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Fractional Sliding Mode Control of Underwater ROVs Subject to Non-differentiable Disturbances

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Abstract: Some hydrodynamic phenomena of an underwater Remotely Operated Vehicle (*ROV*), such as turbulence, cavitation, and multi-phase fluidic regimes, are associated to continuous but nowhere differentiable functions. These disturbances stand as complex forces potentially influencing the *ROVs* during typical navigation tasks. In this paper, the tracking control of a *ROV* subject to nonsmooth Hölder disturbances is proposed based on a fractional-order robust controller that ensures exponential tracking. Notably, the controller gives rise to a closed-loop system with the following characteristics: a) continuous control signal that alleviates chattering effects; b) the fractional sliding motion is substantiated on a proposed resetting memory principle; c) the control is robust to model uncertainties; and d) exact rejection of Hölder disturbances in finite-time. A representative simulation study reveals the feasibility of the proposed scheme.

Keywords: Fractional-order control, fractional sliding modes, Hölder functions, remotely operated vehicle.

1. INTRODUCTION

An underwater Remotely Operated Vehicle (*ROV*) constitutes a nonlinear coupled dynamical system, [1], subject to a wide class of exogenous disturbances as well as parametric uncertainties and unmodeled dynamics, such as added mass, cavitation, turbulence due to thrusters and interface to rigid plates, and stream jets, to name a few, [2]. Thus, the design of a robust controller that withstands any, or all, of those uncertain forces is of interest for designing the next generation of *ROVs*.

Several control approaches for *ROVs* have been proposed, from typical inverse dynamics that are prone to instability for any uncertainty, [3], to H_∞ schemes based on linearized models, [4], and sliding mode structures [5–7]. However, classical sliding mode schemes compromise the performance due to harmful chattering, [8], meanwhile high-order sliding mode structures assure robustness with chattering alleviation by assuming Lipschitz (weakly differentiable) disturbances, [9]. Additionally, underwater disturbances may exhibit whimsical properties, which are due to the complex interaction of turbulence fluid to the *ROV* rigid body, and to the biphasic maritime environment, [10], whose transition from laminar to turbulent regime has been studied using multifractal analysis, [11]. In the latter, the regularity condition has been studied by means of integral Wavelet transforms, resorting on Lipschitz/Hölder exponents, [12]. In addition, Hölder exponents reveal an intriguing relation between the maximum Hölder exponent (with respect to the singularity spectrum) and the Reynolds number, [11], showing that these functions are difficult to handle by using the conventional notion of integer differentiability, because their Hölder exponents are less than one. Furthermore, it is known that there exist singular functions with no integer derivatives but with well-posed fractional order ones, [13]. Also, it has been reported in [14] that some velocity components of a turbulent fluid can be described by continuous but nowhere differentiable fractal functions such as the Weierstrass-Mandelbrot function.

The synthesis of fractional continuous controllers for integer-order plants subject to non-Liptchitz disturbances has been overlooked despite the enormous body of literature on applications of fractional-order control. Clearly, the disturbance rejection of these Hölder (non-Liptchitz) functions stands for a problem that requires unconventional control schemes, such as fractional-order control. This suggests that the fundamentals of Fractional Calculus can be considered to study this problem by using the diverse topological and robustness properties in addition to the integer-order case, [15]. The pioneer work of [16] considers fractional-order reaching dynamics for a system

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without disturbances; however, general results on stability, robustness and finite-time convergence have been omitted. Also, there has been proposed even a fractional PI controller for *ROVs*, showing a better performance with respect to the integer-order counterpart, however, without any analysis of the stability properties, [17]. Since the ROV is subject to complex hydrodynamic disturbances, fractional control structures are considered to address robustness with a formal stability analysis. A step forward has been established in [18,19], wherein the advantageous structural properties of fractional differintegral operators, such as memory and heritage, have been synthesized to propose a robust fractional-order control for integer-order dynamical robots. Nevertheless, the rejection of Hölder disturbances for *ROVs* requires additional developments to design a robust fractional-order controller.

In this paper, and motivated by our previous results in [18,19], it is proposed a fractional sliding mode controller for the fully actuated *ROV*. Assuming that all the state is accessible and known, the following salient contributions of our approach can be enlisted:

- *A sliding mode is enforced in finite-time*
- *The controller is continuous*
- *Robustness against non-differentiable disturbances*
- *Exact rejection of Hölder disturbances*
- *Exponential tracking during the sliding motion*

This suite of characteristics equips the proposal with a convenient control framework for *ROV*s since the exact knowledge of the complex dynamic model is not required, yet the controller rejects anomalous disturbances without the expenditure of a high control frequency.

This paper is organized as follows: Section 2. addresses the preliminaries on Fractional Calculus, kinematic and dynamic model of the *ROV*, and the control problem statement. Section 3. presents the control design and the stability analysis. Section 4. exposes a simulation study to show the viability of the theoretical contribution, and finally, Section 5. discusses the main conclusions.

2. PRELIMINARIES

The background on fractional differintegral operators and Hölder functions are presented in this section, as well as kinematic and dynamic *ROV* models are introduced.

