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Dynamic Analysis of a 5D Fractional-order Hyperchaotic System

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Abstract: In this paper, the fractional-order 5D hyperchaotic system is proposed based on the hyperchaotic Lorenz system. Fractional-order chaotic systems are often three- or four-dimensional. There are few results about high dimension fractional-order systems. For this 5D hyperchaotic system, the stability of equilibrium points is analyzed by means of the stability theory of fractional systems. Then the fractional bifurcation is investigated. It is found that the system admits bifurcations with varying fractional-order and parameters, respectively. Under different bifurcation parameters, some conditions ensuring the bifurcations are presented. Finally, numerical simulations are presented to confirm the given analytical results.

Keywords: Bifurcation, 5D fractional system, hyperchaotic, stability.

1. INTRODUCTION

In recent years, there are too many results about two-, three-, and four-dimensional chaotic systems. So it is important and imperative for us to study the dynamic behaviors of higher-dimensional chaotic systems. Note the Lorenz model as the first chaotic model has great importance in nonlinear sciences and some variants have been constructed. A 5D system was proposed by Yang and Chen [1]. It was constructed from a 4D hyperchaotic system [2] by adding a nonlinear controller to the first equation. As we know, some real practical problems can be described by three- and four-dimensional autonomous systems. Now some five-dimensional (5D) systems have been constructed from three- and four-dimensional autonomous systems. Since the 5D autonomous systems have much higher unpredictability than 3D and 4D systems, they may have a good application value in the field of information technology such as secure communication and encryption.

Now fractional calculus has been attracting increasing interests from researchers and introduced to integer-order models. The idea of fractional calculus has been known since the development of the regular calculus, having almost the same history. Due to the difficulty, fractional calculus has not gained much attention from many scholars. Recent days, the applications of fractional order dynamical systems play a more vital role in real life problems. Such as the affine cipher using date of birth [3], fuzzy fractional integral sliding mode control [4], a novel fractional order King Cobra chaotic system [5], digital cryptography [6], authenticated encryption scheme [7]. Thus, investigation of fractional-order dynamical systems is not only meaningful in the theory, but also significant in practice. Now we will carry out the study of five-dimensional fractional-order system. Generally, the results about higher-dimensional ones are not too many. With higher-dimensions, it will has much complex dynamics and potential applications in practice. Note in higher systems, hyperchaos could happen, which is characterized with more than one positive Lyapunov exponents. Historically, hyperchaos was firstly reported by Rössler in 1979 [8]. The hyperchaotic systems have more complex dynamical behaviors than chaotic systems [9–15]. Such complex behaviors could be helpful in secure communications, signal processing, image processing, etc. In recent years, applications of hyperchaos have become a central topic in research. But the dynamics of hyperchaotic systems have not completely studied by some scholars until now. So it is necessary to explore the higher-dimensional hyperchaotic systems.

Now some interesting hyperchaotic systems were presented in the past two decades, and their dynamics have been investigated extensively. These complex dynamics could be explored via bifurcation analysis of systems with varying parameters. Bifurcation is one of most active research topics in the field of nonlinear science [16–18]. Up to now, the research of fractional-order hyperchaotic systems has not been studied completely. This requires us to construct more higher-dimensional fractional-order hyperchaotic systems and to study its more features. In this

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paper, we mainly discussed the dynamic behaviors of the 5D fractional-order system. It is found that the system admits complex dynamical behaviors with varying parameters, such as bifurcation, chaos. Now its basic dynamical behaviors are analyzed, especially its bifurcation is investigated in detail via theoretical as well as numerical analysis. The bifurcated limit cycle will be created under some parameter conditions. It is worth noting that the limit cycle may be not a solution of the fractional system, but it does attract nearby solutions, which is different from the integer-order system.

The paper is organized as follows: In Section 2, some basic definitions of fractional calculus and some useful stability theorems of fractional-order systems are briefly recalled. The 5D fractional-order system is proposed based on the hyperchaotic Lorenz system. In Section 3, the local stability and chaotic dynamics of the fractional-order system are studied. In Section 4, the bifurcations versus varying parameters are given. Numerical simulation are also performed to verify the theoretical results. Some conclusions are drawn in Section 5.

