J}_2^T(x)R\mathscr{J}_2(x),\end{aligned}$$

where $\mathscr{J}_1(x)=\int_a^b x(s)ds-\frac{2}{b-a}\int_a^b\int_a^s x(u)duds$ and $\mathscr{J}_2(x)=-\int_a^b\int_s^b x(u)duds+\frac{3}{b-a}\int_a^b\int_s^b\int_u^b x(v)dvduds$.

Lemma 2: For any vectors $\{x_i\in\mathbb{R}^n\}_{i=1}^2$, matrices $R\in\mathbb{S}^n_+$, $M\in\mathbb{R}^{n\times n}$, real scalars $\{\alpha_i\ge 0\}_{i=1}^2$ satisfying $\Psi=\begin{bmatrix}R & M\\ \star & R\end{bmatrix}>0$, $\alpha_1+\alpha_2=1$, and $x_i=0$ if $\alpha_i=0$, the following inequality holds:

i) First-order reciprocally convex lemma [15]

$$\frac{1}{\alpha_1}x_1^TRx_1+\frac{1}{\alpha_2}x_2^TRx_2\ge\begin{bmatrix}x_1\\ x_2\end{bmatrix}^T\Psi\begin{bmatrix}x_1\\ x_2\end{bmatrix},$$

ii) Second-order reciprocally convex lemma [16]

$$\frac{1}{\alpha_1^2}x_1^TRx_1+\frac{1}{\alpha_2^2}x_2^TRx_2\ge\begin{bmatrix}x_1\\ x_2\end{bmatrix}^T\Psi\begin{bmatrix}x_1\\ x_2\end{bmatrix}.$$

Lemma 3: Let $x\in\mathbb{R}^n$, $A\in\mathbb{S}^n$, $B\in\mathbb{R}^{m\times n}$ such that $\mathtt{rank}\{B\}<n$, $C\in\mathbb{S}^m_+$ and any matrix $D\in\mathbb{R}^{n\times m}$. The following statements are equivalent:

i) Equivalence 1 (Finsler's lemma) [36]

$$x^TAx<0,\ \forall Bx=0,\ x\neq 0\ \ \Leftrightarrow\ \ {B^{\perp}}^TAB^{\perp}<0,$$

ii) Equivalence 2 [9]

$$A-B^TCB<0\ \ \Leftrightarrow\ \ \begin{bmatrix}A+B^TD & \star\\ +D^TB & \\ D & -C\end{bmatrix}<0.$$

3. MAIN RESULTS

In this section, two stability conditions for system (1) are presented. For simplicity of matrix and vector notations, some scalars and matrices are defined as follows:

$$
\begin{aligned}
\zeta(t) &= \mathtt{col}\{x(t), x(t-h(t)), x(t-h_L), x(t-h_U),\\
&\quad \dot{x}(t), \dot{x}(t-h_L), \dot{x}(t-h_U), \tfrac{1}{h_L}\textstyle\int_{t-h_L}^{t} x(s)ds,\\
&\quad \tfrac{1}{h(t)-h_L}\textstyle\int_{t-h(t)}^{t-h_L} x(s)ds, \tfrac{1}{h_U-h(t)}\textstyle\int_{t-h_U}^{t-h(t)} x(s)ds,\\
&\quad \tfrac{1}{h_L}\textstyle\int_{t-h_L}^{t}\int_{s}^{t} x(u)duds, \tfrac{1}{h(t)-h_L}\textstyle\int_{t-h(t)}^{t-h_L}\int_{s}^{t-h_L} x(u)duds,\\
&\quad \tfrac{1}{h_U-h(t)}\textstyle\int_{t-h_U}^{t-h(t)}\int_{s}^{t-h(t)} x(u)duds\}\\
\Pi_{1,1[h(t)]} &= [e_1, e_3, e_4, h_L e_8,\\
&\quad (h(t)-h_L)e_9 + (h_U-h(t))e_{10}],\\
\Pi_{1,2} &= [e_5, e_6, e_7, e_1-e_3, e_3-e_4],\\
\Pi_2 &= [e_5, e_1, e_6, e_3, e_7, e_4],\\
\Pi_{3,1} &= [e_1, 0_{13n\cdot n}, e_3, e_1-e_3],\\
\Pi_{3,2} &= [e_3, 0_{13n\cdot n}, e_2, e_3-e_2],\\
\Pi_4 &= [e_1-e_3, h_L e_8, -e_1-e_3+2e_8, h_L e_8 - 2e_{11}],\\
\Pi_{5,1[h(t)]} &= [e_3-e_2, (h(t)-h_L)e_9,\\
&\quad -e_3-e_2+2e_9, (h(t)-h_L)e_9 - 2e_{12}],\\
\Pi_{5,2[h(t)]} &= [e_2-e_4, (h_U-h(t)e_{10},\\
&\quad -e_2-e_4+2e_{10}, (h_U-h(t))e_{10} - 2e_{13}],\\
\Xi_{1[h(t)]} &= \mathtt{sym}\{\Pi_{1,1[h(t)]} P \Pi_{1,2}^T\},\\
\Xi_2 &= \Pi_2 \mathtt{diag}\{N_1, N_2-N_1, -N_2\}\Pi_2^T,\\
\Xi_{3,1} &= \Pi_{3,1}\mathtt{diag}\{G_1, -G_1\}\Pi_{3,1}^T\\
&\quad + \mathtt{sym}\{h_L[e_8, e_1-e_8]G_1[0_{13n\cdot n}, e_5]^T\},\\
\Xi_{3,2[h(t)]} &= \Pi_{3,2}\mathtt{diag}\{G_2, -(1-h_D)G_2\}\Pi_{3,2}^T\\
&\quad + \mathtt{sym}\{(h(t)-h_L)[e_9, e_3-e_9]G_2[0_{13n\cdot n}, e_6]^T\},\\
\Xi_4 &= h_L^2[e_5, e_1]Q_1[e_5, e_1]^T - \Pi_4 \mathtt{diag}\{Q_1, 3Q_1\}\Pi_4^T,\\
\Xi_5 &= (h_U-h_L)^2[e_6, e_3]Q_2[e_6, e_3]^T,\\
\Xi_{ze} &= (h_U-h_L)[e_3, e_2, e_4]\mathtt{diag}\{P_1, P_2-P_1, -P_2\}\\
&\quad \times [e_3, e_2, e_4]^T,\\
\tilde{\Xi}_{[h(t)]} &= \Xi_{1[h(t)]} + \Xi_2 + \Xi_{3,1} + \Xi_{3,2[h(t)]}\\
&\quad + \Xi_4 + \Xi_5 + \Xi_{ze},\\
\Gamma &= Ae_1^T + A_d e_2^T - I_n e_5^T,\\
\Lambda_{1[h(t)]} &= [\Pi_{5,1[h(t)]}, \Pi_{5,2[h(t)]}],\\
\mathscr{Q}_i &= Q_2 + \begin{bmatrix} 0_n & P_i \\ P_i & 0_n \end{bmatrix} \ (i=1,2),\\
\Omega_1 &= \begin{bmatrix} \mathtt{diag}\{\mathscr{Q}_1, 3\mathscr{Q}_1\} & M_1 \\ \star & \mathtt{diag}\{\mathscr{Q}_2, 3\mathscr{Q}_2\} \end{bmatrix}, \quad (3)
\end{aligned}
$$