2.1. On fractional operators

Consider the following differintegral operators, [20]:

· *Riemann-Liouville fractional integral*

$$ {}_{t_0}I_t^{\eta} f(t) = \frac{1}{\Gamma(\eta)} \int_{t_0}^{t} \frac{f(\tau)}{(t-\tau)^{1-\eta}} d\tau. \tag{1}$$

· *Caputo fractional derivative*

$$ {}_{t_0}^{C}D_t^{\eta} f(t) = {}_{t_0}I_t^{\lceil\eta\rceil-\eta} \frac{d^{\lceil\eta\rceil}}{dt^{\lceil\eta\rceil}} f(t) \tag{2}$$

where $\lceil\eta\rceil = \min\{x\in\mathbb{Z}:\ x\geq\eta\}$ is the ceil function, $\Gamma(x)=\int_0^\infty z^{x-1}e^{-z}dz$ is the Gamma function and w_k^{η} are fractional-binomial coefficients.

2.2. On fractional differentiability

The Hölder exponent is related to the regularity of a function, [12], which in turns is strongly related to the notion of fractional differentiability, [13,21]. Hence, consider the following useful definition.

Definition 1: A function $f:\Omega\subset\mathbb{R}\to\mathbb{R}$ is Hölder continuous on Ω for the exponent $\eta\in[0,1]$ if $\exists H\in\mathbb{R}$ such that $\forall\, t_1,t_2\in\Omega$,

$$ |f(t_1)-f(t_2)| \leq H|t_2-t_1|^{\eta}. \tag{3}$$

The maximum number η that complies with (3) is called the critical exponent of the function $f(t)$.

Some algebraic properties of Hölder continuous functions are summarized in the following proposition.

Proposition 1: Let $f(t)$ and $g(t)$ be two Hölder continuous functions on a bounded interval $\Omega\subset\mathbb{R}$, whose critical orders are $\mu,\eta\in(0,1)$, respectively, with $\mu<\eta$. Then, $f(t)+g(t), f(t)g(t)$ are Hölder continuous functions of critical order μ on Ω. In addition, $f(g(t)), g(f(t))$ are Hölder functions having the critical order $\mu\eta$ on Ω.

To analyze the fractional differentiability of a Hölder continuous function, it is needed to extend the Caputo fractional derivative from the space of differentiable functions to the space of those functions without a well-posed integer-order derivative, but being Hölder continuous for some positive exponent less than one. To this end, integrating by parts (2), one obtains the following

$$ {}_{t_0}^{C}D_t^{\eta} f(t) = \frac{f(t)-f(t_0)}{\Gamma(1-\eta)(t-t_0)^{\eta}} + \frac{\eta}{\Gamma(1-\eta)} \int_{t_0}^{t} \frac{f(t)-f(\tau)}{(t-\tau)^{\eta+1}}\, d\tau, \tag{4}$$

which is a well-posed operator for Hölder continuous functions $f(t)$ of critical exponents grater than η. In addition, operators (2) and (4) coincide not just for differentiable functions but also for Hölder continuous functions with critical exponents grater than η, [13,21]. In addition, the Hölder continuity is a necessary condition for a function to be fractional differentiable, as it is stated in the following proposition.

Proposition 2: Let $f(t):\Omega\to\mathbb{R}$ be a real valued function, with Ω a convex compact set of real numbers. If $\sup_{t_1,t_2\in\Omega}\left|{}_{t_1}^{C}D_t^{\eta} f(t)\right|_{t=t_2} = k_f\in\mathbb{R}$, with $\eta\in(0,1)$, then $|f(t_2)-f(t_1)|\leq\frac{k_f}{\Gamma(\eta+1)}|t_2-t_1|^{\eta}$.

Moreover, it is worth to point out that the fractional integral of a bounded function is a Hölder continuous function with a critical order at least as large as the order of integration, (Corollary 2 pp. 56 of [21]). This observation

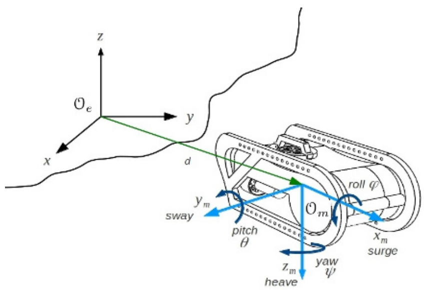


Fig. 1. Inertial and body frames for ROV modeling.

enables us to analytically deal with Hölder disturbances by means of a uniformly continuous controller, assuring chattering alleviation while preserving robustness to a wide class of physical hydrodynamic disturbances occurring in the maritime environment.

2.3. Differential Kinematics of the ROV

Consider the inertial reference frame $\mathcal{O}_e$ and the body frame $\mathcal{O}_m$ of a given ROV, see Fig. 1. Aiming at relating the *extended* velocity vector $\boldsymbol{\nu} \in \mathbb{R}^6$ to the time derivative of the *ROV* pose $\boldsymbol{x} \in \mathbb{R}^{3+l}$, for $l \geq 3$ the number of attitude parameters, let $\boldsymbol{x} = [\boldsymbol{d}^T\ \boldsymbol{\Theta}^T]^T$ and $\boldsymbol{\nu} = [\boldsymbol{v}^T\ \boldsymbol{\omega}^T]^T$ denoting the inertial position $\boldsymbol{d} \in \mathbb{R}^3$ and the attitude parameters $\boldsymbol{\Theta} \in \mathbb{R}^l$, where $\boldsymbol{v} \in \mathbb{R}^3$ and $\boldsymbol{\omega} \in \mathbb{R}^3$ stand for the linear and the angular velocities, respectively, expressed in the vehicle reference frame $\mathcal{O}_m$. Then, $\boldsymbol{\nu} = J_v(\boldsymbol{\Theta})\dot{\boldsymbol{x}}$ represents the forward differential kinematic. Assuming the roll-pitch-yaw attitude representation, $\boldsymbol{\Theta} = [\phi\ \theta\ \psi]^T$. Consider the transformation matrix

