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Chaotic Synchronization of Regular and Irregular Complex Networks with Fractional Order Oscillators

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Abstract: Synchronization of N-coupled fractional-order (FO) chaotic oscillators arranged in regular and irregular topologies is numerically studied. Synchronization is achieved based on the coupling matrix from the complex systems theory. In particular, we consider complex dynamical networks composed by Lorenz, Volta, Duffing and Financial FO chaotic oscillators, where the interaction of the nodes is defined by coupling only one state of each FO oscillator.

Keywords: Chaos, complex networks, fractional-order oscillator, synchronization.

1. INTRODUCTION

Fractional calculus is over 300 years old, however in the last twenty years it has attracted attention of different fields. This is due mainly because there are several phenomena in nature that are better represented by using fractional derivatives.

In 1695, L'Hopital wrote a letter to Leibniz posing the question on the meaning of $D^n f$ where n was not an integer in the common concept of derivative. Since then, numerous contributions to the theory of fractional calculus have been made [1]. Nowadays, applications of the fractional calculus theory can be found in many natural and artificial phenomena [2–4]. The interested reader is referred to [5, 6] for a detailed treatment of fractional calculus and its applications.

On the other hand, regarding the use of complex networks; a complex network is defined as a set of coupled interconnected nodes, where each node is a dynamical system. The study of the complex networks has been intensively researched in the last two decades [7]. Some systems of interest are neural networks, the nervous system, genetic and metabolic networks, etc. Complex networks can also be found in technological applications such as telecommunication networks such as the World Wide Web [8].

The ways in which the network nodes are connected correspond to regular or irregular topologies. Under regular topologies, nodes are connected with a definite pattern (ring coupled, global coupled, star coupled); however, in irregular topologies, nodes are connected without a definite link pattern, which leads to higher complex structures, e.g. small-world networks, random networks or scale-free networks [9–13]. In both cases the complexity in the network might be increased if fractional-order (FO) chaotic oscillators are considered as nodes [14] instead of integer-order chaotic oscillators.

Synchronization of complex networks has fascinated researchers for its possible applications [15–18]. Meanwhile, chaotic synchronization has received an increasing attention since the studies of Pecora and Carroll, in which two identical chaotic systems with different initial conditions were synchronized [12]. Chaotic synchronization has been intensively studied in the integer-order systems and recently in the FO systems. At present time, chaotic synchronization applications are focused on data encryption, process control and systems description [13, 14]. In contrast with the well-known integer order oscillators, in a FO oscillator, chaos can exist for different system parameters and different order in the system derivatives. This benefits applications such as encryption, considering that the former property enlarge the key space [15]. The former idea is similar to that used in projective synchronization where the scales between the synchronized states of the network could be used also as encryption parameters [19].

Several methods have been proposed to achieve synchronization between two FO chaotic systems, e.g.,

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Pecora-Carroll (PC) method [12], linear control [20], pinning control [21], H_∞ control [22], and backstepping control [23]. In general, these methods have a complex design procedure and the resulting control law is present in every state of the system.

The main goal of this paper is to synchronize N-coupled FO chaotic oscillators by using only one state in the control law, our purpose is to obtain a "less invasive" control strategy. This objective is achieved by appealing to results from complex systems theory. We demonstrate numerically that these results hold for FO systems. In this work we extend the results shown in [24] to irregular topologies. Furthermore, the results obtained by applying this method to the Volta, Duffing and Financial FO chaotic oscillators are included.

This paper is arranged as follows: In Section 2 we introduce some necessary definitions and notations of fractional calculus; in this section a brief review on synchronization of complex dynamical networks is also included. In Section 3, the problem of synchronization N-coupled FO chaotic systems in regular network is exposed as well as the FO model of Lorenz and Volta systems which will be used as fundamental nodes to compose the regular networks used. Then, in Section 4 we synchronize two irregular networks with Duffing and Financial FO systems. For each worked case, the corresponding simulation results are provided. Finally, some conclusions are given.

2. PRELIMINARIES

2.1. Fractional-order calculus

Fractional calculus is a generalization of integration and differentiation for integers to the non-integer-order fundamental operator ${}_aD_t^\alpha$, where a and t are the bounds of the operation. The continuous integro-differential operator is defined as

$$ {}_aD_t^\alpha = \begin{cases} \dfrac{d^\alpha}{dt^\alpha}, & \alpha > 0, \\ 1, \quad \alpha = 0, & \displaystyle\int_a^t (d\tau)^{-\alpha}, \quad \alpha < 0 \end{cases} \tag{1} $$

with $\alpha \in R$.

There are three common definitions of a FO derivative: the Grünwald-Letnikov (GL) definition, the Caputo definition and the Riemann-Liouville definition. These definitions are equivalent under some conditions. The interested reader can be referred to [1,4].

First, we define the GL definition in non-integer differentiation as

$$ {}_aD_t^\alpha f(t) = \lim_{h\to 0} f^{-\alpha} \sum_{j=0}^{\frac{t-a}{h}} (-1)^j \binom{\alpha}{j} f(t-jh), \tag{2} $$

where

$$ \binom{\alpha}{j} = \frac{\alpha(\alpha-1)\cdots(\alpha-j+1)}{j!} \tag{3} $$

is the relation between the Euler's Gamma function and the factorial.