2. PRELIMINARIES

There are several definitions of fractional derivatives [19–22]. In this paper, we briefly introduce the Riemann-Liouville derivative and Caputo derivative.

Definition 1: Caputo fractional derivative with order α for function $x(t)$ is defined as

$$^{C}D^{\alpha}_{t_0}x(t) = \frac{1}{\Gamma(m-\alpha)}\int_{t_0}^{t}(t-\tau)^{m-\alpha-1}x^{(m)}(\tau)d\tau,$$

where $m-1<\alpha<m$, $m\in Z_+$, and $t=t_0$ is the initial time, $\Gamma(\cdot)$ is Gamma function.

Definition 2: Riemann-Liouville's fractional derivatives with order α for function x: $R^+\to R$ is defined as

$$^{RL}D^{\alpha}_{t_0}x(t) = \frac{1}{\Gamma(m-\alpha)}\frac{d^m}{dt^m}\int_{t_0}^{t}(t-\tau)^{m-\alpha-1}x(\tau)d\tau,$$

where $m-1\leq\alpha<m$, $m\in Z_+$, and $t=t_0$ is the initial time, $\Gamma(\cdot)$ is Gamma function.

In this paper, we mainly use the definition of Caputo fractional derivative.

Theorem 1: [23] The following autonomous system:

$$D^{\alpha}_{t}x = Jx, x(0)=x_0, \tag{1}$$

with $0<\alpha\leq 1$, $x\in R^n$, and $J\in R^{n\times n}$, is asymptotically stable if and only if $|\arg(\lambda)|>\alpha\pi/2$ is satisfied for all eigenvalues of matrix J. Also, this system is stable if and only if $|\arg(\lambda)|\geq\alpha\pi/2$ is satisfied for all eigenvalues of matrix J with those critical eigenvalues satisfying $|\arg(\lambda)|=\alpha\pi/2$ having geometric multiplicity of one. The geometric multiplicity of an eigenvalue λ of the matrix J is the dimension of the subspace of vectors v for which $Jv=\lambda v$.

A new 5D autonomous hyperchaotic system with three positive Lyapunov exponents was presented in Ref. Ref. [1]. The algebraical form of the hyperchaotic system is very similar to the 5D controlled Lorenz system [24]. But they are different, in fact, it has one or three unstable equilibria, very different from all the other 5D Lorenz-like systems. In Ref. [1], the 5D hyperchaotic system based on Lorenz system is given by

$$\begin{aligned} dx/dt &= a(y-x)+u,\\ dy/dt &= cx-xz+w,\\ dz/dt &= -bz+xy,\\ du/dt &= -hu-xz,\\ dw/dt &= -k_1x-k_2y, \end{aligned} \tag{2}$$

where $abh\neq 0$, a, b and c are the constant parameters, h, $k_1$and k_2 are three control parameters.

When $abhk_1(k_1+k_2)\leq 0$ and $k_1^2+k_2^2\neq 0$, or $abhk_1^2\leq 0$ and $k_2\neq 0$, the origin $O(0,0,0,0,0)$ is the unique equilibrium.

When $abhk_1(k_1+k_2)>0$ and $k_2\neq 0$, system (2) has three equlibria:

$$\begin{aligned} &O(0,0,0,0,0),\\ &E_{\pm}(\pm x_0, \mp\frac{k_1}{k_2}x_0, -hm, \pm m x_0, \pm(hmx_0+cx_0)), \end{aligned}$$

where $x_0=\sqrt{abh(1+\frac{k_2}{k_1})}, m=a(\frac{k_1}{k_2}+1)$.