where $e_i = [0_{n\cdot(i-1)n}, I_n, 0_{n\cdot(13-i)n}]^T \in \mathbb{R}^{13n\times n}$ $(i=1,2,\ldots,13)$ are the block entry matrices, *e.g.*, $e_2^T\zeta(t) = x(t-h(t))$. Now, the first result is given by the following theorem:

Theorem 1: For given scalars h_L, h_U and h_D satisfying (2), the system (1) is asymptotically stable, if there exist matrices $R \in \mathbb{S}_+^{5n}$, $N_i \in \mathbb{S}_+^{2n}$, $G_i \in \mathbb{S}_+^{2n}$, $Q_i \in \mathbb{S}_+^{2n}$, $P_i \in \mathbb{S}^n$ $(i=1,2)$, $M_1 \in \mathbb{R}^{4n\times 4n}$ and $F_1 \in \mathbb{R}^{8n\times 12n}$ satisfying the following LMIs:

$$
\begin{bmatrix} {\Gamma^{\perp}}^T \tilde{\Xi}_{[h_L]}\Gamma^{\perp} + \mathtt{sym}\{{\Gamma^{\perp}}^T \Lambda_{1[h_L]}^T F_1\} & \star \\ F_1 & -\Omega_1 \end{bmatrix} < 0, \quad (4)
$$

$$
\begin{bmatrix} {\Gamma^{\perp}}^T \tilde{\Xi}_{[h_U]}\Gamma^{\perp} + \mathtt{sym}\{{\Gamma^{\perp}}^T \Lambda_{1[h_U]}^T F_1\} & \star \\ F_1 & -\Omega_1 \end{bmatrix} < 0. \quad (5)
$$

Proof: Consider the Lyapunov-Krasovskii functional candidate given by

$$
V(t) = \sum_{i=1}^{5} V_i, \quad (6)
$$

where

$$
\begin{aligned}
V_1 &= \varpi_1^T(t) R \varpi_1(t),\\
V_2 &= \int_{t-h_L}^{t} \varpi_2^T(s)N_1\varpi_2(s)ds + \int_{t-h_U}^{t-h_L} \varpi_2^T(s)N_2\varpi_2(s)ds,\\
V_3 &= \int_{t-h_L}^{t} \varpi_3^T(t,s)G_1\varpi_3(t,s)ds\\
&\quad + \int_{t-h(t)}^{t-h_L} \varpi_3^T(t-h_L,s)G_2\varpi_3(t-h_L,s)ds,\\
V_4 &= h_L\int_{t-h_L}^{t}\int_{s}^{t} \varpi_2^T(u)Q_1\varpi_2(u)duds,\\
V_5 &= (h_U-h_L)\int_{t-h_U}^{t-h_L}\int_{s}^{t-h_L} \varpi_2^T(u)Q_2\varpi_2(u)duds
\end{aligned}
$$

with $\varpi_1(t) = \mathtt{col}\{x(t),\, x(t-h_L),\, x(t-h_U),\, \int_{t-h_L}^{t} x(s)ds,\, \int_{t-h_U}^{t-h_L} x(s)ds\}$, $\varpi_2(t) = \mathtt{col}\{\dot{x}(t), x(t)\}$ and $\varpi_3(t,s) = \mathtt{col}\{x(s), \int_s^t \dot{x}(u)du\}$.
In numerical order, time-differentiating $V(t)$ leads to