For a numerical solution of FO differentiation we can use the relation derived from GL definition given by the following expression [1,4]

$$ {}_{k-Lm/h}D_t^\alpha f(t) \approx h^{-q} \sum_{j=0}^{k} (-1)^j \binom{\alpha}{j} f(t_k - j). \tag{4} $$

The general numerical solution of (4) is

$$ {}_aD_t^\alpha f(t) = f(y(t), t), \tag{5} $$

and can be expressed as

$$ y(t_k) = f(y(t_k), t_k) h^q - \sum_{j=v}^{k} c_j^{(q)} y(t_k - j), \tag{6} $$

where the calculation of the binomial coefficients are given by

$$ \begin{aligned} c_0^{(q)} &= 1, \\ c_j^{(q)} &= \left(1 - \frac{1+q}{j}\right) c_{j-1}^{(q)}. \end{aligned} \tag{7} $$

2.2. Summary on synchronization of complex systems

A complex network can be defined as an interconnected set of oscillators (two or more), where each oscillator is a fundamental unit, with its dynamic depending of the nature of the network.

Each oscillator represents a node of the network, The i-th oscillator N_i will be defined as follows:

$$ \begin{aligned} {}_aD_t^{\alpha_r} x_{i,r}(t) = f_{i,r}(x_{i,r}, t) + u_{i,r}; i &= 1, 2, \ldots, N, \\ r &= 1, 2, \ldots, n, \end{aligned} \tag{8} $$

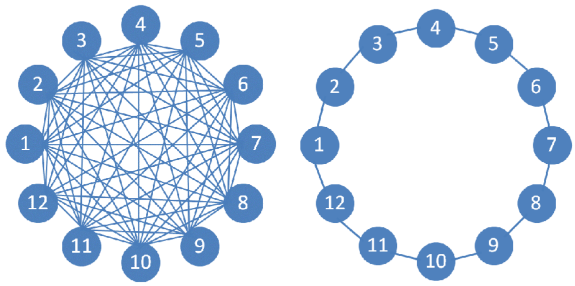
where $x_i = (\ x_{i,1} \quad x_{i,2} \quad \cdots \quad x_{i,n}\ )^T \in R^n$ are the state variables of the N_i node.

In order to synchronize the network, the signal $u_{i,r} \in R$ is added to only one state of each node. For example, if the network is synchronized through the k-th state of each node, the control law achieves $u_{i,r} = 0$ for $r \neq k$, and

$$ u_{i,r} = C \sum_{j=1}^{n} a_{i,j} \Gamma x_j, \quad i = 1, \ldots, n, \tag{9} $$

for $r = k$ [9, 10], where $C > 0$ represents the *coupling strength*; Γ is a constant matrix which links the state variables. Assume $\Gamma = \mathrm{diag}(\gamma_1, \gamma_2, \ldots, \gamma_n)$ a diagonal matrix with $\gamma_k = 1$ for a particular k and $\gamma_j = 0$ for $j \neq k$. Then the matrix Γ links the nodes through their k-th state variables.

The matrix $\mathbf{A} = (a_{i,j}) \in R^{n\times n}$ is named the *coupling matrix*. This matrix reflects the network topology and is defined as follows: If there is a connection between node i



(a) (b)

Fig. 1. Regular network topologies: (a) Globally coupled network, (b) Ring coupled network.

and j, then , otherwise . For $i = j$ the diagonal elements of $\mathbf{A}$ are defined as

$$\begin{aligned} a_{i,i} &= -\sum_{j=1, j\neq i}^{n} a_{i,j} \\ &= -\sum_{j=1, j\neq i}^{n} a_{j,i}, \quad i = 1, 2, \ldots, n. \end{aligned} \tag{10}$$

The dynamical network composed by (8)-(9) is said to achieve complete synchronization if

$$x_1 = x_2 = \cdots = x_n, \qquad \text{as } t \to \infty. \tag{11}$$

The complexity of the network depends on the topology or the oscillators used as nodes. The topology may follow a regular or irregular pattern. In this work, we consider regular and irregular networks with N identical oscillators with a FO chaotic dynamics.

In Fig. 1 the two types of regular networks that will be used in this paper are shown: global and ring connection. The irregular networks used in this work will be presented later in Section 4. In order to guarantee the synchronization of the network (8)-(9) we use the results from [9] and [10] to compute the appropiate coupling strength for all the cases presented. A brief summary of [9] and [10] is presented in appendix in order to make this document as self-contained as possible.

3. SYNCHRONIZATION OF REGULAR NETWORKS WITH N-COUPLED FO OSCILLATORS VIA COUPLING MATRIX

Regular coupled networks are one of the configurations more studied in synchronization of complex networks (globally coupled and ring coupled networks for example). Here, we present the synchronization of these networks using FO chaotic Lorenz system as nodes in the globally coupled case and the FO chaotic Volta system in the ring connection.