Using the Matlab software, it is easy to verify that the system (2) has one unique equilibrium $O(0,0,0,0,0)$ when $(a,b,c,h,k_1,k_2)=(10,8/3,28,-2,0,12.2)$, the hyperchaotic attractor is shown in Fig. 1.(a). The five Lyapunov exponents are

$$\begin{aligned} &\lambda_{LE_1}=0.4930,\ \ \lambda_{LE_2}=0.3665,\ \ \lambda_{LE_3}=0.0692,\\ &\lambda_{LE_4}=0.0000,\ \ \lambda_{LE_5}=-11.5955, \end{aligned}$$

when$(a,b,c,h,k_1,k_2)=(10,8/3,28,-2,-0.09,8)$, system (2) has three equilibria and the hyperchaotic attractor is shown in Fig. 1.(b). The five Lyapunov exponents are

$$\begin{aligned} &\lambda_{LE_1}=0.5555,\ \ \lambda_{LE_2}=0.3886,\ \ \lambda_{LE_3}=0.0452,\\ &\lambda_{LE_4}=0.0000,\ \ \lambda_{LE_5}=-11.6555. \end{aligned}$$

Now introduce the corresponding 5D fractional-order system

$$\begin{aligned} D^{\alpha}x &= a(y-x)+u,\\ D^{\alpha}y &= cx-xz+w,\\ D^{\alpha}z &= -bz+xy,\\ D^{\alpha}u &= -hu-xz, \end{aligned} \tag{3}$$

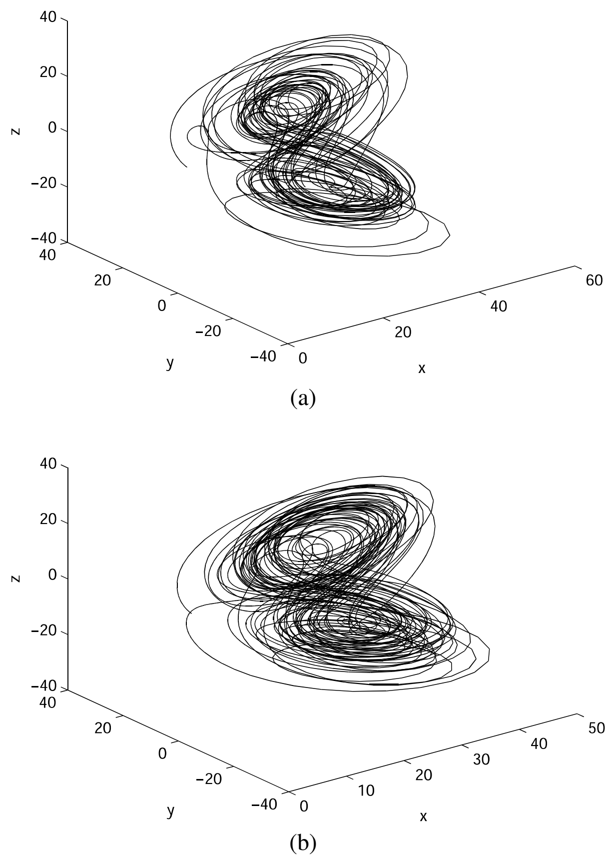


Fig. 1. Hyperchaotic attractor of system (2).

$$D^{\alpha}w = -k_1x - k_2y,$$

where $abh \neq 0$, a, b and c are the constant parameters, h, k_1 and k_2 are three control parameters. The fractional-order α is supposed to lie in $(0,1)$ in this paper.

3. LOCAL STABILITY AND CHAOS IN THE NEW FRACTIONAL-ORDER SYSTEM

When $abhk_1(k_1+k_2) \leq 0$ and $k_1^2+k_2^2 \neq 0$, or $abhk_1^2 \leq 0$ and $k_2 \neq 0$, it is easy to obtain that system (3) has only one equilibrium point $O(0,0,0,0,0)$. If the system has three equilibria, we can discuss the results in the similar way. So the Jacobian matrix J at the equilibrium point $O(0,0,0,0,0)$ is

$$J = \begin{pmatrix} -a & a & 0 & 1 & 0 \\ c & 0 & 0 & 0 & 1 \\ 0 & 0 & -b & 0 & 0 \\ 0 & 0 & 0 & -h & 0 \\ -k_1 & -k_2 & 0 & 0 & 0 \end{pmatrix} \tag{4}$$

and its characteristic equation

$$f(\lambda) = (\lambda+b)(\lambda+h)[\lambda^3 + a\lambda^2 + (k_2 - ac)\lambda + a(k_1+k_2)] = 0. \tag{5}$$