$$J_v(\boldsymbol{\Theta}) = \begin{bmatrix} R^T(\boldsymbol{\Theta}) & 0_{3\times 3} \\ 0_{3\times 3} & R^T(\boldsymbol{\Theta})J_\theta(\boldsymbol{\Theta}) \end{bmatrix} \in \mathbb{R}^{6\times 6},$$

for $R(\boldsymbol{\Theta}) \in \mathbb{R}^{3\times 3}$ the rotation matrix between the inertial frame and the vehicle reference frame, and $J_\theta(\boldsymbol{\Theta}) \in \mathbb{R}^{3\times 3+l}$ given by

$$J_\theta(\boldsymbol{\Theta}) = \begin{bmatrix} C_\theta C_\psi & -S_\psi & 0 \\ C_\theta S_\psi & C_\psi & 0 \\ -S_\theta & 0 & 1 \end{bmatrix},$$

for $S_\alpha = \sin(\alpha)$ and $C_\alpha = \cos(\alpha)$. Notice that the latter transforms the angular velocity $\boldsymbol{\omega}_e$ expressed in the inertial frame $\mathcal{O}_e$ to the time derivative of the attitude parameters, such that, $\boldsymbol{\omega}_e = J_\theta(\boldsymbol{\Theta})\dot{\boldsymbol{\Theta}}$.

2.4. Dynamic model of the ROV

The dynamic model of the *ROV* can be obtained by using the Kirchhoff formulation, [1],

$$\frac{d}{dt}\frac{\partial K}{\partial \boldsymbol{v}} + \boldsymbol{\omega} \times \frac{\partial K}{\partial \boldsymbol{v}} = \boldsymbol{f}, \tag{5}$$

$$\frac{d}{dt}\frac{\partial K}{\partial \boldsymbol{\omega}} + \boldsymbol{v} \times \frac{\partial K}{\partial \boldsymbol{v}} + \boldsymbol{\omega} \times \frac{\partial K}{\partial \boldsymbol{\omega}} = \boldsymbol{n}, \tag{6}$$

where $K = K_a + K_I$ represents the total kinetic energy, $\boldsymbol{f}$ and $\boldsymbol{n}$ are the non inertial force and torque control inputs, respectively. The important added mass effect is modeled through $K_a = \frac{1}{2}\boldsymbol{\nu}_r^T M_a \boldsymbol{\nu}_r$ that stands for the kinetic energy of the fluid around the *ROV*, where $\boldsymbol{\nu}_r \triangleq \boldsymbol{\nu} - \mathcal{R}^T\boldsymbol{\zeta}_e$ models the relative fluid velocity, for $\boldsymbol{\zeta}_e$ the non-rotational velocity of the fluid expressed in the inertial frame $\mathcal{O}_e$,

$$\mathcal{R} = \begin{bmatrix} R(\boldsymbol{\Theta}) & 0_{3\times 3} \\ 0_{3\times 3} & R(\boldsymbol{\Theta}) \end{bmatrix} \in \mathbb{R}^{6\times 6},$$

and M_a stands for the positive definite added mass matrix. The term $K_I = \frac{1}{2}\boldsymbol{\nu}^T M_I \boldsymbol{\nu}$ models the kinetic energy of the *ROV* rigid body, where M_I stands for the constant, symmetric and positive definite inertia matrix given by

$$M_I = \begin{bmatrix} mI_{3\times 3} & -m[\boldsymbol{r}_c\times] \\ m[\boldsymbol{r}_c\times] & I_g \end{bmatrix} \tag{7}$$

for m and $\boldsymbol{r}_c$ the total mass and position of the center of mass of the *ROV*, respectively, and $I_g \triangleq I_c - m[\boldsymbol{r}_c\times]^2$ is the displaced inertia moments matrix, for I_c the inertia tensor with respect to the center of mass of the body in the local frame $\mathcal{O}_m$. To obtain the more familiar Lagrangian formulation, firstly consider the extended momentum, [22],

$$\boldsymbol{P} \triangleq \frac{\partial K}{\partial \boldsymbol{\nu}} = M_b\boldsymbol{\nu} - M_{as}\mathcal{R}^T\boldsymbol{\zeta}_e \tag{8}$$

then, the Kirchhoff formulation (5)-(6) can be written in the following reduced form

$$\dot{\boldsymbol{P}} - \Omega^T(\boldsymbol{\nu})\boldsymbol{P} = \boldsymbol{F}_u + \boldsymbol{F}_g(\boldsymbol{x}) + \boldsymbol{F}_d - D_b(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r \tag{9}$$

where

$$\boldsymbol{F}_u = B\boldsymbol{u} \tag{10}$$

$$\boldsymbol{F}_g(\boldsymbol{x}) = \begin{pmatrix} (m - \rho V)\boldsymbol{g}(\boldsymbol{\theta}) \\ \boldsymbol{r}_c \times m\boldsymbol{g}(\boldsymbol{\theta}) - \boldsymbol{r}_b \times \rho V \boldsymbol{g}(\boldsymbol{\theta}) \end{pmatrix} \tag{11}$$