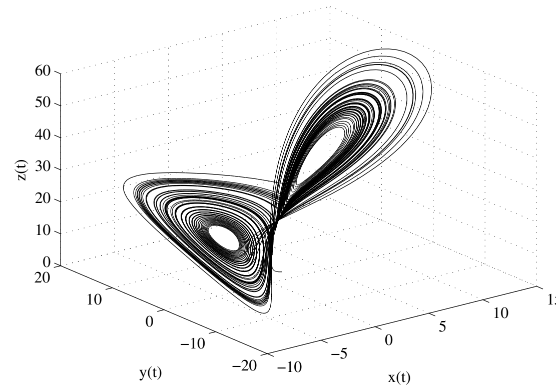


Fig. 2. Chaotic attractor of a FO Lorenz oscillator with $\sigma = 10$, $\rho = 29$, $\beta = 8/3$ and $q_1 = q_2 = q_3 = 0.995$.

3.1. Case 1: Globally coupled network with FO Lorenz oscillators

First, we synchronize complex networks of identical globally coupled FO Lorenz oscillators. This globally coupled network topology is shown in Fig. 1(a).

In 1963 Edward N. Lorenz found a chaotic attractor based on the mathematical equations that model the weather, formerly named after him. Later on, it was found chaotic behavior with different FOs in the derivatives of the system. We use FO Lorenz system in each oscillator. Thus, the i-th node of the network N_i is defined as follows:

$$N_i = \begin{cases} {}_0D_t^{q_1} x_i(t) = \sigma\left(y_i(t) - x_i(t)\right) + u_{i,1}, \\ {}_0D_t^{q_2} y_i(t) = x_i(t)\left(\rho - z_i(t)\right) - y_i(t), \\ {}_0D_t^{q_3} z_i(t) = x_i(t)\, y_i(t) - \beta z_i(t), \end{cases} \tag{12}$$

where $\sigma = 10$, $\rho = 29$, $\beta = 8/3$ and the FO when $q \in [0,1]$ is given by $q_1 = q_2 = q_3 = 0.995$. Under these assumptions the oscillator is chaotic [25]. The chaotic attractor is shown in Fig. 2.

According to (10), the coupling matrix for the configuration shown in Fig. 1(a) is given by

$$\mathbf{A}_{gc} = \begin{bmatrix} -11 & 1 & 1 & \cdots & 1 \\ 1 & -11 & 1 & \cdots & 1 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & -11 \end{bmatrix}. \tag{13}$$

Note from (11) that Γ is defined as $\Gamma = \mathrm{diag}(1,\ 0,\ 0)$ this means the synchronization is achieved by the first state of each oscillator. According to (9), the first law of control is given by the first row of the $\mathbf{A}_{gc}$ matrix

$$\begin{aligned} u_{1,1} = &C(-11x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \\ &+ x_7 + x_9 + x_{10} + x_{11} + x_{12}). \end{aligned} \tag{14}$$

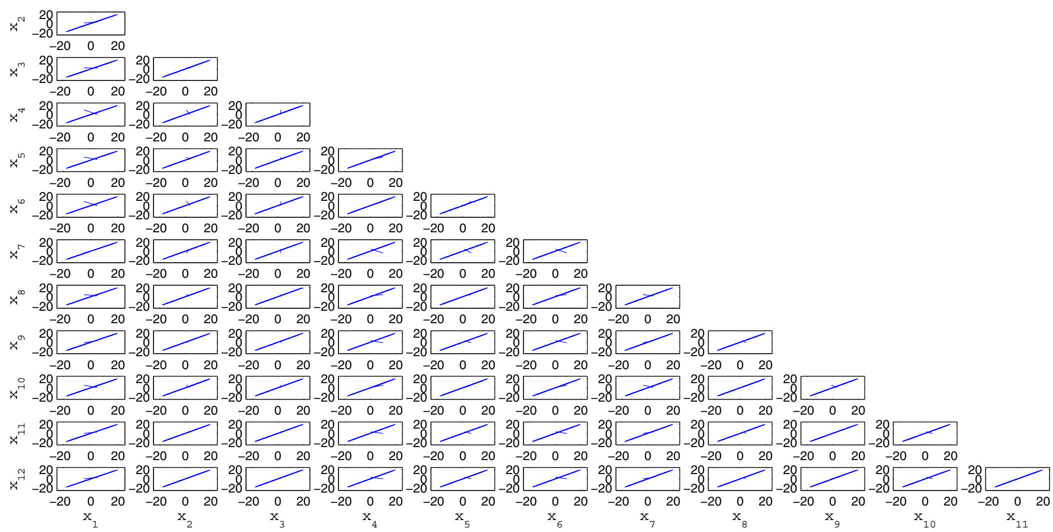


Fig. 3. Phase portrait of the first states (, where i=1,2,...,12) of the twelve FO Lorenz oscillators.

The next oscillator of the network is defined as follows:

$$N_2 = \begin{cases} {}_0D_t^{q_1} x_2(t) = \sigma\left(y_2(t) - x_2(t)\right) + u_{1,2}, \\ {}_0D_t^{q_2} y_2(t) = x_2(t)\left(\rho - z_2(t)\right) - y_2(t), \\ {}_0D_t^{q_3} z_2(t) = x_2(t)\, y_2(t) - \beta z_2(t), \end{cases} \quad (15)$$

and the second control law as

$$\begin{aligned} u_{2,1} = & C(x_1 - 11x_2 + x_3 + x_4 + x_5 + x_6 \\ & + x_7 + x_8 + x_9 + x_{10} + x_{11} + x_{12}), \end{aligned} \quad (16)$$

up to the last node,

$$N_{12} = \begin{cases} {}_0D_t^{q_1} x_{12}(t) = \sigma\left(y_{12}(t) - x_{12}(t)\right) + u_{1,12}, \\ {}_0D_t^{q_2} y_{12}(t) = x_{12}(t)\left(\rho - z_{12}(t)\right) - y_{12}(t), \\ {}_0D_t^{q_3} z_{12}(t) = x_{12}(t)\, y_{12}(t) - \beta z_{12}(t) \end{cases} \quad (17)$$

with control law $u_{12,1} = C(x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} + x_{11} - 11x_{12})$.

