International Journal of Control, Automation and Systems 15(2) (2017) 481-488
http://dx.doi.org/10.1007/s12555-015-0372-9
ISSN:1598-6446 eISSN:2005-4092
http://www.springer.com/12555

Stability Analysis for Switched Positive Linear Systems under State-dependent Switching

Xiuyong Ding* and Xiu Liu

Abstract: This paper addresses the state-dependent stability problem of switched positive linear systems. Some exponential stability criteria are established on the given partitions of the nonnegative state space. First, a exponential stability of systems without delays is established with the help of a single linear co-positive Lyapunov function. When this does not seem possible, we also prove the stability by using multiple linear co-positive Lyapunov functions. Moreover, we extend this result to the delayed systems in terms of the single and multiple linear co-positive Lyapunov functionals respectively. The proposed results can be applied to the general systems without any special restriction. Some numerical examples are given to illustrate the effectiveness of our results.

Keywords: Positive systems, stability, state-dependent switching, switched systems.

1. INTRODUCTION

Switched systems have numerous applications in the control of mechanical systems, the automotive industry, aircraft and air traffic control, switching power converters and many other fields. For a discussion of various issues related to switched systems, see the survey article [1]. A switched system is a dynamical system which consists of a finite of subsystems and a rule orchestrating the switch among them. A switched system is called positive if the states of the subsystems are restricted to be non-negative [2]. Recently, the importance of switched positive systems has been highlighted by many researchers because of finding broad application in communication systems [3], formation flying [4], and other areas.

The stability issues of switched positive systems, especially switched positive linear systems (SPLSs), have drawn a lot of attentions in recent decade. The first question is whether the SPLS is stable under arbitrary switching signals (see, e.g., [5–9]); The other is stability analysis of SPLSs under restricted switching which may be either time domain restrictions (time-dependent) or state space restrictions (state-dependent). Recently, the stability of SPLSs under time-dependent switching captured wide attention from researchers (see [10–12] and some references therein). However, stability of SPLSs under state-dependent is a topic only partially explored [13–16]. In [13], the authors pointed that if there is a Hurwitz convex combination of the system matrices, then the SPLS is state-dependently stable. Notice that the existence of a Hurwitz convex combination, for stability, is only sufficient, but not necessary in general [13] except for some special cases, such as second order [13], two subsystems with rank one difference [14]). Similarly, for discrete-time SPLSs, the existence of a Schur convex combination of the system matrices implies state-dependent stability [15]. The necessity is only for some special cases (for example, second order [15], cyclic monomial matrix, and circulant matrix [16]). It should be emphasized that all these results are essentially based on some special restrictions, such as low dimension, Hurwitz convex combination, etc., hence leading to deeper insights into the state-dependent stability must be endowed with.

This note is devoted to exploit some new treatments for the stability of the given state-dependent SPLSs. It is worth noting that the proposed results are applicable for the general SPLSs which are without any special restriction, in contrast to others in the literature. The layout of this note is as follows: Section 2. recall some necessary background of switched systems and positive systems. Section 3. is concerned with the SPLSs which are composed of linear time-invariat (LTI) subsystems. We first present a sufficient condition for state-dependent stability by using a single linear co-positive Lyapunov function. When this seems impossible, we also prove the stability by using multiple linear co-positive Lyapunov func-

Manuscript received October 7, 2015; revised March 30, 2016; accepted April 25, 2016. Recommended by Associate Editor Do Wan Kim under the direction of Editor Yoshito Ohta. This work was supported by the Foundation of National Nature Science of China (Grant No.61473239) and supported by the Fundamental Research Funds for the Central Universities (Grant No.2682014BR038 and 2682015CX058).

Xiuyong Ding and Xiu Liu are with the School of Mathematics, Southwest Jiaotong University, Chengdu, Sichuan 611756, P. R. China (e-mails: dingxingzhi@outlook.com, xiuliu@outlook.com).
* Corresponding author.

tion. This result, based on the single and multiple linear co-positive Lyapunov functional methods, is then generalized to SPLSs with time-delay in Section 4.. Finally, in Section 5., we present some brief concluding remarks.

Notation: Throughout, $\mathbb{R}^n$ ($\mathbb{R}^n_+$) stands for the n-dimensional (non-negative) real vector space and $\mathbb{R}^{n\times n}$ is the space of $n\times n$ matrices. For matrix A,B or vector x,y, $A\succeq B(A\succ B)$ or $x\succeq y(x\succ y)$ means that all elements of matrix $A-B$ or vector $x-y$ are non-negative (positive). Also, $A\preceq B(A\prec B)$ or $x\preceq y(x\prec y)$ means that all elements of matrix $A-B$ or vector $x-y$ are non-positive (negative). $x\neq 0$ means that there exists at least one non-zero entry in vector x. A^T represents the transpose of matrix A. The notation $\|\cdot\|$ refers to the Euclidean vector norm. For a continuous function $f(x)$ for x in a closed region Ω, $\min_{x\in\Omega} f(x)$ and $\max_{x\in\Omega} f(x)$ denote the minimum and maximum value in Ω, respectively.

2. BACKGROUND AND PRELIMINARIES

First of all, we recall some facts which are relevant for this paper. A matrix is a Metzler matrix if its off-diagonal entries are non-negative. A matrix is a Hurwitz matrix if all its eigenvalues lie in the open left half of the complex plane. The LTI system $\dot{x}(t)=Ax(t)$ is positive if and only if its system matrix A is Metzler [2]. The linear delayed system $\dot{x}(t)=Ax(t)+Bx(t-h)$ is positive if and only if its matrix A is Metzler and $B\succeq 0$ [17]. There are a number of elegant characters for the Metzler Hurwitz matrix, we now recall a classical result which are relevant for the work of this paper as follows:

Theorem 1 [5, 18]**:** Let $A\in\mathbb{R}^{n\times n}$ is a Metzler matrix. Then A is a Hurwitz matrix (or equivalently, the system $\dot{x}(t)=Ax(t)$ is exponentially stable) if and only if there exists a vector $v\succ 0$ in $\mathbb{R}^n$ such that $A^Tv\prec 0$.