By Theorem 1, it is easy to obtain that

Theorem 2 [23]**:** The equilibrium $O(0,0,0,0,0)$ of fractional-order system (3) is locally asymptotically stable if and only if

$$\min_{1\leq i\leq 5} \mid arg(\lambda_i) \mid > \alpha\pi/2. \tag{6}$$

Remark 1: According to Theorem 2, the equilibrium $O(0,0,0,0,0)$ of fractional-order system (3) is unstable when the following condition holds

$$\min_{1\leq i\leq 5} \mid arg(\lambda_i) \mid \leq \alpha\pi/2. \tag{7}$$

Remark 2: Since the Hartman-Grobsman theorem is now known to be only applicable to the integer order systems, that is to say, Hartman-Grobsman theorem is not still showed to hold for fractional order systems. So the linear approximation is used to investigate the local dynamical stability of nonlinear fractional systems.

According to (5), it is clear that the minimun of $\mid arg(\lambda_i) \mid, (i = 1,2,3,4,5)$ depends on the roots of the following cubic equation, since the parameter b, h is positive.

$$P(\lambda) = \lambda^3 + a\lambda^2 + (k_2 - ac)\lambda + a(k_1+k_2) = 0. \tag{8}$$

Thus, the local stability of the equilibrium $O(0,0,0,0,0)$ is absolutely decided by (8) in terms of Theorem 2 and Remark 1.

Define the discriminant $D(P)$ of Eq.(8) by

$$D(P) = 18a_1a_2a_3 + (a_1a_2)^2 - 4a_3a_1^3 - 4a_2^3 - 27a_3^2 \tag{9}$$

in which $a_1 = a, a_2 = k_2 - ac, a_3 = a(k_1+k_2)$.

According to [25], we have the following fractional-order Routh-Hurwitz conditions:

(a) If $D(P) > 0$, then the necessary and sufficient conditions for the equilibrium point to be locally asymptotically stable, is $a_1 > 0, a_3 > 0, a_1a_2 > a_3$.

(b) If $D(P) < 0, a_1 \geq 0, a_2 \geq 0, a_3 > 0$, and $\alpha < 2/3$,then the equilibrium point is locally asymptotically stable , while the equilibrium point is unstable if $D(P) < 0$, $a_1 < 0, a_2 < 0$, and $\alpha > 2/3$.

(c) If $D(P) < 0$, $a_1 > 0, a_2 > 0, a_1a_2 = a_3$ then the equilibrium point is locally asymptotically stable for all $0 < \alpha < 1$.

It is easy to obtain that $D(P) > 0, a_1 > 0, a_3 > 0, a_1a_2 < a_3$, when $(a,b,c,h,k_1,k_2) = (10,8/3,28,-2,0,12.2)$. Then the unique equilibrium $O(0,0,0,0,0)$ is unstable according to the result (a).

When $\alpha = 0.9$, $(a,b,c,h,k_1,k_2) = (10,8/3,28,-2,0,$ $12.2)$, and the initial condition $(x_0,y_0,z_0,u_0,w_0) = (2.0,$ $3.5,4.0,5.0,7.5)$, the hyperchaotic attractor of fractional-order system (3) is shown in Fig. 2. Furthermore, the Lyapunov exponent spectrum is shown in Fig. 3. If we choose fractional-order $\alpha = 0.9$, k_2 as the bifurcation parameter with $a = 10, b = 8/3, c = 28, h = -2, k_1 = 0$, the bifurcation diagram is shown in Fig. 4.