$$
\begin{aligned}
\dot{V}_1 &= 2\varpi_1^T(t) R \underbrace{\begin{bmatrix} \dot{x}(t) \\ \dot{x}(t-h_L) \\ \dot{x}(t-h_U) \\ x(t)-x(t-h_L) \\ x(t-h_L)-x(t-h_U) \end{bmatrix}}_{\dot{\varpi}_1(t)}\\
&= \zeta^T(t)\Xi_{1[h(t)]}\zeta(t),\\
\dot{V}_2 &= \varpi_2^T(t)N_1\varpi_2(t) - \varpi_2^T(t-h_L)N_1\varpi_2(t-h_L)\\
&\quad + \varpi_2^T(t-h_L)N_2\varpi_2(t-h_L)\\
&\quad - \varpi_2^T(t-h_U)N_2\varpi_2(t-h_U)\\
&= \zeta^T(t)\Xi_2\zeta(t),\\
\dot{V}_3 &= \varpi_3^T(t,t)G_1\varpi_3(t,t) - \varpi_3^T(t,t-h_L)G_1\varpi_3(t,t-h_L)\\
&\quad + 2\int_{t-h_L}^{t} \varpi_3^T(t,s)G_1\frac{\partial \varpi_3(t,s)}{\partial t}ds\\
&\quad + \varpi_3^T(t-h_L,t-h_L)G_2\varpi_3(t-h_L,t-h_L)\\
&\quad - (1-\dot{h}(t))\varpi_3^T(t-h_L,t-h(t))G_2
\end{aligned}
$$

$$
\begin{aligned}
&\times \varpi_3(t-h_L,t-h(t)) \\
&+2\int_{t-h(t)}^{t-h_L}\varpi_3^T(t-h_L,s)G_2\frac{\partial \varpi_3(t-h_L,s)}{\partial t}ds, \\
\le& \begin{bmatrix} x(t) \\ 0_n \end{bmatrix}^T G_1 \begin{bmatrix} x(t) \\ 0_n \end{bmatrix} \\
&-\begin{bmatrix} x(t-h_L) \\ x(t)-x(t-h_L) \end{bmatrix}^T G_1 \begin{bmatrix} x(t-h_L) \\ x(t)-x(t-h_L) \end{bmatrix} \\
&+2\begin{bmatrix} \int_{t-h_L}^{t} x(s)ds \\ h_L x(t)-\int_{t-h_L}^{t} x(s)ds \end{bmatrix}^T G_1 \begin{bmatrix} 0_n \\ \dot{x}(t) \end{bmatrix} \\
&+\begin{bmatrix} x(t-h_L) \\ 0_n \end{bmatrix}^T G_2 \begin{bmatrix} x(t-h_L) \\ 0_n \end{bmatrix} \\
&-(1-h_D)\begin{bmatrix} x(t-h(t)) \\ x(t-h_L)-x(t-h(t)) \end{bmatrix}^T G_2 \\
&\times \begin{bmatrix} x(t-h(t)) \\ x(t-h_L)-x(t-h(t)) \end{bmatrix} \\
&+2\begin{bmatrix} \int_{t-h(t)}^{t-h_L} x(s)ds \\ (h(t)-h_L)x(t-h_L)-\int_{t-h(t)}^{t-h_L} x(s)ds \end{bmatrix}^T \\
&\times G_2 \begin{bmatrix} 0_n \\ \dot{x}(t-h_L) \end{bmatrix} \\
=&\zeta^T(t)(\Xi_{3,1}+\Xi_{3,2[h(t)]})\zeta(t),
\end{aligned}
$$

$$
\begin{aligned}
\dot{V}_4 =& h_L^2\varpi_2^T(t)Q_1\varpi_2(t)-h_L\int_{t-h_L}^{t}\varpi_2^T(s)Q_1\varpi_2(s)ds, \\
\dot{V}_5 =& (h_U-h_L)^2\varpi_2^T(t-h_L)Q_2\varpi_2(t-h_L) \\
&-(h_U-h_L)\int_{t-h_U}^{t-h_L}\varpi_2^T(s)Q_2\varpi_2(s)ds. \qquad (7)
\end{aligned}
$$

Moreover, by combining the zero equality inspired by the work [10]

$$
\begin{aligned}
&x^T(t-h_L)P_1x(t-h_L)-x^T(t-h(t))P_1x(t-h(t)) \\
&-2\int_{t-h(t)}^{t-h_L}x^T(s)P_1\dot{x}(s)ds+x^T(t-h(t))P_2x(t-h(t)) \\
&-x^T(t-h_U)P_2x(t-h_U)-2\int_{t-h_U}^{t-h(t)}x^T(s)P_2\dot{x}(s)ds \\
=&0 \qquad (8)
\end{aligned}
$$

with $\dot{V}_5$, we get

$$
\begin{aligned}
\dot{V}_5 =& \zeta^T(t)(\Xi_5+\Xi_{ze})\zeta(t) \\
&-(h_U-h_L)\int_{t-h(t)}^{t-h_L}\varpi_2^T(s)\mathscr{Q}_1\varpi_2(s)ds \\
&-(h_U-h_L)\int_{t-h_U}^{t-h(t)}\varpi_2^T(s)\mathscr{Q}_2\varpi_2(s)ds. \qquad (9)
\end{aligned}
$$

Here, by Lemma 1 i), three integral terms in $\dot{V}_4$ and $\dot{V}_5$ of (9) can be bounded as follows:

• The integral term in $\dot{V}_4$:

$$
\begin{aligned}
&h_L\int_{t-h_L}^{t}\varpi_2^T(s)Q_1\varpi_2(s)ds \\
\ge& \left(\int_{t-h_L}^{t}\begin{bmatrix} \dot{x}(s) \\ x(s) \end{bmatrix}ds\right)^T Q_1 \left(\int_{t-h_L}^{t}\begin{bmatrix} \dot{x}(s) \\ x(s) \end{bmatrix}ds\right) \\
&+3\left(\int_{t-h_L}^{t}\begin{bmatrix} \dot{x}(s) \\ x(s) \end{bmatrix}ds - \frac{2}{h_L}\int_{t-h_L}^{t}\int_{s}^{t}\begin{bmatrix} \dot{x}(u) \\ x(u) \end{bmatrix}duds\right)^T Q_1(\star) \\
=& \begin{bmatrix} x(t)-x(t-h_L) \\ \int_{t-h_L}^{t} x(s)ds \end{bmatrix}^T Q_1 \begin{bmatrix} x(t)-x(t-h_L) \\ \int_{t-h_L}^{t} x(s)ds \end{bmatrix} \\
&+3\begin{bmatrix} -x(t)-x(t-h_L)+\frac{2}{h_L}\int_{t-h_L}^{t}x(s)ds \\ \int_{t-h_L}^{t}x(s)ds-\frac{2}{h_L}\int_{t-h_L}^{t}\int_{s}^{t}x(u)duds \end{bmatrix}^T Q_1(\star) \\
=& \zeta^T(t)\Pi_4\mathtt{diag}\{Q_1,3Q_1\}\Pi_4^T\zeta(t). \qquad (10)
\end{aligned}
$$

• The two integral terms in $\dot{V}_5$ of (9):

$$
\begin{aligned}
&(h_U-h_L)\int_{t-h(t)}^{t-h_L}\varpi_2^T(s)\mathscr{Q}_1\varpi_2(s)ds \\
\ge& \frac{1}{\alpha(t)}\left(\int_{t-h(t)}^{t-h_L}\begin{bmatrix} \dot{x}(s) \\ x(s) \end{bmatrix}ds\right)^T \mathscr{Q}_1(\star) \\
&+\frac{3}{\alpha(t)}\left(\int_{t-h(t)}^{t-h_L}\begin{bmatrix} \dot{x}(s) \\ x(s) \end{bmatrix}ds - \frac{2}{h(t)-h_L}\int_{t-h(t)}^{t-h_L}\int_{s}^{t-h_L}\begin{bmatrix} \dot{x}(u) \\ x(u) \end{bmatrix}duds\right)^T \mathscr{Q}_1(\star) \\
=& \frac{1}{\alpha(t)}\begin{bmatrix} x(t-h_L)-x(t-h(t)) \\ \int_{t-h(t)}^{t-h_L}x(s)ds \end{bmatrix}^T \mathscr{Q}_1[\star] \\
&+\frac{3}{\alpha(t)}\begin{bmatrix} -x(t-h_L)-x(t-h(t)) + \frac{2}{h(t)-h_L}\int_{t-h(t)}^{t-h_L}x(s)ds \\ \int_{t-h(t)}^{t-h_L}x(s)ds - \frac{2}{h(t)-h_L}\int_{t-h(t)}^{t-h_L}\int_{s}^{t-h_L}x(u)duds \end{bmatrix}^T \\
&\times \mathscr{Q}_1[\star] \\
=& \frac{1}{\alpha(t)}\zeta^T(t)\Pi_{5,1}\mathtt{diag}\{\mathscr{Q}_1,3\mathscr{Q}_1\}\Pi_{5,1}^T\zeta(t) \qquad (11)
\end{aligned}
$$

and

$$
\begin{aligned}
&(h_U-h_L)\int_{t-h_U}^{t-h(t)}\varpi_2^T(s)\mathscr{Q}_2\varpi_2(s)ds \\
\ge& \frac{1}{1-\alpha(t)}\left(\int_{t-h_U}^{t-h(t)}\begin{bmatrix} \dot{x}(s) \\ x(s) \end{bmatrix}ds\right)^T \mathscr{Q}_2(\star) \\
&+\frac{3}{1-\alpha(t)}\left(\int_{t-h_U}^{t-h(t)}\begin{bmatrix} \dot{x}(s) \\ x(s) \end{bmatrix}ds - \frac{2}{h_U-h(t)}\int_{t-h_U}^{t-h(t)}\int_{s}^{t-h(t)}\begin{bmatrix} \dot{x}(u) \\ x(u) \end{bmatrix}duds\right)^T \mathscr{Q}_2(\star) \\
=& \frac{1}{1-\alpha(t)}\begin{bmatrix} x(t-h(t))-x(t-h_U) \\ \int_{t-h_U}^{t-h(t)}x(s)ds \end{bmatrix}^T \mathscr{Q}_2[\star] \\
&+\frac{3}{1-\alpha(t)}\begin{bmatrix} -x(t-h(t))-x(t-h_U) + \frac{2}{h_U-h(t)}\int_{t-h_U}^{t-h(t)}x(s)ds \\ \int_{t-h_U}^{t-h(t)}x(s)ds - \frac{2}{h_U-h(t)}\int_{t-h_U}^{t-h(t)}\int_{s}^{t-h(t)}x(u)duds \end{bmatrix}^T
\end{aligned}
$$

$$\times \mathscr{Q}_2 [\ \star\ ]$$
$$= \frac{1}{1-\alpha(t)} \zeta^T(t) \Pi_{5,2} \mathtt{diag}\{\mathscr{Q}_2, 3\mathscr{Q}_2\} \Pi_{5,2}^T \zeta(t), \quad (12)$$

where $\alpha(t) = \frac{h(t)-h_L}{h_U-h_L}$.
As a result, the upper bounds of $\dot{V}_4$ and $\dot{V}_5$ are obtained as