Related to this result, we recall a definition which is a powerful research tool for positive systems. The function $V(x)=x^Tv$ is called a linear co-positive Lyapunov (LCL) function of the positive LTI system $\dot{x}(t)=Ax(t)$ if $V(x)>0$ and $\dot{V}(x)=x^TA^Tv<0$ for all non-zero $x\succeq 0$.

The S-procedure for linear version as follows:

Theorem 2 (See [19], Section 2.6.3) ("S-procedure for linear forms")**:** Let vectors x,w_0, and w_1 be in $\mathbb{R}^n$. Consider the two conditions: (i) $x^Tw_0\geq 0$ whenever $x^Tw_1\geq 0$. (ii) There exists a constant $\tau\geq 0$ such that $x^T(w_0-\tau w_1)\geq 0$. Condition (ii) always implies condition (i). The converse holds, provided that there is one x_0 such that $x_0^Tw_1>0$.

The following lemma which is straightforward from Theorem 2 will play a key role in deriving the results of this paper.

Lemma 1: Let vectors x,w_0, and w_1 be in $\mathbb{R}^n$. If there is a constant $\tau>0$ such that $w_0-\tau w_1\succ 0$, then $x^Tw_0>0$ whenever $x^Tw_1\geq 0$, where $x\succeq 0$ and $x\neq 0$.

In what follows, we discuss the problem of verifying stability of a given state-dependent SPLS. As we all know, when referring to the state-dependent problem, a reasonable partition of state space is first required. Specifically, assume that the state space $\mathbb{R}^n_+$ has a partition given by disjoint regions $\{\Omega_1,\cdots,\Omega_m\}$, i.e., $\mathbb{R}^n_+=\cup_{i=1}^m\Omega_i$, and these regions Ω_i are defined a priori as restriction of the possible switching signals. Especially, in this paper we assume that, from the characteristics of positive systems, the regions Ω_i are given by the linear forms

$$\Omega_i=\{x\in\mathbb{R}^n_+|x^Tw_i\geq 0\},i\in\underline{m}. \tag{1}$$

In addition, use $\Omega_{i,j}$ to denote the boundary $\bar{\Omega}_i\cap\bar{\Omega}_j$, where $\bar{\Omega}_i$ and $\bar{\Omega}_j$ denote the closure of Ω_i and Ω_j, respectively. Formally,

$$\Omega_{i,j}=\{x\in\mathbb{R}^n_+|x^Tw_{i,j}=0\},i,j\in\underline{m}, \tag{2}$$

where $w_i,w_{i,j}\in\mathbb{R}^n$.

Now a proposed switching rule is described as: the i-th subsystem can only be active for states within Ω_i and a switching event can occur only when the trajectory crosses a boundary region $\Omega_{i,j}$ which stands for the switching surface where the trajectory passes from region Ω_i to Ω_j.

It should be emphasized that such a switching rule implies that each individual subsystem are of concern only in the regions where this system is active, and the behavior of this system in other parts of the state space has no influence on the switched system. Notice that if at least one of the individual is asymptotically stable, this problem is trivial (just keep to activate the stable subsystems). We hence, in the sequel, steadily make a assumption that none of the individual subsystems of SPLS is stable.

3. SWITCHED LINEAR POSITIVE SYSTEMS COMPOSED OF LTI SUBSYSTEMS

In this section we consider the SPLS which is composed of LTI subsystems as the form:

$$\dot{x}(t)=A_{\sigma(x)}x(t), \tag{3}$$

where $x(t)\in\mathbb{R}^n$ is the state vector. $\sigma(x):\mathbb{R}^n_+\to\underline{m}:=\{1,2,\cdots,m\}$ is the so-called switching signal which is state-dependent. $A_i\in\mathbb{R}^{n\times n}$ are Metzler matrices for all $i\in\underline{m}$ such that the system (3) is positive.

3.1. Single LCL function method

First, we try to derive stability of SPLS (3) by using a single LCL function $V(x)=x^Tv$. For this aim, this LCL function needs to satisfy the following conditions:

(**H1**): For all $x\in\mathbb{R}^n_+$ and $x\neq 0$, $V(x)=x^Tv>0$, or, equivalently, $v\succ 0$.

This condition requires LCL function $V(x)=x^Tv$ must be positive definite on the whole positive orthant $\mathbb{R}^n_+$.

(**H2**): For all $x \in \Omega_i$ and $x \neq 0$, $\dot{V}(x) < 0$.

This requirement implies that the LCL function associated the i-th subsystem should be decreasing along the trajectories inside Ω_i. This decreasing property can be represented as: there is a vector v in $\mathbb{R}^n$ such that

$$x^T A_i^T v < 0, \ \ \forall x \in \Omega_i. \tag{4}$$

Based on the above conditions, we present the following lemma.

Lemma 2: If there is a LCL function $V(x) = x^T v$ to satisfy (H1) and (H2). Then the SPLS (3) is exponentially stable.

Proof: Suppose that the i-th mode is active on $[t_k, t_{k+1})$ for some integer $k > 0$, i.e., $x(t) \in \Omega_{i_k}$ for $t \in [t_k, t_{k+1})$, the derivative of V along the solution of (3) is

$$\dot{V}(x(t)) = x^T(t) A_{i_k}^T v \leq -\delta_{i_k} x^T(t) v = -\delta_{i_k} V(x(t)),$$

where $\delta_{i_k} = \min_{x \in \Omega_{i_k}} \frac{x^T(t) A_{i_k}^T v}{V(x)}$. This implies that $V(x(t)) \leq \exp(-\delta_{i_k}(t - t_k)) V(x(t_k))$. Furthermore, observe that the values of the Lyapunov function certainly matches on the switching instants, setting $\delta = \min_{i_k \in \underline{m}} \delta_{i_k}$ we can deduce that

$$\begin{aligned} V(x(t)) &\leq \exp(-\delta(t - t_k)) V(x(t_k)) \\ &= \exp(-\delta(t - t_k^-)) V(x(t_k^-)) \\ &\leq \cdots \\ &\leq \exp(-\delta(t - t_0)) V(x(t_0)). \end{aligned}$$

Note that the LCL function $V(x(t))$ is positive definite from (H1), we can sufficiently conclude that the solution $x(t)$ converges exponentially to zero as $t \to \infty$. This completes the proof. □

Note that the above inequality (4) is constrained by the switching regions (or equivalently, by the states). By applying Lemma 1, condition (4) can be replaced by the following stronger condition without constraints

$$A_i^T v + c_i w_i \prec 0, \ i \in \underline{m},$$

where $c_i > 0$ are constant scalars for all i, and then we get the following stability result.