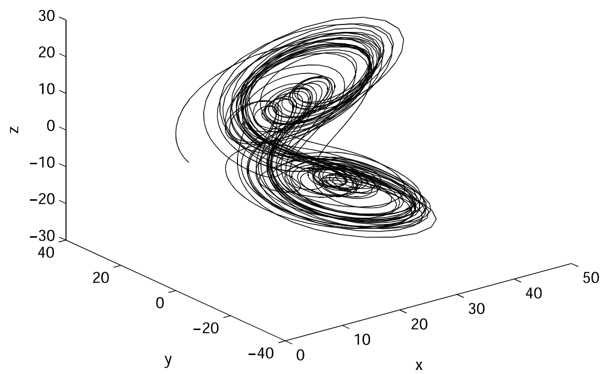
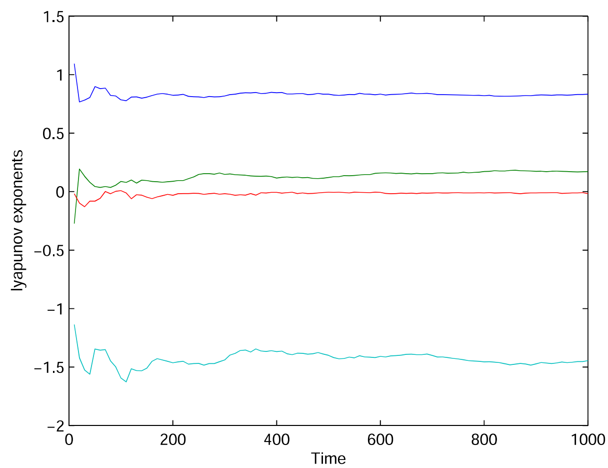
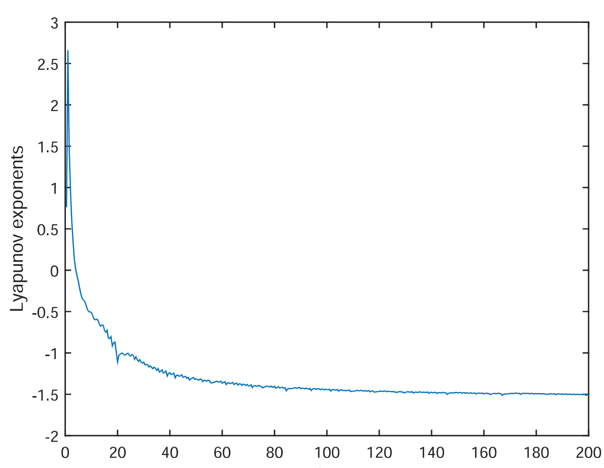


Fig. 2. Hyperchaotic attractor of system (3) virus $\alpha = 0.9$.



(a) The four Lyapunov exponents of system (3).



(b) The minimum Lyapunov exponent of system (3).

Fig. 3. The Lyapunov exponent spectrum of system (3).

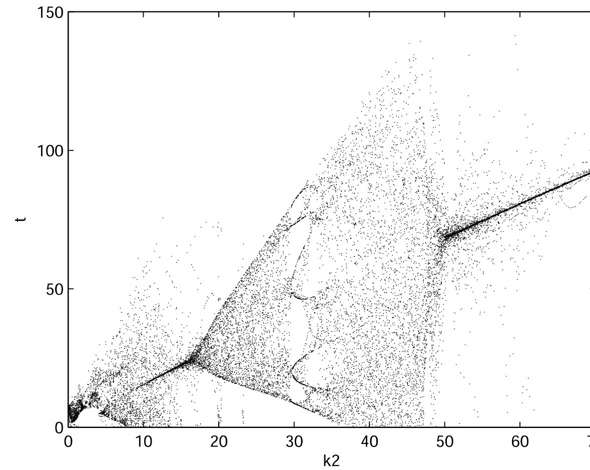


Fig. 4. Bifurcation diagram of system (3) virus k_2, $k_2 \in (0, 70)$.

4. BIFURCATION IN THE 5D FRACTIONAL-ORDER SYSTEM

In this section, bifurcation of system (3) will be further investigated based on stability theory of fractional-order system, by choosing a proper bifurcation parameter. Here the bifurcation for fractional order systems refers to the change of stability of equilibrium points, just similar to the Hopf case in integer order systems, however, since it does not have a normal form, we only explore the dynamical behavior when the stability of equilibrium point changes.

4.1. Bifurcation analysis versus the fractional-order α

According to Theorem 1, it is known that the fractional-order α has an effect on the stability of fractional system. Hence similar to Hopf bifurcation, the fractional-order α can be also chosen as the bifurcation parameter in fractional-order system [26]. In this section, we will study bifurcation in system (3) by choosing the fractional-order α as the bifurcation parameter.