$$\Omega(\boldsymbol{\nu}) \triangleq \begin{bmatrix} [\boldsymbol{\omega}\times] & [\boldsymbol{v}\times] \\ 0 & [\boldsymbol{\omega}\times] \end{bmatrix} \tag{12}$$

for $M_b \triangleq M_I + M_{as}$, $M_{as} \triangleq \frac{M_a + M_a^T}{2}$, ρ the density of the fluid, V the volume of the *ROV*, $\boldsymbol{r}_b$ the buoyancy center of the underwater robot, $\boldsymbol{g}(\boldsymbol{\theta}) = \mathcal{R}^T\boldsymbol{g}_e$, with $\boldsymbol{g}_e \in \mathbb{R}^6$, the inertial gravity vector, $\boldsymbol{u} \in \mathbb{R}^n$ the vector of control forces of the l thrusters that power the *ROV*, $\boldsymbol{F}_d \in \mathbb{R}^6$ the vector of exogenous disturbances that are assumed continuous but not necessarily differentiable, *i.e.* those are Hölder disturbances, $B \in \mathbb{R}^{6\times n}$ the thruster configuration input matrix, and $D_b(\boldsymbol{\nu}_r) \in \mathbb{R}^{6\times 6}$ the hydrodynamic damping matrix. Finally, there arises the *ROV* Lagrangian representation with $\boldsymbol{q}$ as the generalized coordinates, [22],

$$H_q(\boldsymbol{q})\ddot{\boldsymbol{q}} + C_q(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + D_q(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{g}_q(\boldsymbol{q}) = \boldsymbol{\tau}_q + \boldsymbol{f}_d, \tag{13}$$

where

$$H_q(\boldsymbol{q}) = J_v^T(\boldsymbol{q})M_T J_v(\boldsymbol{q}) \quad (14)$$
$$C_q(\boldsymbol{q},\dot{\boldsymbol{q}}) = J_v^T(\boldsymbol{q})M_T\dot{J}_v(\boldsymbol{q}) + J_v^T(\boldsymbol{q})C_b J_v(\boldsymbol{q}), \quad (15)$$
$$D_q(\boldsymbol{q},\dot{\boldsymbol{q}}) = J_v^T(\boldsymbol{q})D_b(\boldsymbol{q},\dot{\boldsymbol{q}})J_v(\boldsymbol{q}), \quad (16)$$
$$\boldsymbol{g}_q(\boldsymbol{q}) = J_v^T(\boldsymbol{q})\boldsymbol{g}_b, \quad (17)$$
$$\boldsymbol{\tau}_q = J_v^T(\boldsymbol{q})\boldsymbol{F}_u, \quad (18)$$
$$\boldsymbol{f}_d = J_v^T(\boldsymbol{q})\boldsymbol{F}_d, \quad (19)$$

with $M_T \triangleq M_I + M_a > 0$, $C_b(\boldsymbol{\nu})\boldsymbol{\nu} = -\Omega^T(\boldsymbol{\nu})M_b\boldsymbol{\nu}$ represents the Coriolis matrix, and the net balance of gravitational buoyant forces is given by $\boldsymbol{g}_b(\boldsymbol{x}) = -\boldsymbol{F}_g(\boldsymbol{x})$. Thus, the control problem can be stated as follows:
"Assuming the ROV dynamic model is uncertain and subject to additive Hölder disturbances, design a continuous controller τ_q, such that exponential tracking is assured provided that kinematic maps and full state are available.

2.5. Structural properties of the Lagrangian model

There exists $c_i > 0, i = 0,\ldots,4$, such that, [22, 23],
1) $c_0 < \lambda_m(H_q(\boldsymbol{q})) \leq \| H_q(\boldsymbol{q}) \| \leq \lambda_M(H_q(\boldsymbol{q})) < c_1$
2) $\| C_q(\boldsymbol{q},\dot{\boldsymbol{q}}) \| \leq c_2 \| \dot{\boldsymbol{q}} \|$
3) $\| D_q(\boldsymbol{q},\dot{\boldsymbol{q}}) \| \leq c_3$
4) $\| \boldsymbol{g}_q(\boldsymbol{q}) \| \leq c_4$
for $\lambda_{m(M)}(A)$ the minimum (maximum) eigenvalue of the matrix A. Then, $u \in \mathbb{R}^6$ implies that the *ROV* is fully actuated and realizable since B and J_v are invertible. Notice that since f_d is an additive matched Hölder hydrodynamic disturbance, it makes unfeasible to rely on known sliding mode controllers for rejection of Liptchitz disturbances.

3. FRACTIONAL CONTROL DESIGN

The control problem is to design a continuous τ_q such that $q \to q_d$, for $q_d \in \mathcal{C}^3$. To this end, firstly consider the componentwise control approach [8], and let the $i-$th entry of the coupled nonlinear dynamic model of (13) be the nominal model, with the off-diagonal terms considered as unmodeled exogenous terms. Then, the error manifold is designed, and the controller is given; finally, the stability analysis is shown.