$$\dot{V}_4 \leq \zeta^T(t) \Xi_4 \zeta(t), \quad (13)$$
$$\begin{aligned} \dot{V}_5 \leq & \zeta^T(t)(\Xi_5 + \Xi_{ze})\zeta(t) \\ & - \frac{1}{\alpha(t)} \zeta^T(t) \Pi_{5,1} \mathtt{diag}\{\mathscr{Q}_1, 3\mathscr{Q}_1\} \Pi_{5,1}^T \zeta(t) \\ & - \frac{1}{1-\alpha(t)} \zeta^T(t) \Pi_{5,2} \mathtt{diag}\{\mathscr{Q}_2, 3\mathscr{Q}_2\} \Pi_{5,2}^T \zeta(t). \end{aligned} \quad (14)$$

Furthermore, from (12), by Lemma 2 i), if $\Omega_1 > 0$, then the $\dot{V}_5$ can be rebounded as for any matrix M_1

$$\begin{aligned} \dot{V}_5 \leq & \zeta^T(t)(\Xi_5 + \Xi_{ze})\zeta(t) \\ & - \zeta^T(t) \Lambda_{1[h(t)]} \Omega_1 \Lambda_{1[h(t)]}^T \zeta(t). \end{aligned} \quad (15)$$

Hence, an upper bound of $\dot{V}(t)$ is obtained as follows:

$$\dot{V}(t) \leq \zeta^T(t) \tilde{\Xi}_{[h(t)]} \zeta(t) - \zeta^T(t) \Lambda_{1[h(t)]} \Omega_1 \Lambda_{1[h(t)]}^T \zeta(t), \quad (16)$$

where $\tilde{\Xi}_{[h(t)]} = \Xi_{1[h(t)]} + \Xi_2 + \Xi_{3,1} + \Xi_{3,2[h(t)]} + \Xi_4 + \Xi_5 + \Xi_{ze}$.
Here, the following condition is the stability condition for system (1):

$$\begin{aligned} & \zeta^T(t) \tilde{\Xi}_{[h(t)]} \zeta(t) - \zeta^T(t) \Lambda_{1[h(t)]} \Omega_1 \Lambda_{1[h(t)]}^T \zeta(t) < 0 \\ & \text{s.t. } \Gamma \zeta(t) = 0, \end{aligned} \quad (17)$$

where $\Gamma = A e_1^T + A_d e_2^T - I_n e_5^T$, which is equivalent to from Lemma 3 i)

$$\Gamma^{\perp T} \tilde{\Xi}_{[h(t)]} \Gamma^{\perp} - \Gamma^{\perp T} \Lambda_{1[h(t)]} \Omega_1 \Lambda_{1[h(t)]}^T \Gamma^{\perp} < 0. \quad (18)$$

In succession, since the inequality (18) is not affinely dependent on $h(t)$, using Lemma 3 ii) changes from (18) to the following LMI form

$$\begin{bmatrix} \Gamma^{\perp T} \tilde{\Xi}_{[h(t)]} \Gamma^{\perp} + \mathtt{sym}\{\Gamma^{\perp T} \Lambda_{1[h(t)]} F_1\} & \star \\ F_1 & -\Omega_1 \end{bmatrix} < 0 \quad (19)$$

for any matrix F_1 with an appropriate dimension.
Therefore, if the LMI (19) holds then the condition (17) is satisfied, which means that system (1) is asymptotically stable. Hence, in order to hold the LMI (19), we only have to solve the LMIs (4) and (5), which correspond to two vertices of LMI (19). It should be noted that if LMIs LMIs (4) and (5), then $\Omega_1 > 0$ is satisfied. This completes our proof. □

Remark 1: By constructing the integral terms (V_4 and V_5) of augmented state quadratic form in Lyapunov-Krasovskii functional $V(t)$, their time-derivative values were estimated by utilizing Wirtinger-based integral inequality, respectively. Very recently, further improved inequality than Wirtinger-based integral inequality was proposed in [31] which is the same with Remark 4 in [30]. However, the constructed double integral terms in Lyapunov-Krasovsii functional are based on single-state quadratic form. Furthermore, the terms $\dot{x}(t-h_L)$ and $\dot{x}(t-h_U)$ have not been considered as elements of augmented vector $\zeta(t)$. In next section, it will be shown that Theorem 1 can provide larger delay bounds than those of [31], which shows the effectiveness in reducing the conservatism of stability criteria when the double integral terms of augmented state quadratic form are chosen even though Wirtinger-based integral inequality which provide more loose bound than the integral inequality [31] is utilized.

To improve the result of the stability criterion of system (1) more, the following two functionals

$$\begin{aligned} V_6 = & \frac{h_L^2}{2} \int_{t-h_L}^{t} \int_{s}^{t} \int_{u}^{t} \dot{x}^T(v) Q_3 \dot{x}(v) dv du ds, \\ V_7 = & \frac{(h_U-h_L)^2}{2} \\ & \times \int_{t-h_U}^{t-h_L} \int_{s}^{t-h_L} \int_{u}^{t-h_L} \dot{x}^T(v) Q_4 \dot{x}(v) dv du ds, \end{aligned} \quad (20)$$

will be considered in addition to the Lyapnov-Krasovskii functional (6) employed in Theorem 1. Then, the following theorem is introduced as the result with the functional (20).