Now, define a function with respect to α

$$m(\alpha) = \alpha\pi/2 - \min_{1 \le i \le 5} |\, arg(\lambda_i)\,| \,. \tag{10}$$

So if $m(\alpha) < 0$, then the equilibrium point is locally asymptotically stable; if $m(\alpha) > 0$, then the equilibrium point is unstable. Next, we will use the function $m_i(\alpha)$ to investigate bifurcation in the 5D hyperchaotic system versus the fractional-order α.

Theorem 3: When bifurcation parameter α passes through the critical value $\alpha^* \in (0, 1)$, fractional-order system (3) undergoes a bifurcation at the equilibrium point $O(0, 0, 0, 0, 0)$ if the following conditions hold

(a) the corresponding characteristic equation (5) of system (3) has a pair of complex conjugate roots $\lambda_{1,2} = \theta \pm iw$(wh-ere $\theta > 0$), and three negative real roots λ_3, λ_4 and λ_5.

(b) $m(\alpha^*) = \alpha^*\pi/2 - \min_{1\le i\le 5} \mid arg(\lambda_i) \mid = 0$,

(c) $\frac{dm(\alpha)}{d\alpha} \mid_{\alpha=\alpha^*} \neq 0$.

Proof: In order to meet the condition (a), we must have some restrictive conditions. It is easy to find the remaining three roots could be obtained by solving Eq.(8).

Now some conditions will be derived, under which Eq.(8) has a pair of complex conjugate roots $\lambda_{1,2} = \theta \pm iw$(where $\theta > 0$), and one negative real root λ_3. Using the results of [27], if $D(P) < 0$, then Eq.(8) has a pair of complex conjugate roots $\lambda_{1,2}$ and one real root λ_3. Note that$\lambda_1\lambda_2\lambda_3 = -a(k_1+k_2)$. Hence, if $\lambda_3 < 0$, we have $a(k_1+k_2) > 0$ and $\lambda_{1,2}$ are a pair of complex conjugate roots. In addition, it is not difficult to obtain that Eq.(8) has a pair of purely imaginary conjugate roots if and only if $k_1 = -ac$. Moreover, according to Routh-Hurwitz theorem, the roots of Eq.(8) have negative real parts if and only if $a > 0, a(k_1+k_2) > 0, a(k_2-ac) - a(k_1+k_2) > 0$. Therefore, under conditions given below, condition (a) will be guaranteed.

$$
\begin{aligned}
&a(k_1+k_2) > 0, D(P) < 0,\\
&k_1 + ac \neq 0,\\
&a(k_2-ac) - a(k_1+k_2) \le 0.
\end{aligned}
$$

Therefore, according to condition (a), we have $\min_{1\le i\le 5} \mid arg(\lambda_i) \mid = \arctan \mid \frac{w}{\theta} \mid \in (0, \pi/2)$. Then $m(\alpha^*) = \alpha^*\pi/2 - \arctan \mid \frac{w}{\theta} \mid = 0$, so $\alpha^* = \frac{2}{\pi}\arctan \mid \frac{w}{\theta} \mid$. Clearly $\alpha^* \in (0,1)$.

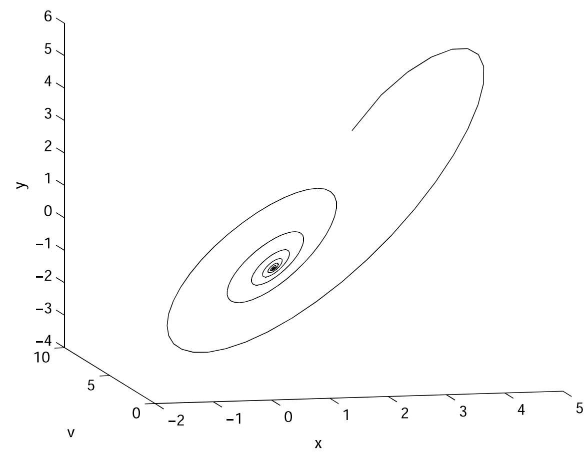
Finally, condition (c) ensures that the sign of $m(\alpha)$ can change when the bifurcation parameter α passes through the critical value α^*, i.e., the equilibrium point $O(0,0,0,0,0)$ is asymptotically stable for $\alpha \in (0,\alpha^*)$, and is unstable when $\alpha \in (\alpha^*, 1)$. Therefore, one can assert that bifurcation in system (3) occurs at $\alpha = \alpha^*$. □

Remark 3: Also note that condition (a) can ensure that the existence of α^*satisfying condition (b). From the proof, one can obtain the critical value of bifurcation parameter $\alpha^* = \frac{2}{\pi}\arctan \mid \frac{w}{\theta} \mid$.