Theorem 3: The SPLS (3) is exponentially stable if there exist a vector $v, w_i \in \mathbb{R}^n$ and positive scalars c_i such that the following conditions are satisfied

$$\begin{aligned} & v \succ 0, \\ & A_i^T v + c_i w_i \prec 0, \ i \in \underline{m}. \end{aligned} \tag{5}$$

Proof: First of all, the positive definite condition (H1) results from $v \succ 0$. Moreover, for non-zero $x \in \mathbb{R}^n_+$, $A_i^T v + c_i w_i \prec 0$ implies $x^T(A_i^T v + c_i w_i) < 0$. By Lemma 1, it follows that $x^T A_i^T v < 0$ whenever $x^T w_i \geq 0$, namely, the decreasing property (H2) is satisfied, and then the exponential stability can be obtained by Lemma 2. □

Specially, we consider a classic assumption when there exists a Hurwitz convex combination $\sum_{i=1}^m \alpha_i A_i$ of the system matrices $A_i (i \in \underline{m})$ [20]. Under this assumption, the state-dependent stability of (3) can be achieved by the LCL function method.

Theorem 4: If there exist constants $\alpha_i \in [0, 1]$ such that matrix $\sum_{i=1}^m \alpha_i A_i$ is Hurwitz with $\sum_{i=1}^m \alpha_i = 1$, then the SPLS (3) is exponentially stable by selecting a switching strategy $\sigma(x) = \arg\min_{i \in \underline{m}} x^T A_i^T v$.

Proof: Since $\sum_{i=1}^m \alpha_i A_i$ is Hurwitz, we can find according to Theorem 1 a vector $v \succ 0$ with $\sum_{i=1}^m \alpha_i A_i^T v \prec 0$, which leads to, for any non-zero $x \succeq 0$, $\sum_{i=1}^m \alpha_i x^T A_i^T v < 0$. This implies that there exists at least one i such that $x^T A_i^T v < 0$ for non-zero $x \succeq 0$. Now define the associated switching regions $\Omega_i = \{x \in \mathbb{R}^n_+ | -x^T A_i^T v > 0, i \in \underline{m}\}$, it is easy to check that the conditions (H1) and (H2) are satisfied by choosing a LCL function $V(x(t)) = x^T v$, and hence the exponential stability of (3) follows from Lemma 2. □

3.2. Multiple LCL function method

In the above subsection, the stability analysis was carried out with the help of a single LCL function. When this does not seem possible, one can try to find a prove stability by using the multiple LCL functions $V_i(x) = x^T v_i, i \in \underline{m}$. For stability, the following conditions are required:

(**H3**): The Lyapunov functions $V_i(x) > 0$ for all non-zero $x \in \Omega_i$, or, equivalently,

$$x^T v_i > 0, \quad \forall x \in \Omega_i, \ \ i \in \underline{m}. \tag{6}$$

Notice that this condition, different from (H1), implies that each function $V_i(x)$ is positive definite only in the associated switching region Ω_i (there is no requirement on positive definiteness outside Ω_i).

(**H4**): For all $x \in \Omega_i$ and $x \neq 0$, $\dot{V}_i(x) < 0$ for $i \in \underline{m}$.

This restriction can be rewritten as: for each i, there is a vector v_i in $\mathbb{R}^n$ such that

$$x^T A_i^T v_i < 0, \ \ \forall x \in \Omega_i. \tag{7}$$

In addition, on the switching surfaces $\Omega_{i,j}$, it is required that the LCL functions' values are non-increasing.

(**H5**): For all $x \in \Omega_{i,j}$, the adjacent LCL functions satisfy $V_j(x) \leq V_i(x)$ for $i, j \in \underline{m}$, i.e.,

$$x^T v_j \leq x^T v_i, \ \ \forall x \in \Omega_{i,j}. \tag{8}$$

Lemma 3: If there exist some multiple LCL functions $V_i(x) = x^T v_i, i \in \underline{m}$ to satisfy (H3), (H4), and (H5). Then the SPLS (3) is exponentially stable.

Proof: The proof is very similar to that in Lemma 2 and thus omitted. □

Also, by Lemma 1 we can remove the region restriction in conditions (6), (7), and (8) by the following stronger conditions:

$$v_i - a_i w_i \succ 0, \ \ i \in \underline{m},$$
$$A_i^T v_i + c_i w_i \prec 0, \ \ i \in \underline{m},$$

and

$$v_j + d_{i,j} w_{i,j} \preceq v_i, \ \ i,j \in \underline{m},$$

respectively, where $a_i > 0, c_i > 0$, and $d_{i,j}$ are constants.