The numerical simulation presented was obtained by using a step time of size 0.005, considering the mentioned parameter set. The initial conditions used for this, and the three other cases presented in this document, are shown in Table 1.

In all cases, we apply Lemma 1 of Appendix to calculate C. Resulting in a value of $C = 20$ for this case. Under this coupling strength, each state synchronizes with the corresponding state of each system as we can see in phase portraits shown in Fig. 3. Note that each subfigure shows the appearance of a line of 45° reflecting the synchronization of the network.

3.2. Case 2: Ring coupled network with FO Volta oscillators

In this type of coupling, only the nearest neighbors are connected. In addition the last oscillator is connected to the first one. In the proposed case, a ring coupled network with twelve FO Volta oscillators was synchronized. This topology is illustrated in Fig. 1(b).

The systems used as nodes were discovered by Volta in 1984. The FO Volta oscillator is defined as follows:

$$N_i = \begin{cases} {}_0D_t^{q_1} x_i(t) = -x_i(t) - a y_i(t) - y_i(t)\, z_i(t), \\ {}_0D_t^{q_2} y_i(t) = -y_i(t) - b x_i(t) - x_i(t)\, z_i(t), \\ {}_0D_t^{q_3} z_i(t) = 1 + c z_i(t) + x_i(t)\, y_i(t). \end{cases} \quad (18)$$

The system (17) is chaotic under the parameters $a = 19$, $b = 11$, $c = 0.73$ and a FO equal to $q = 2.97$, where $q_1 = q_2 = q_3 = 0.99$ [25]. The chaotic attractor related to (17) is shown in the Fig. 4.

For this case, the oscillators of the dynamical network are arranged as follows:

$$N_i = \begin{cases} {}_0D_t^{q_1} x_i(t) = -x_i(t) - a y_i(t) - y_i(t)\, z_i(t), \\ {}_0D_t^{q_2} y_i(t) = -y_i(t) - b x_i(t) - x_i(t)\, z_i(t) + u_{i,2}, \\ {}_0D_t^{q_3} z_i(t) = 1 + c z_i(t) + x_i(t)\, y_i(t), \end{cases} \quad (19)$$

where $i = 1, 2, \ldots, 12$ represents the node number. The coupling matrix for the ring connection is given by

$$\mathbf{A}_{nc} = \begin{bmatrix} -k & 1 & 0 & \cdots & 1 \\ 1 & -k & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 1 & -k \end{bmatrix}, \quad (20)$$

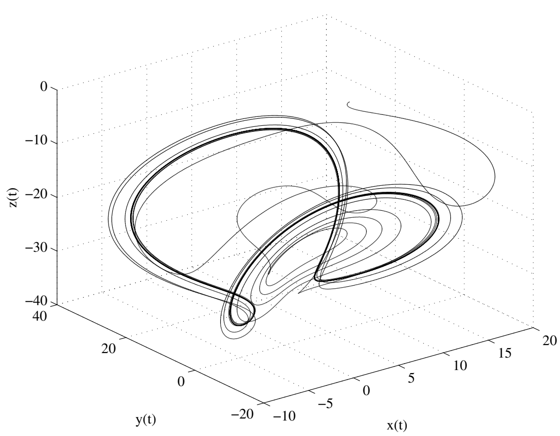


Fig. 4. Chaotic attractor of the FO Volta oscillator, with $a = 19$, $b = 11$, $c = 0.73$ and $q_1 = q_2 = q_3 = 0.99$.

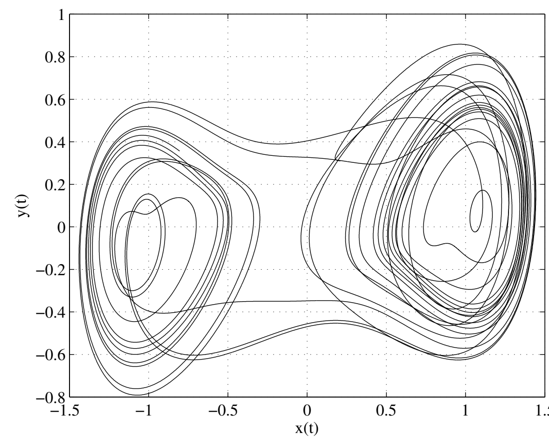


Fig. 5. FO Duffing attractor for $\alpha = 0.15$, $\delta = 0.3$, $\omega = 1$, and $q_1 = 0.9$, $q_2 = 1$.

where $k = 2$ since each oscillator is adjacent to the neighboring oscillators.