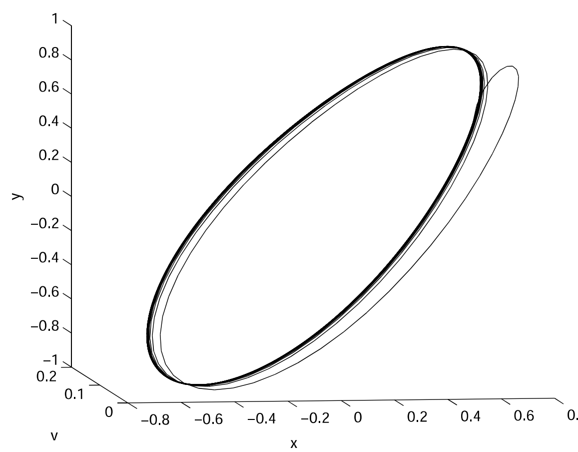
Remark 4: It is worth noting that the limit set of a trajectory of fractional-order system could be not a solution of this system [28], which is different from the integer-order system. In [29], it was proved the nonexistence of periodic solutions in time-invariant fractional-order system. In [30], an example was presented where the solutions of the system are not periodic, but they converge to periodic signals. In this paper, we focus on the final state of the trajectory. Hence, the limit cycle may not be a solution of the fractional system, but it does attract nearby solutions.

4.2. Bifurcation analysis versus the parameter k_2

In this section, the fractional-order α is fixed and the parameter k_2 is considered as a control parameter. A similar method is adopted to analyze the bifurcation in system (3).



(a) Phase portrait with $\alpha = 0.85$



(b) Phase portrait with $\alpha = 0.92$

Fig. 5. Phase portrait.

Define a function of k_2

$$
m(k_2) = \alpha\pi/2 - \min_{1\le i\le 5} \mid arg(\lambda_i(k_2)) \mid . \tag{11}
$$

Theorem 4: When bifurcation parameter k_2 passes through the critical value k_2^*, fractional-order system (3) undergoes a bifurcation at the equilibrium point $O(0,0,0,0,0)$ if the following conditions hold

(a) the corresponding characteristic equation (5) of system (3) has a pair of complex conjugate roots $\lambda_{1,2} = \theta(k_2) \pm iw(k_2)$(where $\theta(k_2) > 0$), and three negative real roots λ_3, λ_4 and λ_5.

(b) $m(k_2^*) = \alpha^*\pi/2 - \min_{1\le i\le 5} \mid arg(\lambda_i(k_2^*)) \mid = 0$,

(c) $\frac{dm(k_2)}{dk_2} \mid_{k_2=k_2^*} \neq 0$.

Proof: It can be proved in the way similar to that of Theorem 4. So it is omitted here. □

4.3. Numerical simulations

Simulation results are presented in this section, which are carried out by virtue of the Adams-Bashforth-Moulton scheme in [31].

Case 1: In this case, choose fractional-order α as the bifurcation parameter with $a=10, b=8/3, c=28, h=4, k_1=-199, k_2=350$.

When $abhk_1(k_1+k_2)\le 0$ and $k_1^2+k_2^2\ne 0$, then the origin $O(0,0,0,0,0)$ is the unique equilibrium.

It is easy to obtain $D(P)=-49458700<0, k_1+k_2=151>0, k_1+ac>0$. Then using the Matlab software, one can calculate the five roots of the characteristic equation (8) of system (3), which are $\lambda_{1,2}=1.6429\pm 10.5336i, \lambda_3=-13.2858, \lambda_4=-2.6667, \lambda_5=-4.0000$, when $a=10, b=8/3, c=28, h=4, k_1=-199, k_2=350$. By condition (b) in Theorem 3, one can get the critical value of bifurcation parameter

$$\alpha^* = \frac{2}{\pi}\arctan\left|\frac{w}{\theta}\right| = \frac{2}{\pi}\arctan\left|\frac{10.5336}{1.6429}\right| = 0.9015.$$

Finally,we have $\frac{dm(\alpha)}{d\alpha}\mid_{\alpha=\alpha^*} = \pi/2\ne 0$, so condition (c) hold.