Theorem 5: For $i,j \in \underline{m}$, the SPLS (3) is exponentially stable if there exist vectors $v_i, w_i \in \mathbb{R}^n$ and scalars $a_i > 0, c_i > 0$, and $d_{i,j}$ such that the following conditions hold:

$$\begin{aligned} & v_i - a_i w_i \succ 0, \\ & A_i^T v_i + c_i w_i \prec 0, \\ & v_j + d_{i,j} w_{i,j} \preceq v_i. \end{aligned} \tag{9}$$

Proof: For non-zero $x \in \mathbb{R}_+^n$, conditions $v_i - a_i w_i \succ 0$ and $A_i^T v_i + c_i w_i \prec 0$ imply $x^T(v_i - a_i w_i) > 0$ and $x^T(A_i^T v_i + c_i w_i) < 0$, respectively. By Lemma 1 one can obtain $x^T v_i > 0$ and $x^T A_i^T v_i < 0$ whenever $x^T w_i \geq 0$. That is, the conditions (H3) and (H4) are satisfied. In addition, $x^T(v_j + d_{i,j} w_{i,j}) \leq x^T v_i$ follows from the last inequality, or, equivalently, $x^T v_j \leq x^T v_i$ whenever $x^T w_{i,j} = 0$, this is the condition (H5). Hence all conditions in Lemma 3 are satisfied, proving this theorem. □

Remark 1: If $v_i = v$ for all $i \in \underline{m}$, then Theorem 5 is reduced to Theorem 3.

3.3. Examples: non-delay case

Example 1: Define a state-dependent switched linear system as follows:

$$\dot{x} = \begin{cases} A_1 x, & if \ \ x \in \Omega_1 = \{x \in \mathbb{R}_+^n | x^T w_1 \geq 0\} \\ A_2 x, & if \ \ x \in \Omega_2 = \{x \in \mathbb{R}_+^n | x^T w_2 \geq 0\}, \end{cases} \tag{10}$$

where $\omega_1 = [1 \ -1]^T, \omega_2 = [-1 \ 1]^T$. This switching rule is shown in Figure 1. Let the coefficient matrices be given by

$$A_1 = \begin{bmatrix} -8.5 & 1 \\ 0.2 & 3 \end{bmatrix}, \ A_2 = \begin{bmatrix} 4 & 0.8 \\ 0.5 & -7.8 \end{bmatrix}. \tag{11}$$

It is clear that system (10) with coefficient matrices (11) is positive since A_1, A_2 are Metzler, and both subsystems are unstable.

According to Theorem 3, one feasible solution of Linear Programming Problem (5) provides $v = [1.5529 \ \ 2.2441]^T$ and $c_1 = 8.2863, c_2 = 7.7481$, and the state evolution can be seen in Figure 2 which confirms that the exponential stability of the system (10) can be achieved by means of the single LCL function $V(x) = x^T[1.5529 \ 2.2441]^T$.

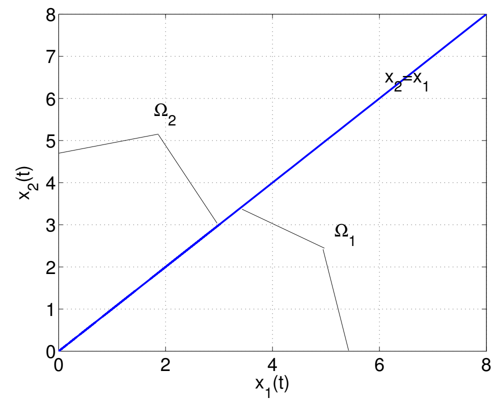


Fig. 1. Switching rule: when $x(t) \in \Omega_1$, $\sigma(x) = 1$; when $x(t) \in \Omega_2$, $\sigma(x) = 2$.

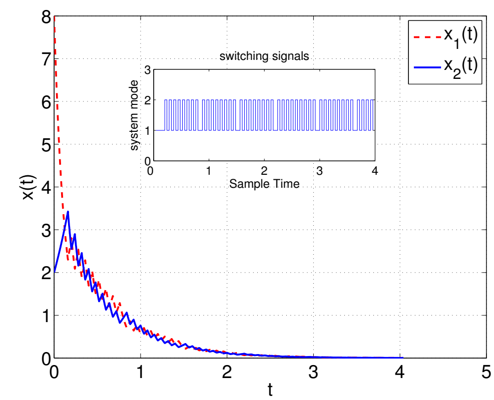


Fig. 2. State responses and switching signals of the SPLS in Example 1, where the initial conditions are $x(0) = [8 \ 2]^T$.

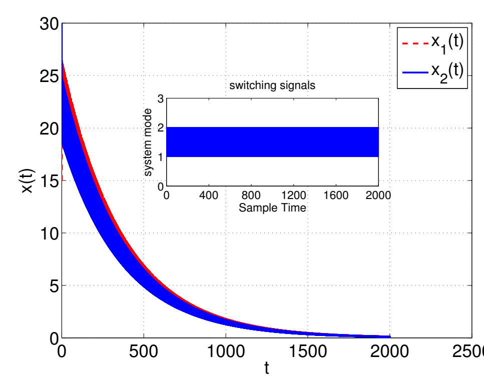


Fig. 3. State responses and switching signals of SPLS in Example 2, where the initial conditions are $x(0) = [15 \ 30]^T$.

Example 2: Let us define the same state-dependent switched linear system (10), where the system matrices (11) are replaced with

$$A_1 = \begin{bmatrix} -4.75 & 1 \\ 0.2 & 3 \end{bmatrix}, \ A_2 = \begin{bmatrix} 4 & 0.8 \\ 0.5 & -4.58 \end{bmatrix} \tag{12}$$

Obviously, the associated switched system is positive and unstable.

First of all, we solve the Linear Programming Problem (5), the resulting index output $exitflag = -2$. Namely, there is no feasible solution of Linear Programming Problem (5). In this case, we can not, according to Theorem 3, check the stability of the system (10) with the matrices (12) via the single LCL function method. However, one can try to prove the stability by using multiple LCL function method. By solving Linear Programming Problem (9), one feasible solution provides

$$v_1 = [64.1750\ 26.1038]^T, v_2 = [22.1234\ 68.1554]^T,$$
$$c_1 = 199.0451, c_2 = 187.6236,$$

and

$$d_{1,2} = -d_{2,1} = -42.0516.$$

Therefore, by Theorem 5 the SPLS (10) with matrices (12) is exponentially stable, and the simulation is shown in Figure 3.