Theorem 2: For given scalars h_L, h_U and h_D satisfying (2), the system (1) is asymptotically stable, if there exist matrices $R \in \mathbb{S}_+^{5n}$, $N_i \in \mathbb{S}_+^{2n}$, $G_i \in \mathbb{S}_+^{2n}$, $Q_i \in \mathbb{S}_+^{2n}$, $Q_{i+2} \in \mathbb{S}_+^{n}$, $P_i \in \mathbb{S}^n$ $(i = 1,2)$, $M_1 \in \mathbb{R}^{4n \times 4n}$, $M_2 \in \mathbb{R}^{2n \times 2n}$, $F_1 \in \mathbb{R}^{8n \times 12n}$ and $F_2 \in \mathbb{R}^{4n \times 12n}$ satisfying the following LMIs:

$$\begin{bmatrix} \Gamma^{\perp T} \hat{\Xi}_{[h_L]} \Gamma^{\perp} + \mathtt{sym}\{\Gamma^{\perp T} \Lambda_{1[h_L]}^T F_1\} + \mathtt{sym}\{\Gamma^{\perp T} \Lambda_{2[h_L]}^T F_2\} & \star & \star \\ F_1 & -\hat{\Omega}_{1[h_L]} & \star \\ F_2 & 0_{4n \cdot 8n} & -\Omega_2 \end{bmatrix} < 0, \quad (21)$$

$$\begin{bmatrix} \Gamma^{\perp T} \hat{\Xi}_{[h_U]} \Gamma^{\perp} + \mathtt{sym}\{\Gamma^{\perp T} \Lambda_{1[h_U]}^T F_1\} + \mathtt{sym}\{\Gamma^{\perp T} \Lambda_{2[h_U]}^T F_2\} & \star & \star \\ F_1 & -\hat{\Omega}_{1[h_U]} & \star \\ F_2 & 0_{4n \cdot 8n} & -\Omega_2 \end{bmatrix} < 0, \quad (22)$$

where

$$\begin{aligned} \Pi_{6,1[h(t)]} = & [(h(t)-h_L)e_3 - (h(t)-h_L)e_9, \\ & \frac{h(t)-h_L}{2} e_3 + (h(t)-h_L)e_9 - 3e_{12}], \end{aligned}$$

$$\Pi_{6,2[h(t)]} = [(h_U - h(t))e_2 - (h_U - h(t))e_{10}, \frac{h_U - h(t)}{2} e_2 + (h_U - h(t))e_{10} - 3e_{13}],$$

$$\Xi_6 = \left(\frac{h_L^2}{2}\right)^2 e_5 Q_3 e_5^T - (h_L e_1 - h_L e_8) Q_3 (h_L e_1 - h_L e_8)^T - 2\left(\frac{h_L}{2} e_1 + h_L e_8 - 3e_{11}\right) Q_3 \left(\frac{h_L}{2} e_1 + h_L e_8 - 3e_{11}\right)^T,$$

$$\Xi_7 = \left(\frac{(h_U - h_L)^2}{2}\right)^2 e_6 Q_4 e_6^T,$$

$$\hat{\Xi}_{[h(t)]} = \tilde{\Xi}_{[h(t)]} + \Xi_6 + \Xi_7,$$

$$\Lambda_{2[h(t)]} = [\Pi_{6,1[h(t)]}, \Pi_{6,2[h(t)]}],$$

$$\hat{\Omega}_{1[h(t)]} = \Omega_1 + \mathtt{diag}\left\{\frac{(h_U - h_L)(h_U - h(t))}{2} Q_4, 0_{7n}\right\},$$

$$\Omega_2 = \begin{bmatrix} \mathtt{diag}\{\mathscr{Q}_3, 2\mathscr{Q}_3\} & M_2 \\ \star & \mathtt{diag}\{\mathscr{Q}_4, 2\mathscr{Q}_4\} \end{bmatrix}.$$

Proof: With Lemmas 1 ii) and 2 ii), the following processes driving the upper bounds of $\dot{V}_6$ and $\dot{V}_7$ are similar to the processes for $\dot{V}_4$ and $\dot{V}_5$ in Theorem 1:

$$\begin{aligned} \dot{V}_6 =& \left(\frac{h_L^2}{2}\right)^2 \dot{x}^T(t) Q_3 \dot{x}(t) - \frac{h_L^2}{2} \int_{t-h_L}^{t} \int_s^t \dot{x}^T(u) Q_3 \dot{x}(u) du ds \\ \le& \left(\frac{h_L^2}{2}\right)^2 \dot{x}^T(t) Q_3 \dot{x}(t) - \left(\int_{t-h_L}^{t} \int_s^t \dot{x}(u) du ds\right)^T \times Q_3(\star) - 2\left(-\int_{t-h_L}^{t} \int_s^t \dot{x}(u) du ds + \frac{3}{h_L} \int_{t-h_L}^{t} \int_s^t \int_u^t \dot{x}(v) dv du ds\right)^T Q_3(\star) \\ =& \left(\frac{h_L^2}{2}\right)^2 \dot{x}^T(t) Q_3 \dot{x}(t) - \left(h_L x(t) - \int_{t-h_L}^{t} x(s) ds\right)^T \times Q_3(\star) - 2\left(\frac{h_L}{2} x(t) + \int_{t-h_L}^{t} x(s) ds - \frac{3}{h_L} \int_{t-h_L}^{t} \int_s^t x(u) du ds\right)^T Q_3(\star) \\ =& \zeta^T(t) \Xi_6 \zeta(t) \end{aligned} \tag{23}$$