3.1. Control design

Let $\boldsymbol{f}_d$ attain bounded fractional derivatives for every order less than the critical order $\eta_c \in (0,1)$. Since the diagonal of $H_q(\boldsymbol{q})$ is positive definite and bounded, the diagonal entrywise dynamics of (13) can be written as follows

$$\begin{aligned} \ddot{q}_i &= h_{ii}^{-1}\tau_{qi} + \bar{\varphi}_i, \quad (20)\\ \bar{\varphi}_i &\triangleq h_{ii}^{-1}\left(-\sum_{i\neq j} h_{ij}\ddot{q}_j + \sum_j (c_{ij}+d_{ij})\dot{q}_j + g_i + f_{di}\right), \end{aligned}$$

where arguments are omitted to avoid cumbersome notation. Let the linear error manifold be

$$S_i \triangleq \Delta\dot{q}_i + \beta_i\Delta q_i, \quad (21)$$

with $\Delta q_i = q_i - q_{di}$ representing the tracking error, and $\beta_i > 0$ a feedback gain. Let us define the nominal reference velocity as

$$\dot{q}_{ri} \triangleq \dot{q}_d - \beta_i\Delta q_i + S_i(t_0)e^{-\kappa(t-t_0)},$$

for $\kappa > 0$ a desired convergence rate, [22], and let the following sliding error manifold be

$$S_{qi} \triangleq \dot{q}_i - \dot{q}_{ri} \quad (22)$$

whose derivative of (22) along (20) becomes

$$\dot{S}_{qi} = h_{ii}^{-1}\tau_{qi} + \varphi_i, \quad (23)$$

where $\varphi_i = \bar{\varphi}_i - \ddot{q}_{di} + \beta_i\Delta\dot{q}_i + \kappa S(t_0)e^{-\kappa(t-t_0)}$. Since φ_i has bounded derivatives of every order $\eta < \eta_c$, (23) is η-times fractionally differentiable, hence, there exists a bounded positive scalar c such that $\sup_{a_i,t\in\mathbb{R}} \left|{}_{a_i}^{C}D_t^{\eta}\varphi_i\right|_t = c$. In this condition, consider the fractional sliding mode controller

$$\tau_{qi}(t) \triangleq \tau_{qi}(a_i) - k_i \, {}_{a_i}I_t^{\eta}\operatorname{sign}(S_{qi}), \quad (24)$$

where a_i is a lower terminal based on a principle of dynamic memory resetting, addressed in Theorem 1, and k_i is a feedback gain. Substituting controller (24) into the open-loop error equation (23), the following closed-loop error equation arises

$$\dot{S}_{qi}(t) = \dot{S}_{qi}(a_i) - h_{ii}^{-1}k_i \, {}_{a_i}I_t^{\eta}\operatorname{sign}(S_{qi}(t)) + \varsigma_i(t), \quad (25)$$

with $\varsigma_i(t) = [h_{ii}(t)^{-1} - h_{ii}(a_i)^{-1}]\tau_{qi}(a_i) + {}_{a_i}I_t^{\eta} \, {}_{a_i}^{C}D_t^{\eta}\varphi_i$.

3.2. On exponential tracking

There arises a major concern related to the memory of the fractional operators, which is simultaneously and surprisingly, a main advantage over integer-order structures, [24]. To proof robustness, stability and finite-time convergence, a *resetting memory principle* is proposed, which consists in resetting the memory of the differintegral operator at each time $t = t_{ni}$ $(ni \in \mathbb{N}_0)$ when $S_{qi}(t_{ni}) = 0$.

For a simpler notation, let $\sigma = S_{qi}$ and $\bar{k} = h_{ii}^{-1}k_i$ for an arbitrary i. Also, define the lower terminal as $a_i = t_n$, thus (25) becomes

$$\dot{\sigma}(t) = \sigma(t_n) - \bar{k} \, {}_{t_n}I_t^{\eta}\operatorname{sign}(\sigma(t)) + \varsigma(t). \quad (26)$$

Henceforth, consider the following main result.

Theorem 1: Consider the closed-loop fractional-order system (26) with $\eta \in (0,1)$, $\sigma(t_0) = 0$, $\dot{\sigma}(t_0)$ any real number, and $c = \sup_{a,b\in\mathbb{R}} \left|{}_{a}^{C}D_t^{\eta}\varsigma_i\right|_{t=b}$. Let $\{t_n\}$ and $\{t_n'\}$ be the strictly increasing sequences of non-negative real numbers such that $\sigma(t_n)=0$ and $\dot\sigma(t'_n)=0$, respectively. Then, for $\bar k(t,\sigma(t))\in[k_m,\alpha k_m]$, with

$$k_m > \frac{3+\eta}{2-\alpha(1+\eta)}c \quad \text{and} \quad 1\le\alpha<\frac{2}{1+\eta}, \tag{27}$$

$\exists t_s\in\mathbb{R}$ such that $\dot\sigma(t)=\sigma(t)=0\ \forall t\ge t_s$, guarantying the exponential convergence of tracking errors.