4. SWITCHED LINEAR POSITIVE SYSTEMS WITH TIME-DELAY

Consider the SPLS with time-delay as the form:

$$\dot{x}(t) = A_{\sigma(x)}x(t) + B_{\sigma(x)}x(t-h), \tag{13}$$

where $h > 0$ is the constant delay, switching signal $\sigma(x) : \mathbb{R}^n_+ \to \underline{m} := \{1,2,\cdots,m\}$ is state-dependent. Matrices A_i are Metzler and $B_i \succeq 0$ for all $i \in \underline{m}$ such that the system is positive.

4.1. Single LCL functional method

Consider the following LCL functional candidate

$$V(t) = V(t,x_t) = e^{\delta_0 t}x^T(t)v + e^{\delta_0 t}\int_{t-h}^{t} x^T(s)\gamma ds, \tag{14}$$

where x_t denotes the continuous function on $[-h,0]$ given by $x_t(\theta) = x(t+\theta)$ for $\theta \in [-h,0]$, δ_0 is a constant, and $v,\gamma \in \mathbb{R}^n$. For stability, $V(t)$ needs to satisfy the following conditions:

(**H6**): For all $x \in \mathbb{R}^n_+$ and $x \neq 0$, $V(t) > 0$, or, equivalently, $v \succ 0$ and $\gamma \succ 0$.

(**H7**): For $x \in \Omega_i$ and $x \neq 0$,

$$\begin{aligned}&\dot{V}(t) - \delta_0 V(t)\\&\quad= e^{\delta_0 t}x^T(t)(A_i^T v+\gamma) + e^{\delta_0 t}x^T(t-h)(B_i^T v-\gamma)\\&\quad< 0, \quad i \in \underline{m}.\end{aligned}$$

This requirement can be rewritten as: there are vectors $v,\gamma \in \mathbb{R}^n$ such that

$$\begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix}^T \begin{bmatrix} A_i^T v+\gamma \\ B_i^T v-\gamma \end{bmatrix} < 0, \ \ \forall x \in \Omega_i. \tag{15}$$

Lemma 4: If there is a LCL functional as the form (14) to satisfy the conditions (H6) and (H7). Then the SPLS (13) is exponentially stable.

Proof: First, for the LCL functional (14) it follows from (H6) that

$$\begin{aligned}&\kappa_1 e^{\delta_0 t}\|x(t)\| \le V(t)\\&\qquad \le \kappa_2 e^{\delta_0 t}\|x(t)\| + \kappa_3 e^{\delta_0 t}\int_{t-h}^{t}\|x(s)\|ds,\end{aligned} \tag{16}$$

with $\kappa_1 = \min_{1\le l\le n}\{v_l\}, \kappa_2 = \max_{1\le l\le n}\{v_l\}, \kappa_3 = \max_{1\le l\le n}\{\gamma_l\}$, where v_l and γ_l denote the l-th entry of vectors v and γ, respectively,

Now suppose that the i-th mode is active on $[t_k,t_{k+1})$ for some integer $k > 0$, i.e., $x(t) \in \Omega_{i_k}$ for $t \in [t_k,t_{k+1})$, along the solution of (13) we have from (H7) that

$$\begin{aligned}&\dot{V}(t) - \delta_0 V(t)\\&\quad= e^{\delta_0 t}x^T(t)(A_{i_k}^T v+\gamma) + e^{\delta_0 t}x^T(t-h)(B_{i_k}^T v-\gamma)\\&\quad< 0.\end{aligned}$$

Now set

$$\lambda_{1i_k} = -\max_{1\le l\le n}\{[A_{i_k}^T v+\gamma]_l\}, \lambda_{2i_k} = -\max_{1\le l\le n}\{[B_{i_k}^T v-\gamma]_l\},$$

where $[A_{i_k}^T v+\gamma]_l$ and $[B_{i_k}^T v-\gamma]_l$ denote the l-th entry of $A_{i_k}^T v+\gamma$ and $B_{i_k}^T v-\gamma$, respectively, we further have

$$\dot{V}(t) - \delta_0 V(t) \le -e^{\delta_0 t}(\lambda_{1i_k}\|x(t)\| + \lambda_{2i_k}\|x(t-h)\|) < 0.$$

By (16) we can derive

$$\begin{aligned}\dot{V}(t) &\le \delta_0 V(t) - e^{\delta_0 t}(\lambda_{1i_k}\|x(t)\| + \lambda_{2i_k}\|x(t-h)\|)\\&= e^{\delta_0 t}\Big[(\kappa_2\delta_0 - \lambda_{1i_k})\|x(t)\| - \lambda_{2i_k}\|x(t-h)\|\\&\qquad\qquad + \kappa_3\delta_0\int_{t-h}^{t}\|x(s)\|ds\Big]\\&\le e^{\delta_0 t}\left[(\kappa_2\delta_0 - \lambda_{1i_k})\|x(t)\| + \kappa_3\delta_0\int_{t-h}^{t}\|x(s)\|ds\right].\end{aligned}$$

Now integrating both sides of above inequality from t_k to t gives

$$\begin{aligned}&V(t) \le V(t_k) +\\&\int_{t_k}^{t} e^{\delta_0 s}\left[(\kappa_2\delta_0 - \lambda_{1i_k})\|x(s)\| + \kappa_3\delta_0\int_{s-h}^{s}\|x(\theta)\|d\theta\right]ds.\end{aligned}$$

Notice that the Lyapunov values certainly matches on the switching instants and set $\lambda_1 = \min_{i_k\in\underline{m}}\lambda_{1i_k}$, then we can easily deduce that

$$\begin{aligned}V(t) \le& V(0) + (\kappa_2\delta_0 - \lambda_1)\int_0^t e^{\delta_0 s}\|x(s)\|ds\\&+ \kappa_3\delta_0\int_0^t e^{\delta_0 s}\int_{s-h}^{s}\|x(\theta)\|d\theta ds\end{aligned}$$