and

$$\begin{aligned} \dot{V}_7 =& \left(\frac{(h_U - h_L)^2}{2}\right)^2 \dot{x}^T(t-h_L) Q_4 \dot{x}(t-h_L) - \frac{(h_U - h_L)^2}{2} \int_{t-h(t)}^{t-h_L} \int_s^{t-h_L} \dot{x}^T(u) Q_4 \dot{x}(u) du ds \\ &- \frac{(h_U - h_L)^2}{2} \int_{t-h_U}^{t-h(t)} \int_s^{t-h(t)} \dot{x}^T(u) Q_4 \dot{x}(u) du ds - \frac{(h_U - h_L)^2}{2} (h_U - h(t)) \int_{t-h(t)}^{t-h_L} \dot{x}^T(s) Q_4 \dot{x}(s) ds \\ \le& \left(\frac{(h_U - h_L)^2}{2}\right)^2 \dot{x}^T(t-h_L) Q_4 \dot{x}(t-h_L) - \left(\frac{1}{\alpha(t)}\right)^2 \psi_1(t) - \left(\frac{1}{1-\alpha(t)}\right)^2 \psi_2(t) \\ &- \frac{(h_U - h_L)^2}{2} (h_U - h(t)) \int_{t-h(t)}^{t-h_L} \dot{x}^T(s) Q_4 \dot{x}(s) ds \\ \le& \zeta^T(t) \Xi_7 \zeta(t) - \zeta^T(t) \Lambda_{2[h(t)]} \Omega_2 \Lambda_{2[h(t)]}^T \zeta(t) - \frac{(h_U - h_L)^2}{2} (h_U - h(t)) \int_{t-h(t)}^{t-h_L} \dot{x}^T(s) Q_4 \dot{x}(s) ds, \end{aligned} \tag{24}$$

where $\alpha(t)$ was defined in (12), and

$$\psi_1(t) = \underbrace{\begin{bmatrix} \int_{t-h(t)}^{t-h_L} \int_s^{t-h_L} \dot{x}(u) du ds \\ -\int_{t-h(t)}^{t-h_L} \int_s^{t-h_L} \dot{x}(u) du ds + \frac{3}{h(t)-h_L} \times \int_{t-h(t)}^{t-h_L} \int_s^{t-h_L} \int_u^{t-h_L} \dot{x}(v) dv du ds \end{bmatrix}^T}_{\zeta^T(t) \Pi_{6,1[h(t)]}} \times \mathtt{diag}\{Q_4, 2Q_4\} \Pi_{6,1[h(t)]}^T \zeta(t),$$

$$\psi_2(t) = \underbrace{\begin{bmatrix} \int_{t-h_U}^{t-h(t)} \int_s^{t-h(t)} \dot{x}(u) du ds \\ -\int_{t-h_U}^{t-h(t)} \int_s^{t-h(t)} \dot{x}^T(u) du ds + \frac{3}{h_U-h(t)} \times \int_{t-h_U}^{t-h(t)} \int_s^{t-h(t)} \int_u^{t-h(t)} \dot{x}(v) dv du ds \end{bmatrix}^T}_{\zeta^T(t) \Pi_{6,2[h(t)]}} \times \mathtt{diag}\{Q_4, 2Q_4\} \Pi_{6,2[h(t)]}^T \zeta(t).$$

By incorporating the last term $-\frac{(h_U-h_L)^2}{2}(h_U - h(t)) \times \int_{t-h(t)}^{t-h_L} \dot{x}^T(s) Q_4 \dot{x}(s) ds$ of (24) into the inequality (9) and utilizing the result (16), an upper bound of $\dot{V}(t) = \sum_{i=1}^{6} V_i$ is newly estimated as follows:

$$\begin{aligned} \dot{V}(t) \le& \zeta^T(t) \hat{\Xi}_{[h(t)]} \zeta(t) - \zeta^T(t) \Lambda_{1[h(t)]} \hat{\Omega}_{1[h(t)]} \Lambda_{1[h(t)]}^T \zeta(t) \\ &- \zeta^T(t) \Lambda_{2[h(t)]} \Omega_2 \Lambda_{2[h(t)]}^T \zeta(t). \end{aligned} \tag{25}$$

Thus, the following inequality is the stability condition for system (1):

$$\begin{aligned} &\zeta^T(t) \hat{\Xi}_{[h(t)]} \zeta(t) - \zeta^T(t) \Lambda_{1[h(t)]} \hat{\Omega}_{1[h(t)]} \Lambda_{1[h(t)]}^T \zeta(t) - \zeta^T(t) \Lambda_{2[h(t)]} \Omega_2 \Lambda_{2[h(t)]}^T \zeta(t) \\ &< 0 \end{aligned} \tag{26}$$

subject to $\Gamma \zeta(t) = 0$.
By Lemma 3 i), the above condition is equivalent to

$$\begin{aligned} &{\Gamma^{\perp}}^T \hat{\Xi}_{[h(t)]} \Gamma^{\perp} - {\Gamma^{\perp}}^T \Lambda_{1[h(t)]} \hat{\Omega}_{1[h(t)]} \Lambda_{1[h(t)]}^T \Gamma^{\perp} - {\Gamma^{\perp}}^T \Lambda_{2[h(t)]} \Omega_2 \Lambda_{2[h(t)]}^T \Gamma^{\perp} \\ &< 0. \end{aligned} \tag{27}$$

In succession, since the inequality (27) is non-LMI form, using Lemma 3 ii) changes from (27) to the following LMI

Table 1. Upper bounds of time-varying delays with $h_D = 0.3$ (Example 1).