The Gamma matrix for this case is defined as $\Gamma = \text{diag}(0,\ 1,\ 0)$, this means that the synchronization is achieved by the second state of each oscillator. The synchronization is achieved by using the following coupling signals, where each $u_{i,2}$ is applied to each node (N_i).

$$\begin{aligned} u_{1,2} &= C(-2y_1 + y_2 + y_{12}), \\ u_{2,2} &= C(y_1 - 2y_2 + y_3), \\ &\vdots \\ u_{11,2} &= C(y_{10} - 2y_{11} + y_{12}), \\ u_{12,2} &= C(y_1 + y_{11} - 2y_{12}). \end{aligned} \tag{21}$$

A numerical simulation where synchronization is achieved using a coupling strength $C = 13$ was performed. The resulting phase portraits are shown in Fig. 6. In this particular case, the synchronization is achieved with a small coupling strength compared with the FO Lorenz oscillators case.

4. IRREGULAR NETWORKS

In addition, we propose to synchronize two irregular coupled networks with the FO Duffing and Financial oscillators. In irregular networks the patterns of the coupling matrix are not defined.

We consider $G = (V, E)$, a graph, where $N = |V|$, with $V = V(G) = (\ v_1 \quad v_2 \quad \cdots \quad v_N\ )$ representing the set of nodes and $M = |E|$ connections between oscillators, where $E = E(G) = (\ e_1 \quad e_2 \quad \cdots \quad e_M\ )$ represents set of connections.

In Fig. 7, the connections among the nodes of both networks considered are shown. There are two important matrices that can be used to synchronize these networks; they are the adjacency matrix and the degree matrix which are defined below.

1) Adjacency matrix $\mathbf{A}(G): N \times N$ matrix. The elements $a_{i,j}$ are defined as follows :

$$a_{i,j} = \begin{cases} 1, & \text{if } (i,j) \in E(G) \\ 0, & \text{otherwise,} \end{cases} \tag{22}$$

where $(i, j) \in E(G)$ means that node i is connected with node j.

2) Degree Matrix $\mathbf{D}(G): N \times N$ matrix. The elements $d_{i,j}$ are defined as

$$d_{i,j} = \begin{cases} d_i, & \text{if } i = j, \\ 0, & \text{otherwise,} \end{cases} \tag{23}$$

where d_i is a degree of node i, and given that node i is connected without a definite pattern, then d_i is the sum of elements of the row i of the Adjacency matrix $\mathbf{A}(G)$.

With the last two matrices we can form the *Laplacian matrix* as $\mathbf{L}(G) = \mathbf{D}(G) - \mathbf{A}(G)$. For N nodes $\mathbf{L}(G)$ is a $N \times N$ matrix with elements.

$$l_{i,j} = \begin{cases} -1, & \text{if } (i,j) \in \mathbf{E}(G) \\ d_{i,j}, & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases} \tag{24}$$

When the network is irregular, the properties of each coupling configuration are different for each case.

4.1. Case 3: Irregular network with FO-Duffing systems

The Duffing oscillator was introduced in 1918 by G. Duffing, this system has negative linear stiffness, damping and periodic excitation. The FO representation of this oscillator is given by

$$N_i = \begin{cases} {}_0D_t^{q_1} x_i(t) = y_i(t), \\ {}_0D_t^{q_2} y_i(t) = x_i(t) - \alpha y_i(t) x_i^3(t) + \delta \cos(\omega). \end{cases} \tag{25}$$

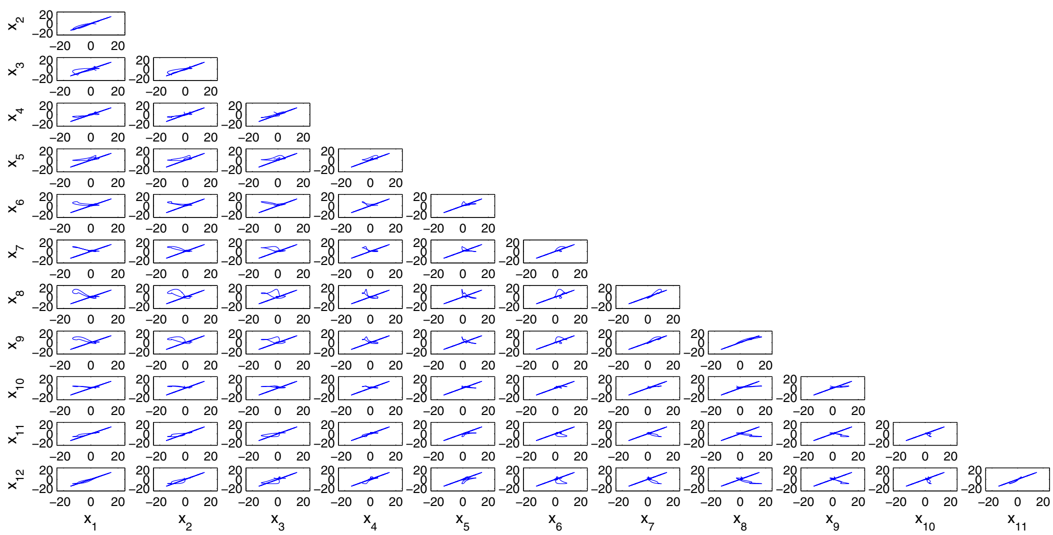


Fig. 6. Phase portrait of the second states (y_i, where $i = 1, 2, \ldots, 12$) of the ring coupled Volta oscillators.