Observe that

$$\int_0^t e^{\delta_0 s}\int_{s-h}^{s}\|x(\theta)\|d\theta ds \le h\tau e^{\delta_0 h}$$
$$\times\int_0^t e^{\delta_0\theta}\|x(\theta)\|d\theta+\frac{1}{\delta_0}\int_{-h}^{0}(e^{\delta_0(\theta+h)}-1)\|x(\theta)\|d\theta,$$

where $\tau \ge 1$ is a constant. This gives

$$V(t)\le V(0)+(\kappa_2\delta_0+\kappa_3\delta_0 h\tau e^{\delta_0 h}-\lambda_1)$$
$$\times\int_0^t e^{\delta_0 s}\|x(s)\|ds+\kappa_3\int_{-h}^{0}(e^{\delta_0(s+h)}-1)\|x(s)\|ds.$$

Now choose sufficiently small δ_0 such that $\lambda_1 \ge \kappa_2\delta_0+\kappa_3\delta_0 h\tau e^{\delta_0 h}$, we have

$$V(t)\le V(0)+\kappa_3\int_{-h}^{0}(e^{\delta_0(s+h)}-1)\|x(s)\|ds.$$

By (16) it follows that

$$\|x(t)\|\le\frac{1}{\kappa_1}e^{-\delta_0 t}\Big[V(0)$$
$$+\kappa_3\int_{-h}^{0}(e^{\delta_0(s+h)}-1)\|x(s)\|ds\Big],$$

therefore $x(t)$ exponentially converges to zero as $t\to\infty$. This completes the proof. □

Since $x(t), x(t-h)$ are non-negative, the constrained inequality (15), by applying Lemma 1, can be replaced by a stronger condition without constraints as follows:

$$\begin{cases} A_i^T v+\gamma+c_i w_i \prec 0,\\ B_i^T v-\gamma\preceq 0.\end{cases}\tag{17}$$

Theorem 6: The delayed SPLS (13) is exponentially stable if there exist vectors $v,\gamma,w_i\in\mathbb{R}^n$ and positive scalars $c_i, i\in \underline{m}$ such that the following conditions are satisfied

$$\begin{aligned} &v\succ 0,\\ &\gamma\succ 0,\\ &A_i^T v+\gamma+c_i w_i\prec 0,\ i\in\underline{m}.\\ &B_i^T v-\gamma\prec 0,\end{aligned}\tag{18}$$

Proof: First of all, the positive definite condition (H6) results from the conditions $v\succ 0$ and $\gamma\succ 0$. On the other hand, for non-zero $x(t)\succeq 0$, condition $A_i^T v+\gamma+c_i w_i\prec 0$ implies $x^T(t)(A_i^T v+\gamma+c_i w_i)<0$. By Lemma 1 we further get that $x^T(t)(A_i^T v+\gamma)<0$ whenever $x^T(t)w_i\ge 0$. Note that, from the last inequality, $x^T(t-h)(B_i^T v-\gamma)<0$ for non-zero $x(t-h)\succeq 0$. Therefore, we can obtain that $x^T(t)(A_i^T v+\gamma)+x^T(t-h)(B_i^T v-\gamma)<0$ whenever $x^T(t)w_i\ge 0$, i.e., condition (H7) follows. All conditions in Lemma 4 are satisfied, and thus the system (13) is exponentially stable. □

Now we consider a special case when the system matrices $A_i (i\in\underline{m})$ share a Hurwitz convex combination, and we shall show that the state-dependent stability of delayed SPLS (13) can be derived by single LCL functional method. In fact, from Theorem 1, there is a vector $v\succ 0$ such that $\sum_{i=1}^m \alpha_i A_i^T v\prec 0$. This implies that there exists at least one i such that $x^T A_i^T v<0$ for some non-zero $x\succeq 0$. We can further find a vector $\gamma\succ 0$ satisfying $x^T(t)A_i^T v\le -x^T\gamma<0$ some non-zero $x\succeq 0$. Now we define $\Omega_i=\{x\in\mathbb{R}_+^n \mid -x^T A_i^T v>0\}$ and require system matrices B_i to satisfy $x^T(t)B_i^T v\preceq x^T\gamma$, which ensures that condition (H7) holds, and then the stability of (13) follows.

Theorem 7: If there exist constants $\alpha_i\in[0,1], i\in\underline{m}$ such that matrix $\sum_{i=1}^m \alpha_i A_i$ is Hurwitz with $\sum_{i=1}^m\alpha_i=1$, and $B_i^T v-\gamma\preceq 0$, where vectors $v,\gamma\succ 0$ are defined above. Then the SPLS (13) is exponentially stable by selecting the switching rule $\sigma(x)=\arg\min_{i\in\underline{m}} x^T A_i^T v$.

4.2. Multiple LCL functional method

When a stability analysis based on a single LCL functional breaks down, one can use multiple LCL functionals as the form

$$V_i(t)=V_i(x_t)=e^{\delta_0 t}x^T(t)v_i+e^{\delta_0 t}\int_{t-h}^{t}x^T(s)\gamma ds,\ \ i\in\underline{m}.\tag{19}$$

We need to make the following restriction on the Lyapunov functionals (19):

(H8): For all $x\in\Omega_i$ and $x\neq 0$, $V_i(t)>0$, or, equivalently, $x^T v_i>0, \forall x\in\Omega_i$ and $\gamma\succ 0, \forall x\in\mathbb{R}_+^n$.

(H9): $\dot{V}_i(t)-\delta_0 V_i(t)<0$ for all non-zero $x\in\Omega_i, i\in\underline{m}$.

An equivalent description is: there are vectors $v_i,\gamma\in\mathbb{R}^n$ such that $\forall x\in\Omega_i,\ i\in\underline{m}$

$$\begin{bmatrix} x(t)\\ x(t-h)\end{bmatrix}^T\begin{bmatrix} A_i^T v_i+\gamma\\ B_i^T v_i-\gamma\end{bmatrix}<0.\tag{20}$$

(H10): $V_j(t)\le V_i(t)$ for $x\in\Omega_{i,j}, i,j\in\underline{m}$.