Proof: Without loss of generality consider $\dot\sigma(t_0)>0$, then $\sigma(t_0^+)>0$. Also, consider

$$\begin{aligned}\xi_1(t) &= \dot\sigma(t_0)(t-t_0)-\frac{\alpha k_m+c}{\Gamma(2+\eta)}(t-t_0)^{1+\eta},\\ \xi_2(t) &= \dot\sigma(t_0)(t-t_0)-\frac{k_m-c}{\Gamma(2+\eta)}(t-t_0)^{1+\eta}.\end{aligned}$$

Then, by using the monotonicity of the fractional integral over $[t_0,t]\subseteq[t_0,t_1]$, it results $\dot\xi_1(t)\le\dot\sigma(t)\le\dot\xi_2(t)$. Integrating again, one obtains $\xi_1(t)\le\sigma(t)\le\xi_2(t)$. To estimate t_1, solve for $\xi_1(t)=0$ and $\xi_2(t)=0$, and consider that $\xi_1(t)$ crosses zero at $t_{\xi_1}<t_1$ (*i.e.* before the instant at which $\sigma(t_1)=0$), and that $\xi_2(t)$ crosses zero at $t_{\xi_2}>t_1$, thus obtaining

$$\frac{\dot\sigma(t_0)\Gamma(2+\eta)}{\alpha k_m+c}\le(t_1-t_0)^\eta\le\frac{\dot\sigma(t_0)\Gamma(2+\eta)}{k_m-c}. \tag{28}$$

Now, one has that $\dot\sigma(t_1)$ is bounded below by $\dot\xi_1(t_{\xi_2})$, where t_{ξ_2} is the time at $\xi_2(t_{\xi_2})=0$. This leads to $-\mu\dot\sigma(t_0)\le\dot\sigma(t_1)$, with $\mu=\frac{\alpha k_m+c}{k_m-c}(1+\eta)-1<1$ since by (27), one gets $c<\frac{2-\alpha(1+\eta)}{3+\eta}k_m$. Also, proceeding by mathematical induction, one has that

$$|\dot\sigma(t_n)|\le\mu|\dot\sigma(t_{n-1})|\ \forall n\in\mathbb{N}, \tag{29}$$

with $\dot\sigma(t_n)\dot\sigma(t_{n-1})\le0$. Hence, $|\dot\sigma(t_n)|\le\mu^n|\dot\sigma(t_0)|$ and consequently

$$\dot\sigma(t_n)\to0 \text{ as } n\to\infty. \tag{30}$$

Notice that the first cross $\dot\sigma(t'_0)=0$ can be estimated in a similar fashion. Then, one has that

$$\frac{\dot\sigma(t_0)\Gamma(1+\eta)}{\alpha k_m+c}\le(t'_0-t_0)^\eta\le\frac{\dot\sigma(t_0)\Gamma(1+\eta)}{k_m-c}. \tag{31}$$

Therefore, from $\sigma(t)\le\xi_2(t)$ for $t\in[t_0,t_1]$, one obtains

$$\sigma(t'_0)\le\dot\sigma(t_0)^{1+1/\eta}\frac{\eta}{1+\eta}\left[\frac{\Gamma(1+\eta)}{k_m-c}\right]^{1/\eta}. \tag{32}$$

The latter establishes an upper bound for $\sigma(t)$. Additionally, in virtue of $|\dot\sigma(t_n)|\le\mu|\dot\sigma(t_{n-1})|$, it follows that

$$|\sigma(t'_n)|\le\mu^{n(1+1/\eta)}|\sigma(t'_0)|, \tag{33}$$

with $t=t'_n$ at $\dot\sigma(t)=0$, and hence

$$\sigma(t'_n)\to0 \text{ as } n\to\infty. \tag{34}$$

The time of convergence is $t_s=t_0+\sum_{n=0}^\infty(t_{n+1}-t_n)$. Also, for each time interval $[t_n,t_{n+1}]$, one can find that $(t_{n+1}-t_n)^\eta\le\mu^n\frac{\dot\sigma(t_0)\Gamma(2+\eta)}{k_m-c}$ holds. Then,

$$t_s\le t_0+\left[\frac{\dot\sigma(t_0)\Gamma(2+\eta)}{k_m-c}\right]^{1/\eta}\sum_{n=0}^\infty(\mu^{1/\eta})^n, \tag{35}$$

and as a consequence,

$$t_s\le t_0+\frac{1}{(1-\mu^{1/\eta})}\left[\frac{\dot\sigma(t_0)\Gamma(2+\eta)}{k_m-c}\right]^{1/\eta}\in\mathbb{R} \tag{36}$$

since $\mu^{1/\eta}<1$. Moreover, from $t_n=t_0+\sum_{k=0}^n(t_{k+1}-t_k)$ and $(t'_n-t_n)^\eta\le\mu^n(t'_0-t_0)^\eta$, one has that

$$\lim_{n\to\infty}t'_n=\lim_{n\to\infty}t_n=t_s. \tag{37}$$

Therefore

$$(\dot\sigma(t),\sigma(t))\to(0,0) \text{ as } t\to t_s. \tag{38}$$

Invariance of $(\dot\sigma(t),\sigma(t))=0$ for $t\ge t_s$, which results of the continuity of $\dot\sigma(t)$, establishes that a sliding regime is sustained afterwards, and this fact leads to conclude that, from (21) and (22), $S_q(t)=0\Leftrightarrow S(t)=S(t_0)e^{\kappa(t-t_0)}$, which finally leads to $\Delta q(t)\to0\Rightarrow q(t)\to q_d(t),\Delta\dot q(t)\to0\Rightarrow\dot q(t)\to\dot q_d(t)$ exponentially after $t_s\in\mathbb{R}$. □

Remark 1: During sliding motion, the invariant $\dot\sigma=0$, shows that from (23), one obtains

$$\boldsymbol{\tau}_q=H_q(\boldsymbol{q})\ddot{\boldsymbol{q}}_r+C_q(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}}_r+D_q(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}}_r+\boldsymbol{g}_q(\boldsymbol{q})-\boldsymbol{f}_d.$$

That is, the controller exactly observes the disturbance without resorting on the equivalent method of the classical integer-order sliding mode scheme, [8].