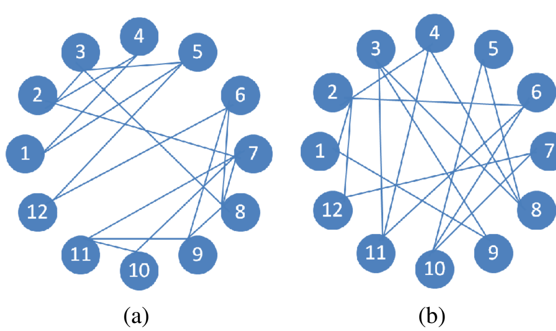


Fig. 7. Irregular coupled network used for (a) FO Duffing oscillator case, (b) FO Financial oscillator case.

Under the parameters $\alpha = 0.15$, $\delta = 0.3$, $\omega = 1$, and a FO equal to $q = 1.9$, where $q_1 = 0.9$ and $q_2 = 1$ the oscillator is chaotic [25]. The attractor related to this system is presented in Fig. 5.

We use the irregular structure shown in Fig. 7(a), with its nodes defined by FO Duffing oscillators as follows:

$$N_1 = \begin{cases} {}_0D_t^{q_1} x_1(t) = y_1(t) + u_{1,1}, \\ {}_0D_t^{q_2} y_1(t) = x_1(t) - \alpha y_1(t) x_1^3(t) + \delta \cos(\omega), \end{cases} \tag{26}$$

up to

$$N_{12} = \begin{cases} {}_0D_t^{q_1} x_{12}(t) = y_{12}(t) + u_{12,1}, \\ {}_0D_t^{q_2} y_{12}(t) = x_{12}(t) - \alpha y_{12}(t) x_{12}^3(t) + \delta \cos(\omega), \end{cases} \tag{27}$$

First, the Laplacian Matrix is computed by using (22)

$$\mathbf{L}(G) = \begin{bmatrix} -2 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -3 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -3 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & -4 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & -4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & -2 \end{bmatrix}. \tag{28}$$

Now, from (26), control laws for the Fig. 7.a case with FO Duffing oscillators as nodes are defined as follows:

$$u_{i,1} = C \sum_{j=1}^{n} l_{i,j} x_j, \quad i = 1, \ldots, 12, \tag{29}$$

where $l_{i,j}$ represents the i-th element of the j-th column of the laplacian matrix shown in equation (22). In Fig. 9 the phase portraits of the twelve FO Duffing systems are shown, we use a coupling strength $C = 15$. It can be seen that all the systems are synchronized with the coupling strength selected. For this case the Gamma matrix is $\Gamma = \text{diag}(1,\ 0)$, it means that the synchronization is achieved by the first states x_i.

4.2. Case 4: Irregular network with FO financial chaotic oscillator

The topology of the second and last irregular network presented is shown in Fig. 7(b). For this case we consider FO Financial oscillators as nodes. This oscillator was pro-

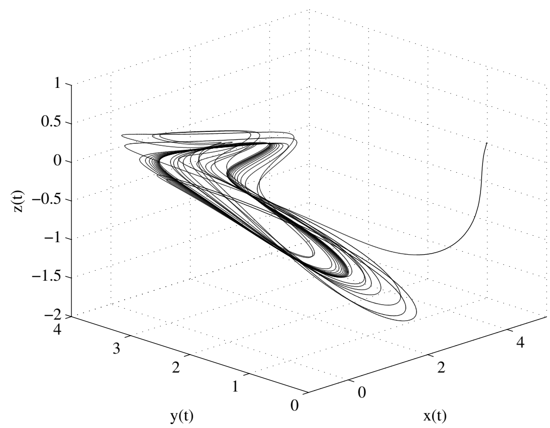


Fig. 8. Attractor of the FO Financial oscillator.

posed in 1985 in macro economics and is described as

$$N_i = \begin{cases} {}_0D_t^{q_1} x_i\,(t) = z_i\,(t) - x(y_i\,(t) - a), \\ {}_0D_t^{q_2} y_i\,(t) = 1 - by_i\,(t) - x_i^2\,(t), \\ {}_0D_t^{q_3} z_i\,(t) = x_i\,(t) - cz_i\,(t), \end{cases} \tag{30}$$

where, under the orders $q_1 = 1$, $q_2 = 0.95$, $q_3 = 0.9$, and the parameters $a = 1$, $b = 0.1$, $c = 1$ the oscillator (30) exhibit chaotic behaivor [25]. The FO chaotic attractor is shown in Fig. 8.

As in the previous example, we need to calculate the Degree and Adjacency matrices to calculate the Laplacian matrix for Fig. 7(b). case as shown below

$$\mathbf{L}(G) = \begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & -4 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -3 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -3 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & -3 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & -3 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -2 \end{bmatrix}. \tag{31}$$

With the $\mathbf{L}(G)$ matrix we obtain the control laws of this irregular network as in (26); for this case the $l_{i,j}$ represents the i-th element of the j-th column of the laplacian matrix shown in (31).