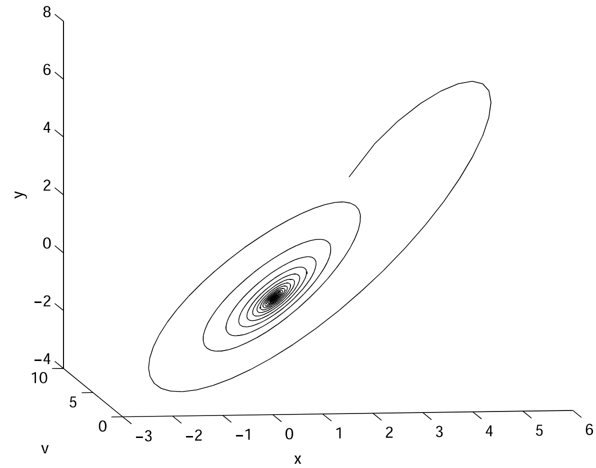
Hence when $0<\alpha<0.9015$, the equilibrium $O(0,0,0,0,0)$ of system (3) is locally asymptotically stable as shown in Fig. 5.(a). When $\alpha=0.9015$, system (3) undergoes a Hopf bifurcation as mentioned above, the fixed point becomes unstable, and the limit cycle appears for $\alpha\in(0.9015,1)$. When $\alpha=0.92$, a limit cycle which attracts nearby solutions appears as shown in Fig. 5.(b).

Case 2: In this case, choose fractional-order $\alpha=0.85$, k_2 as the bifurcation parameter with $a=10, b=8/3, c=28, h=4, k_1=-199$.

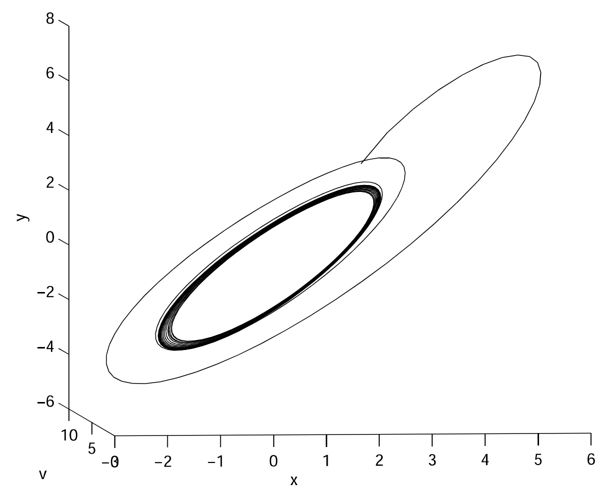
Using the proposed conditions in Theorem 4, the critical bifurcation value is localized at $k_2^*=295.2611$. It is easy to verify that condition (6) holds, when $k_2=295$. Hence, when $k_2=310$, the equilibrium $O(0,0,0,0,0)$ of system (3) is locally asymptotically stable as shown in Fig. 6.(a). When $k_2^*=295.2611$, system (3) undergoes a bifurcation as mentioned above, the fixed point becomes unstable. When $k_2=285$, a limit cycle which attracts nearby solutions appears as shown in Fig. 6.(b).

5. CONCLUSIONS

In this paper, we have proposed the corresponding 5D fractional-order hyperchaotic system based on the hyperchaotic Lorenz system and its basic properties such as stability and bifurcation are investigated. It is found that it admits complex and interesting dynamic system behaviors, such as chaos (hyperchaos) and bifurcation. Some conditions ensuring stability and bifurcation are derived, respectively. Numerical simulations are also carried out to verify the theoretical analysis.



(a) Phase portrait with $k_2=310$



(b) Phase portrait with $k_2=285$

Fig. 6. Phase portrait.

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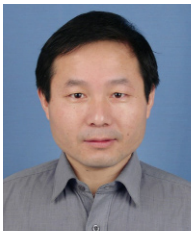

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