4. SIMULATIONS

Consider a fully actuated *ROV*. The simulator runs an Euler integrator at $1KHz$ in Matlab 2014. Differintegrals are computed using the Grünwald-Letnikov operator on a low-end Laptop, where the discontinuous $\mathrm{sign}(x)$ function is approximated by $\tanh(100x)$.

4.1. The plant

Dynamic parameters of the ROV are given in Table 1, with damping parameters in Table 2, for a SNAME notation, [25]. The term $A_{\dot b}\dot b=\frac{\partial A}{\partial \dot b}\frac{d}{dt}b$. X, Y and Z are the forces acting along x, y and z, respectively; similarly, K, M, N are torques acting along ϕ,θ,ψ, respectively. For low velocities, typical in underwater exploration tasks, added mass and damping matrix can be approximated by $M_a=-\mathrm{diag}\{X_{\dot u},\ Y_{\dot v},\ Z_{\dot w},\ K_{\dot p},\ M_{\dot q},\ N_{\dot r}\}$ and $D_b(\boldsymbol{\nu}_r)=-\mathrm{diag}\{X_u,\ Y_v,\ Z_w,\ K_p,\ M_q,\ N_r\}-\mathrm{diag}\{X_{u|u|}|u|,\ Y_{v|v|}|v|,\ Z_{w|w|}|w|,\ K_{p|p|}|p|,\ M_{q|q|}|q|,\ N_{r|r|}|r|\}$. Added mass coefficients are $(X_{\dot u},Y_{\dot v},Z_{\dot w},K_{\dot p},M_{\dot q},N_{\dot r})=$

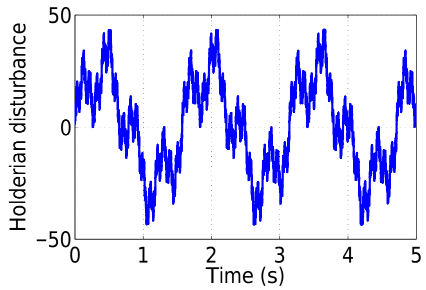


Fig. 2. Hölder disturbance $f_d = 50 \sum_{\gamma=1}^{\infty} 4^{-0.5\gamma} \sin(4^{\gamma} t)$.

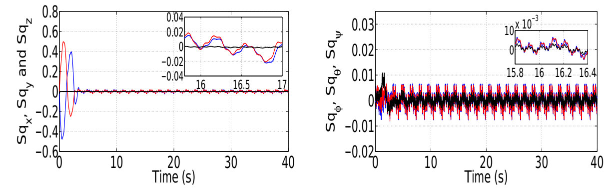
Table 1. Dynamic parameters.

Parameter	Value	Units
m	25	kg
V	0.043	m^3
$\boldsymbol{r}_c$	$[0\ 0\ 0]^T$	m
$\boldsymbol{r}_b$	$[0\ 0\ 0.025]^T$	m
I_{xx}	2.514	kgm^2
I_{yy}	4.069	kgm^2
I_{zz}	3.755	kgm^2
I_{xy}	0.064	kgm^2
I_{xz}	7.94×10^{-7}	kgm^2
I_{yz}	1.061×10^{-5}	kgm^2
d_x	0.35	m
d_y	0.15	m
k_Q	0.01	-
ρ	999	kg/m^3
g	9.81	m/s^2

Table 2. Damping matrix coefficients.

Coefficient	Value	Units
X_u	-18.75	kg/s
Y_v	-27.5	kg/s
Z_w	-31.25	kg/s
K_p	-2.5	kgm/(rad·s)
M_q	-5	kgm/(rad·s)
N_r	-6.25	kgm/(rad·s)
$X_{u\|u\|}$	-10	kg/s
$Y_{v\|v\|}$	-18.75	kg/s
$Z_{w\|w\|}$	-18.75	kg/s
$K_{p\|p\|}$	-3	kgm/(rad·s)
$M_{q\|q\|}$	-3.75	kgm/(rad·s)
$N_{r\|r\|}$	-5	kgm/(rad·s)

$(-45.539$ Kg, -95.402 Kg, -91.538 Kg, -1.4186 Kgm^2, -0.066 Kgm^2, -1.418 Kgm$^2)$. The hydrodynamic disturbance is $f_d = 50 \sum_{\gamma=1}^{p} 4^{-\nu\gamma} \sin(4^{\gamma} t) N$, $p \to \infty$, for each actuator, Fig. 2, with the critical order $\nu = 0.5$, see [13]. For simulation purposes $p = 200$.



(a) Position surfaces. (b) Orientation surfaces.

Fig. 3. Sliding surfaces evolve around the origin.

4.2. The task

The task is to track $x_d = 2\sin\left(\frac{\pi}{10}t\right)$ m, $y_d = 2\cos\left(\frac{\pi}{10}t\right)$ m, $z_d = \frac{t}{100}$ m, with desired attitude $\phi_d = 0$ rad, $\theta_d = 0$ rad, $\psi_d = -\frac{\pi}{10}t$ rad.