Now we apply (A.1) to the set of twelve FO oscillators in irregular connection

$$N_1 = \begin{cases} {}_0D_t^{q_1} x_1\,(t) = z_1\,(t) - x\,(y_1\,(t) - a) + u_{1,1}, \\ {}_0D_t^{q_2} y_1\,(t) = 1 - by_1\,(t) - x_1^2\,(t), \\ {}_0D_t^{q_3} z_1\,(t) = x_1\,(t) - cz_1\,(t), \end{cases} \tag{32}$$

up to

$$N_{12} = \begin{cases} {}_0D_t^{q_1} x_{12}\,(t) = z_{12}\,(t) - x\,(y_{12}\,(t) - a) + u_{12,1}, \\ {}_0D_t^{q_2} y_{12}\,(t) = 1 - by_{12}\,(t) - x_{12}^2\,(t), \\ {}_0D_t^{q_3} z_{12}\,(t) = x_{12}\,(t) - cz_{12}\,(t). \end{cases} \tag{33}$$

In Fig. 10 the phase portrait of the first states of the irregular coupled network is shown. The synchronization is achieved by the first states of each node x_i where $i = 1, 2, \ldots, 12$, as in the other systems, therefore, the $\mathbf{L}(G)$ matrix is the same. The coupling strength used for this case is $C = 10$.

5. CONCLUSIONS AND FUTURE WORKS

In this paper, synchronization of N-coupled FO chaotic oscillators was numerically shown by using complex network theory. Regular and irregular topologies were considered.

The coupling signal was used only in one state of the chaotic oscillators. It was shown that, by computing the coupling strength C, the synchronization was achieved for all cases presented.

We considered FO Lorenz and Volta chaotic oscillators for the regular coupled networks. Additionally, we explore the irregular complex networks and synchronize two networks, conformed by FO Duffing and Financial FO oscillators as nodes. Numerical simulations are provided to verify the effectiveness of this method. Simulations show the effectiveness of the proposed synchronization scheme. Phase portraits are shown for every case in order to confirm the synchronization of the networks. We have extended the results reported in [24] for irregular topologies. In a forthcoming work we will be concerned on a physical implementation using electronic devices.

APPENDIX A

We assume the following condition: Suppose that there are no isolated clusters in the network, then the coupling matrix $\mathbf{A}$, obtained as explained in Section 2.2, is a symmetric irreducible matrix, so one eigenvalue of $\mathbf{A}$ is zero and all the other eigenvalues are strictly negative.

Theorem 1 [9]**:** Consider the dynamical network given by (8) and (9). Let

$$0 = \lambda_1 > \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_N \tag{A.1}$$

be the eigenvalues of the coupling matrix $\mathbf{A}$. Suppose that there exist an $n \times n$ diagonal matrix $\mathbf{D} > 0$ and two constants $\overline{d} < 0$ and $\tau > 0$, such that

$$[D^q f(s(t)) + d\Gamma]^T \mathbf{D} + \mathbf{D}[D^q f(s(t)) + d\Gamma] \leq \tau \mathbf{I}_N \tag{A.2}$$

for all $d \leq \overline{d}$, where $\mathbf{I}_N \in \Re^{n\times n}$ is an unitary matrix. If moreover,

$$C\lambda_2 \leq \overline{d}, \tag{A.3}$$

then the synchronization error is exponentially stable, i.e., $x_1 = x_2 = \cdots = x_n$, as $t \to \infty$.

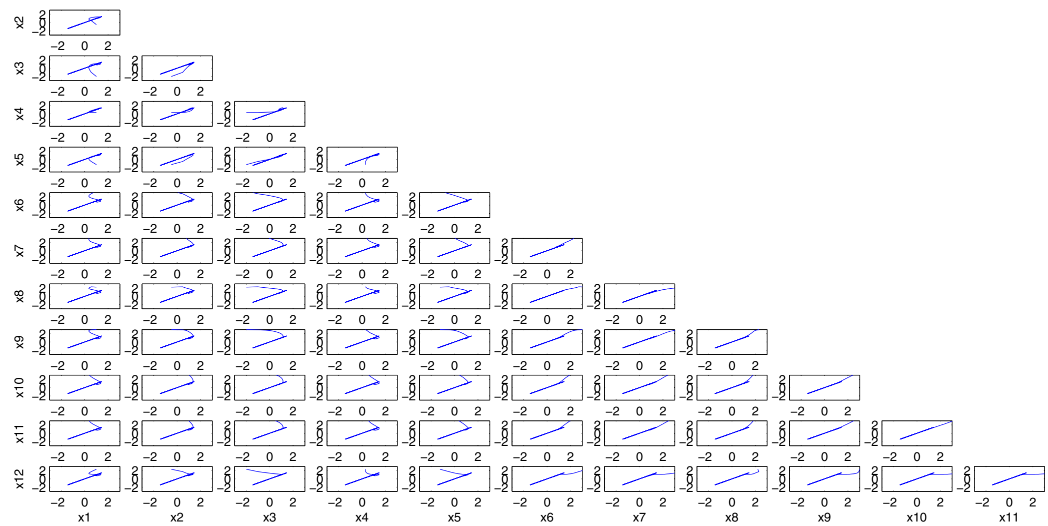


Fig. 9. Phase portrait of the first states (where i=1,2,...,12) of the synchronization with twelve FO Duffing oscillators.