h_L	0.3	0.5	0.8	1
[26]	2.432	2.433	2.430	2.423
[17]	2.41	2.43	2.46	2.47
Theorem 1	2.5198	2.5290	2.5493	2.5705
Theorem 2	2.5198	2.5291	2.5505	2.5730

form

$$\begin{bmatrix} \begin{matrix} {\Gamma^{\perp}}^T \hat{\Xi}_{[h(t)]} \Gamma^{\perp} \\ +\mathtt{sym}\{{\Gamma^{\perp}}^T \Lambda^T_{1[h(t)]} F_1\} \\ +\mathtt{sym}\{{\Gamma^{\perp}}^T \Lambda^T_{2[h(t)]} F_2\} \end{matrix} & \star & \star \\ F_1 & -\hat{\Omega}_{1[h(t)]} & \star \\ F_2 & 0_{4n \cdot 8n} & -\Omega_2 \end{bmatrix} < 0 \tag{28}$$

for any matrices F_1 and F_2 with appropriate dimensions. The rest processes are similar to the proof of Theorem 1, so it is omitted. □

Remark 2: Very recently, based in the result of [27], Wirtinger-based double integral inequality was proposed in [29]. However, this inequality was applied to systems with only time-invariant delays. To show the effectiveness in reducing the conservatism of Wirtinger-based double integral inequality, V_6 and V_7 were considered and their time-derivative were estimated by applying Wirtinger-based double integral inequality. In next section, Theorem 2 can provide slightly larger delay bounds than Theorem 1. by comparing maximum delay bounds.

4. NUMERICAL EXAMPLES

In this section, two illustrative examples are introduced to show the improvements of the proposed methods.

Example 1: Consider the following system

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix} x(t - h(t)). \tag{29}$$

When $h_D = 0.3$, maximum delay bounds obtained by Theorems 1 and 2 are listed in Table 1 for various h_L. In Table 1, comparisons with those of existing results in [17, 26] are conducted. From Table 1, it can be confirmed that Theorem 1 provides larger delay bound than those of [17, 26]. Furthermore, it can also be confirmed that Theorem 2 provides slightly larger delay bounds than those of Theorem 1. When h_D is unknown, Table 2 shows the results of Theorems 1 and 2, and some other results in [16, 17, 25, 31]. Table 2 also shows the superiority of Theorems 1 and 2. Furthermore, as mentioned in Remark 1, our proposed criteria provide larger delay bounds than those of [31], which supports the statements of Remark 1.

Table 2. Upper bounds of time-varying delays with unknown h_D (Example 1).

h_L	0.3	0.5	0.8	1
[25]	1.27	1.39	1.61	1.76
[16]	1.29	1.43	1.64	1.79
[17]	1.31	1.45	1.66	1.81
[31]	1.35	1.47	1.67	1.82
Theorem 1	1.4326	1.5325	1.7135	1.8502
Theorem 2	1.4347	1.5336	1.7140	1.8504

Table 3. Upper bounds of time-varying delays with different conditions of h_D and h_L (Example 2).

Method	h_L	$h_D = 0.1$	$h_D = 0.5$
[32]	1	4.1935	2.3058
[33]		4.4045	2.3513
Theorem 1		4.7560	2.4897
Theorem 2		4.7561	2.4904
[32]	2	4.4932	2.5663
[33]		4.5729	2.6987
Theorem 1		4.7726	2.7994
Theorem 2		4.7746	2.7994
[32]	3	4.3979	3.3408
[33]		4.5406	3.4186
Theorem 1		4.7931	3.4977
Theorem 2		4.8005	3.4977
[32]	4	4.1978	4.1690
[33]		4.2367	4.2097
Theorem 1		4.7554	4.2939
Theorem 2		4.7567	4.2939
[32]	5	5.0275	5.0275
[33]		5.0440	5.0440
Theorem 1		5.1372	5.1372
Theorem 2		5.1372	5.1372

Table 4. Upper bounds of time-varying delays with unknown h_D and various h_L (Example 2).

h_L	0	0.3	0.6
[33]	1.70	1.78	1.89
[35]	1.95	2.02	2.08
[34]	2.02	2.08	2.15
Theorem 1	2.2322	2.2574	2.2718
h_L	0.9	1.2	1.5
[33]	2.04	2.20	2.37
[35]	2.15	2.25	2.38
[34]	2.23	2.34	2.47
Theorem 1	2.3146	2.3999	2.5246

Example 2: Consider the following system

$$\dot{x}(t) = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix} x(t) + \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} x(t - h(t)). \tag{30}$$

For the above system, which is the benchmark example in this research field, the results of the upper bounds of time-delay for different conditions of h_D and h_L are compared with some existing results in Table 3. It can be shown that the proposed criteria for system (30) provide enhanced feasible region for stability. Moreover, for the unknown h_D, the results of the upper bounds of time-delay obtained by Theorem 1 are listed in Table 4. From Table 4, one can also see that our results for this system give larger bounds than the ones in the existing works.

5. CONCLUSIONS

In this paper, the stability problem for linear systems with interval time-varying delays has been investigated. To improve the feasible region of stability criterion for the systems, the compositions of the Lyapunov-Krasovskii functional were pointed out through this work. To achieve this, by constructing the augmented Lyapunov-Krasovskii functional, sufficient conditions for guaranteeing asymptotic stability of the systems have been derived in Theorem 1 within the framework of LMIs. Furthermore, the effectiveness of reducing the conservatism of Wirtinger-based double integral inequality [29] was confirmed in Theorem 2. Two numerical examples have been given to show the effectiveness of the proposed criteria.

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