4.3. Control parameters and tunning procedure

The tunning procedure complies to the stability analysis, however, final feedback gains are tuned heuristically by setting the required convergence rate of S_q based on gains β and κ with $\lambda_M(\beta) < \kappa$. Then, set η and k, starting with a low value of η while tunning k to get an acceptable finite-time convergence regime. Finally, it is suggested to tune η by increasing it to alleviate the chattering with a sufficiently low value to assure robustness against disturbances. Following this tunning procedure, the control parameters for the simulation are $\beta = I_{6\times 6}$, $\kappa = 1$, $k = diag(100\ 100\ 1200\ 50\ 50\ 100)$, $\eta = 0.45$.

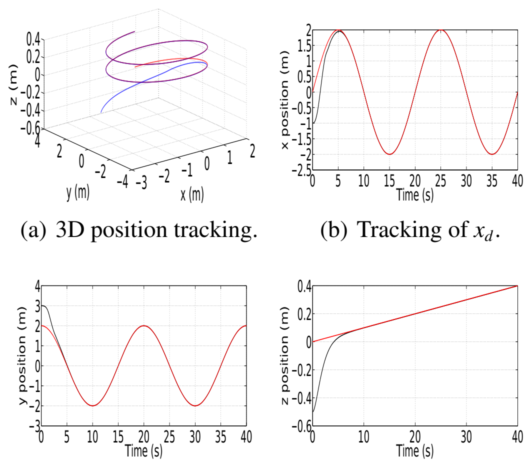
4.4. Results

Sliding surfaces converge in finite-time, as shown in Fig. 3, and remain around zero afterwards. Consequently, tracking errors converge exponentially to yield the 3D Cartesian tracking without overshoot observed in Fig. 4. Notice in Fig. 5 that the orientation angles are smoother than expected, despite of the non-differentiable disturbance and the fact that the controller does not required any information of the hydrodynamic model.

Fig. 6 depict thruster and moment control inputs. The apparent chattering in reality stands simply as the typical performance without chattering, [3], see in particular the time scale shown in each subfigure inset. Notice that the large disturbance amplitude exacerbates actuator activity, which varies according to the amplitude of the disturbance, in virtue of the exact disturbance estimation of the controller.

5. CONCLUSIONS

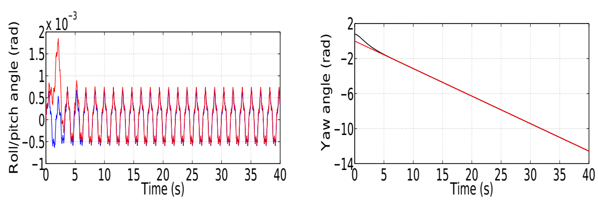
Fractional operators allow to synthesize a continuous controller that outperforms its integer-order counterpart by using the proposed resetting memory principle. In this way, the tracking control of the complete dynamic *ROV* model subject to Hölder hydrodynamic disturbances is proposed by exploiting fractional sliding modes. It is



(a) 3D position tracking. (b) Tracking of x_d.

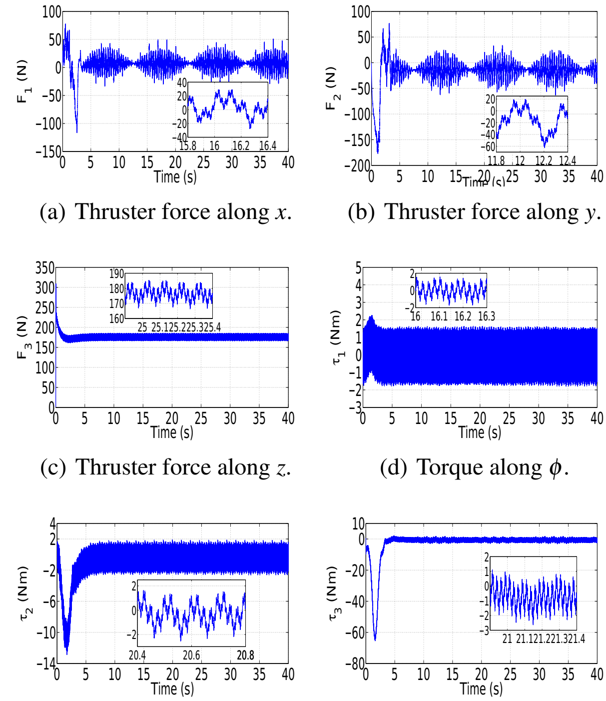
(c) Tracking of y_d. (d) Tracking of z_d.

Fig. 4. Position tracking in Cartesian (x,y,z) coordinates.



(a) Roll and pitch angles. (b) Yaw angle.

Fig. 5. Orientation tracking quickly reacts to variations.



(a) Thruster force along x. (b) Thruster force along y.

(c) Thruster force along z. (d) Torque along ϕ.

(e) Torque along θ. (f) Torque along ψ.

Fig. 6. Thruster forces along Cartesian coordinates and torques along unitary axis, with insets showing continuous control signals.

shown that the establishment of the sliding motion is enforced in finite-time without chattering. Remarkably, the closed-loop system shows that indeed, exact disturbance compensation is obtained thanks to the topological properties of the fractional control signal, which are appealing for typical unknown underwater environments where complex disturbance phenomena are present.

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