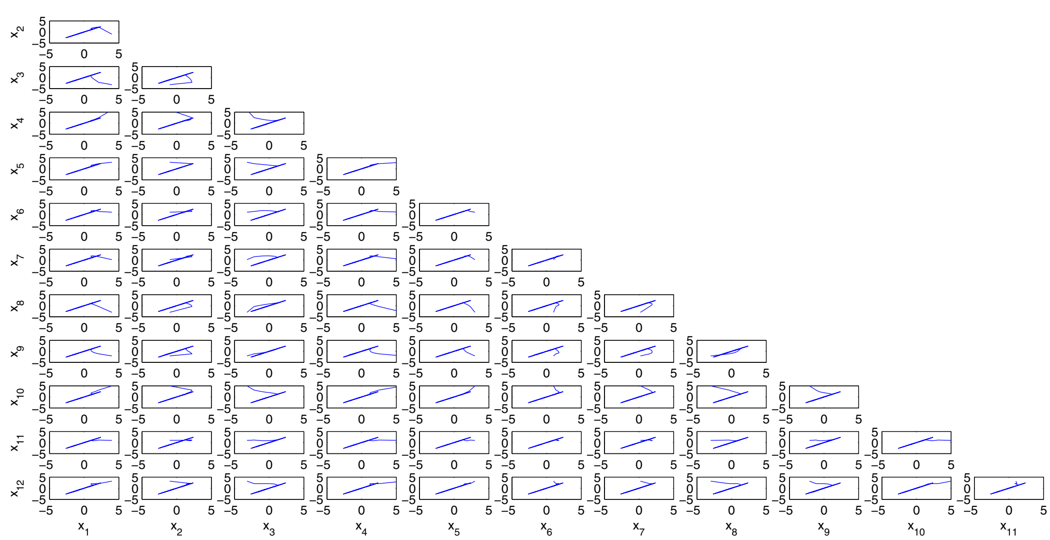


Fig. 10. Phase portrait of the first states (, where i=1,2,...,12) of the irregular coupled network with Financial oscillators.

The coupling strength C, that determines the stability of the synchronization state and that is present in the control law (9), is computed based on the following lemma:

Lemma 1 [10]**:** Consider the network given by (8) and (9). Let λ_2 be the largest nonzero eigenvalue of the coupling matrix **A** of the network. The synchronization state of network (1) defined by $x_1 = x_2 = \cdots = x_n$ is asymptotically stable, if the coupling strength meets the restriction

$$C \le -\frac{T}{\lambda_2}, \tag{A.4}$$

where $C > 0$ denotes the coupling strength and $T > 0$ a positive constant such that zero is an exponentially stable point of the n-dimensional system

$$\begin{aligned} {}_0D_t^{q_1} z_1 &= f_1(z_1, z_2, \ldots, z_n) - Tz_1, \\ {}_0D_t^{q_2} z_2 &= f_2(z_1, z_2, \ldots, z_n), \\ &\vdots \\ {}_0D_t^{q_n} z_n &= f_n(z_1, z_2, \ldots, z_n). \end{aligned} \tag{A.5}$$

Condition (A.4) means that the entire network will synchronize provided that 1) the topology of the network is such that λ_2 is negative enough, and 2) there exist a con-

Table 1. Initial conditions used for the different nodes N_i in the four cases presented.

Case	State	N_1	N_2	N_3	N_4	N_5	N_6	N_7	N_8	N_9	N_{10}	N_{11}	N_{12}
1) Globally coupled Lorenz FO Osc.	$x(0)$	-5.1	0.8	2.3	10	6.5	9	-3.7	4.5	-1	5.7	0	0.19
	$y(0)$	0.2	12	-5	10	9	1.2	-1	8	0.01	7.9	3	4
	$z(0)$	2.5	-1	-2.6	1	3	3.57	4	6	4	3.1	0	7.5
2) Ring coupled Volta FO Osc.	$x(0)$	0.2	8	4	-7	5	-4	3.4	6.1	4.5	2.1	7.2	10.1
	$y(0)$	-0.1	-4	0.1	5	3.2	-2.9	-2	7.7	8.1	-1.1	3	-6
	$z(0)$	0.1	-9	2	5.21	-3	0.12	4.3	-5.2	1.1	0.65	8	1.3
3) Irregular coupled Duffing FO Osc.	$x(0)$	0.5	-2	0.3	0.3	1.25	3.4	7.2	2.2	3	3.1	5	2.3
	$y(0)$	3.2	-3.5	4.2	0.5	0.01	-1.6	4.3	-5	2.2	0.12	1	-2
4) Irregular coupled Financial FO Osc.	$x(0)$	4	-1	-3.2	6	3	1	0.2	-3	-2.1	5	1	3
	$y(0)$	2	0.5	-3	7	1.5	2	-4	-1	-5.3	1.23	3	2.8
	$z(0)$	1	3	1	-5	3	0.5	-5.1	6	3.55	-3	-0.5	-2

stant T so that the self-feedback term $-Tz_1$ can stabilize an isolated oscillator.

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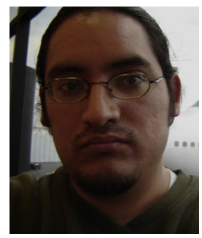

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