This non-increasing property on switching surfaces can be represented as:

$$x^T(t)v_j\le x^T(t)v_i,\ \ \forall x\in\Omega_{i,j},\ i,j\in\underline{m}.\tag{21}$$

Based on the above conditions we give the following technical lemma.

Lemma 5: If there is a LCL functional as the form (19) satisfying (H8), (H9), and (H10). Then SPLS (13) is stable.

Proof: The proof is essentially same as that in Lemma 4, and thus omitted. □

Now by applying Lemma 1 we can replace the inequalities (18) (20), and (21) with the following stronger conditions without constraints, respectively:

$$v_i-a_i w_i\succ 0, \gamma\succ 0,\ \ i\in\underline{m},$$

$$\begin{cases} A_i^T v_i + \gamma + c_i w_i \prec 0, \\ B_i^T v_i - \gamma \preceq 0, \end{cases} \quad i \in \underline{m},$$

and

$$v_j + d_{i,j} w_{i,j} \preceq v_i, \quad i, j \in \underline{m},$$

where $a_i > 0, c_i > 0$, and $d_{i,j}$ are constants.

In summary, the above discussion can be encapsulated the following result.

Theorem 8: For $i, j \in \underline{m}$, the SPLS (13) is exponentially stable if there exist vectors $v_i, w_i, \gamma \in \mathbb{R}^n$ and scalars $a_i > 0, c_i > 0$, and $d_{i,j}$ such that the following conditions hold:

$$\begin{aligned} & v_i - a_i w_i \succ 0, \\ & \gamma \succ 0, \\ & A_i^T v_i + \gamma + c_i w_i \prec 0, \\ & B_i^T v_i - \gamma \preceq 0, \\ & v_j + d_{i,j} w_{i,j} \preceq v_i. \end{aligned} \tag{22}$$

Proof: The proof is very similar to that of Theorem 5 and Theorem 6, and thus is omitted. □

Remark 2: If $v_i = v$ for all $i \in \underline{m}$, then Theorem 8 is reduced to Theorem 6.

Remark 3: Notice that both condition (17) and (22) are delay-independent, irrespective the sizes of delays. Hence Theorem 6 and 8 are easily extended to the SPLS with any bounded time-varying delay

$$\dot{x}(t) = A_{\sigma(x)} x(t) + B_{\sigma(x)} x(t - h(t)),$$

where the time-varying delay $h(t)$ satisfying $0 < h(t) < h$ is bounded.

4.3. Examples: delay case

Example 3: Define a state-dependent switched linear system

$$\dot{x}(t) = \begin{cases} A_1 x(t) + B_1 x(t-h), & if\ x(t) \in \Omega_1, \\ A_2 x(t) + B_2 x(t-h), & if\ x(t) \in \Omega_2, \end{cases} \tag{23}$$

where Ω_1, Ω_2 are same as Example 1. The coefficient matrices are as follows:

$$A_1 = \begin{bmatrix} -8.8 & 2 \\ 1 & 3 \end{bmatrix}, \ B_1 = \begin{bmatrix} 0.2 & 0.3 \\ 0.1 & 0.2 \end{bmatrix}, \\ A_2 = \begin{bmatrix} 2 & 1 \\ 1 & -7.6 \end{bmatrix}, \ B_2 = \begin{bmatrix} 0.4 & 0.1 \\ 0.2 & 0.3 \end{bmatrix}. \tag{24}$$

It is clear that the system (23) with (24) is positive and unstable since A_i are non-Hurwitz Metzler and $B_i \succ 0$ for $i = 1, 2$.

According to Theorem 6, one feasible solution of Linear Programming Problem (18) provides $v = [1.6300\ 1.4483]^T$, $\gamma = [2.0113\ 1.3628]^T$, and $c_1 = 9.1280, c_2 = 7.0650$, which implies that the SPLS (23) with (24) is exponentially stable. The simulation result can be seen in Figure 4, where the switching rule is same as that in Figure 1.

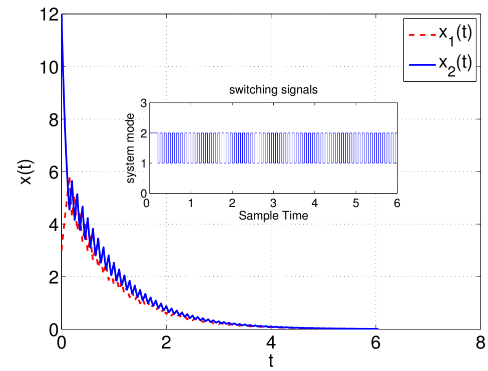


Fig. 4. State responses and switching signals of the SPLS in Example 3, where the initial condition is $x(0) = [3\ 12]^T$ and the delay is $h = 0.05$.

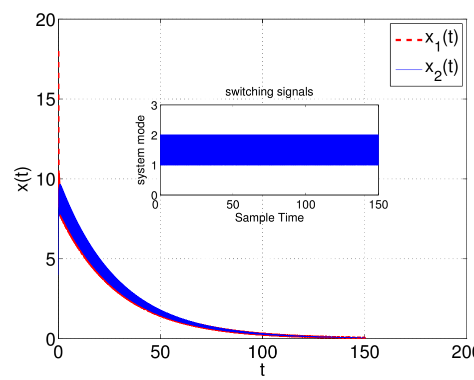


Fig. 5. State responses and switching signals of the SPLS in Example 4, where the initial conditions is $x(0) = [18\ 4]^T$ and the delay is $h = 0.05$.

Example 4: Let us define the same state-dependent switched linear system (23), where we replace the system matrices A_1, A_2 with

$$A_1 = \begin{bmatrix} -6.2 & 2 \\ 1 & 3 \end{bmatrix}, \ A_2 = \begin{bmatrix} 2 & 1 \\ 1 & -5.8 \end{bmatrix}. \tag{25}$$

First of all, we solve the Linear Programming Problem (18), the resulting index output $exitflag = -2$. Namely, there is no feasible solution of Linear Programming Problem (18), this implies that the single LCL functional method is not applicable on checking the system stability. However, the stability analysis can be carried out with the help of the multiple LCL functional method. By Theorem 8, one feasible solution of Linear Programming Problem (22) provides $v_1 = [89.8385\ 11.8505]^T$, $v_2 = [19.1808\ 82.5082]^T$, $\gamma = [96.3139\ 98.8531]^T$, $c_1 = 356.5931, c_2 = 270.3358$, and $d_{1,2} = -d_{2,1} = -70.6578$.

The simulation is shown in Figure 5, from which one can see that SPLS is exponentially stable.

5. CONCLUDING REMARKS

In this note, we discuss the stability problem of SPLSs with a given state-dependent switching. Based on the given partitions of the nonnegative orhant, some exponential stability criteria are derived and formulated as Linear Programming Problems. Such criteria can be applied to the general non-delayed and delayed SPLSs without any special restriction. Future work will consider the case when the state-dependent switching laws are not known in advance. We suspect that the results presented here will be of great value in this future study.

REFERENCES

[1] D. Liberzon and A. S. Morse, "Basic problems in stability and design of switched systems," *IEEE Control Syst. Mag.*, vol. 19, pp. 59-70, 1999.

[2] L. Farina and S. Rinaldi, *Positive Linear Systems: Theory and Applications*, Wiley, New York, 2000.

[3] R. Shorten, D. Leith, J. Foy, and R. Kilduff, "Towards an analysis and design framework for congestion control in communication networks," *Proc. of the 12th Yale Workshop on Adaptive and Learning Systems*, 2003.

[4] A. Jadbabaie, J. Lin, and A. Morse, "Coordination of groups of mobile autonomous agents using nearest neighbor rules," *IEEE Trans. Automat. Control*, vol. 48, no. 6, pp. 988-1001, 2003.

[5] O. Mason and R. Shorten, "On linear copositive Lyapunov functions and the stability of switched positive linear systems," *IEEE Trans. Automat. Control*, vol. 52, no. 7, pp. 1346-1349, 2007.

[6] X. Ding, L. Shu, and X. Liu, "On linear copositive Lyapunov functions for switched positive systems," *J. Franklin Inst.*, vol. 348, pp. 2099-2107, 2011.

[7] X. Xue and Z. Li, "Asymptotic stability analysis of a kind of switched positive linear discrete systems," *IEEE Trans. Automat. Control*, vol. 55, no. 9, pp. 2198-2203, 2010.

[8] Z. Yu, Z. Zhang, S. Cheng, and H. Gao, "Approach to stabilisation of continuous-time switched positive systems," *IET Control Theory Appl.*, vol. 8, no. 13, pp. 1207-1214, 2014.

[9] X. Zhao, X. Liu, S. Yin, and H. Li, "Improved results on stability of continuous-time switched positive linear systems," *Automatica*, vol. 50, no. 2, pp. 614-621, 2014. [click]

[10] X. Liu and C. Dang, "Stability analysis of positive switched linear systems with delays," *IEEE Trans. Automat. Control*, vol. 50, no. 7, pp. 1684-1690, 2011.

[11] X. Zhao, L. Zhang, P. Shi, and M. Liu, "Stability of switched positive linear systems with average dwell time switching," *Automatica*, vol. 48, pp. 1132-1137, 2012. [click]

[12] J.-S. Zhang, Y.-W. Wang, J.-W. Xiao, and Y.-J. Shen, "Stability analysis of switched positive linear systems with stable and unstable subsystems," *Int. J. Syst. Sci.*, vol. 45, no. 12, pp. 2458-2465, 2014. [click]

[13] F. Blanchini, P. Colaneri, and M. E. Valcher, "Copositive Lyapunov functions for the stabilization of positive switched systems," *IEEE Trans. Automat. Control*, vol. 57, no. 2, pp. 3038-3050, 2012.

[14] E. Fornasini and M. E. Valcher, "On the stability of continuous-time positive switched systems with rank one difference," *Control and Cybernetics*, vol. 42, pp. 47-62, 2013.

[15] E. Fornasini and M. E. Valcher, "Stability and stabilizability criteria for discrete-time positive switched systems," *IEEE Trans. Automat. Control*, vol. 57, no. 5, pp. 1208-1221, 2012.

[16] E. Fornasini and M. E. Valcher, "Stability and stabilizability of special classes of discrete-time positive switched systems," *Proc. of 2011 American Control Conference*, San Francisco, CA, USA, pp. 2619-2624, June 29-July 01 2011.

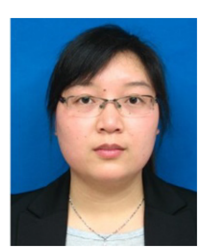
[17] M. Ait Rami, U. Helmke, and F. Tadeo, "Positive observation problem for linear time-delay positive systems," in *2007 Mediterranean Conference on Control and Automation, MED*, Athens, Greece, July 2007.

[18] M. Ait Rami and F. Tadeo, "Controller synthesis for positive linear systems with bounded controls," *IEEE Trans. Circuits Syst. II: Exp. Briefs*, vol. 54, no. 2, pp. 151-155, 2007.

[19] S. Boyd, L. E. Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, SIAM, Philadelphia, PA, 1994.

[20] S. Pettersson and B. Lennartson, "Stabilization of hybrid systems using a min-projection strategy," *Proc. Amer. Control Conf.*, vol. 54, 2001, pp. 308-322.

Xiuyong Ding is a senior lecturer of the School of Mathematics at Southwest Jiaotong University in Chengdu, China. He received his M.S. and Ph.D. degrees in Applied Mathematics from University of Electronic Science and Technology of China, in 2009 and 2012, respectively. His research interests include differential equations, hybrid systems and its applications.



Xiu Liu is a senior lecturer of the School of Mathematics at Southwest Jiaotong University in Chengdu, China. She received her M.S. and Ph.D. degrees in Applied Mathematics from University of Electronic Science and Technology of China, in 2010 and 2013, respectively. Her research interests include switched systems, impulsive systems, and descriptor systems.