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Lyapunov Stability Analysis of Second-Order Sliding-Mode Control and Its Application to Chattering Reduction Design

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Abstract: This paper proposes a new sliding mode control design with reduced control chattering. The proposed new design inherits the design concept from dynamic sliding mode control, in which the first-order time derivative of the control input is treated as the control variable for a chattering control design. Previous dynamic sliding mode designs require an extra uncertainty observer or uncertainty estimator to construct the sliding surface. This paper is able to waive such observer or estimator.

Keywords: Chattering reduction, dynamic sliding mode control, second-order sliding mode.

1. INTRODUCTION

Sliding mode control (SMC) is robust with respect to system uncertainties through the use of switching control [1]. However, chattering of the sliding mode control signal has become the major obstacle to its applications in the real world. The cause of chattering may be due to delayed switching in digital implementation [2], unmodelled dynamics [3, 4], or measurement noise [5]. In practice, the control chattering is undesirable since it can damage the actuator and the system. To resolve this problem, a continuous constant boundary layer [6, 7], or a state-dependent boundary layer [8] has been proposed to replace the switching function so that the control becomes a continuous function of the feedback state. The control signal resulting from the boundary layer design will have no chattering in a noise-free environment. However, even with the boundary layer, the control signal still exhibits chattering when stochastic measurement noise is introduced into the sliding mode control law [5].

The other approach for chattering reduction in sliding mode control is the dynamic sliding mode control design [9, 10]. The idea of dynamic sliding mode control design is to perform a dynamic extension of the control input as shown in Figure 1. In Figure 1, an integrator or a low-pass filter is placed in front of the control system. The variable w is treated as the control variable for the extended system (the system plus the first-order low-pass filter or the integrator). A switching sliding mode control is then designed using w for the extended system; the resultant w thus has high-frequency chattering. Fortunately, with nowadays digital implementation, the control variable w is inside the computer, and its chattering will not do any harm to the hardware system. The real control u will be chattering free since the first-order low-pass filter in front of u will filter out the high frequency oscillations in w.

However, the design of the sliding surface for dynamic sliding mode control is more difficult than conventional sliding mode control. In the literature [11], a one-dimensional observer is proposed to estimate the sliding variable, but the closed-loop stability is guar- anteed only if a differential inequality with bounded coefficients is satisfied. In [10], an LTR observer is successfully proposed to estimate the sliding surface without the assumption in [11]. It is shown [10] that the dynamic sliding mode control can substantially reduce control chattering even in a noisy environment. Another approach to the dynamic sliding mode control is the second-order sliding mode control [12–15]. However, the second-order sliding mode control requires knowledge of the time derivative of the sliding variable. As a result, differentiators must be used to estimate the time derivative of the sliding variable. The limitation of differentiators is that the close-loop stability is ensured only locally, and that measurement noise may become a problem in noisy environments.

This paper follows the second-order sliding mode control approach to design a chattering- free sliding mode control. However, unlike the conventional second-order

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sliding mode control, which requires differentiator as in [12] and [13] to estimate the time derivative of sliding surface, the second-order sliding mode control proposed in this paper uses only information of sliding surface σ, but not information of $\dot{\sigma}$. This paper also proposes for the first time in the literature a rigorous Lyapunov stability analysis for such a control. Based on this second order sliding mode control, a chattering-free dynamic sliding mode control is designed. The resultant dynamic sliding mode control has the advantage that it does not require observer or estimator in [10, 11] to estimate the uncertainty in the sliding variable. The applications of the proposed second-order sliding mode control algorithm can be many folds [16]; this paper demonstrates only one application, namely, the chattering-free sliding mode control design.

This paper is arranged as follows: Section 2 clarifies the difference between conventional first-order sliding mode control and the modern second-order sliding mode control. In Section 3, a rigorous stability analysis based on the Lyapunov function is presented for the proposed second-order sliding mode control algorithm. In Section 4, the proposed second-order sliding mode control algorithm is applied to the design of a chattering-free sliding mode control. Section 5 gives the conclusions.

2. PROBLEM FORMULATION

In the conventional (first-order) sliding mode control design, the sliding surface is driven by the control input so that it satisfies a first order differential equation

$$\dot{\sigma}+a\sigma=-\rho sign(\sigma)+d, \tag{1}$$

where $\sigma\in\Re$ is the sliding surface, $a\in\Re^{+}$ is an arbitrary positive constant, is the switching control gain satisfying

$$\rho\geq|d|, \tag{2}$$

$sign(\cdot)$ is the signum function, and d is a uniformly bounded time-varying disturbance. Regarding the behavior of σ in (1), one has the following theorem.

Theorem 1: The sliding surface σ in (1) converges asymptotically to zero if the switching control gain ρ satisfies the upper bound condition (2).

Proof: The proof is standard, and can be found in for example [2].

In the new second-order sliding mode control design proposed in this paper, the sliding surface σ is driven by the control input so that it satisfies a second order differential equation

$$\ddot{\sigma}+a_1\dot{\sigma}+a_0\sigma=-\rho sign(\sigma)+d, \tag{3}$$

where a_1 and a_0 are two arbitrary positive constants, ρ is the switching control gain satisfying

$$\rho\geq|d|+\frac{1}{\alpha}|\dot{d}|, \tag{4}$$

where $\alpha=\frac{a_1}{(1+a_0)}$, and d is a time-varying disturbance with uniformly bounded d and $\dot{d}$.

Theorem 2: The sliding surface σ and $\dot{\sigma}$ in (3) converge asymptotically to zero if the switching control gain ρ satisfies the upper bound condition (4).

The proof of Theorem 2 will be given in the next section.

3. LYAPUNOV STABILITY ANALYSIS

To prove the stability of the system (3), one first re-write the second-order differential equation into a state space form. Define system state $\mathbf{x}^T=\begin{bmatrix}\sigma & \dot{\sigma}\end{bmatrix}$. The differential equation can be written as

$$\dot{\mathbf{x}}=\mathbf{F}\mathbf{x}+\mathbf{G}(-\rho sign(\sigma)+d), \tag{5}$$

where

$$\mathbf{F}=\begin{bmatrix}0 & 1\\ -a_0 & -a_1\end{bmatrix}\text{ and }\mathbf{G}=\begin{bmatrix}0\\1\end{bmatrix}. \tag{6}$$

Since a_0 and a_1 are positive constants, the system matrix $\mathbf{F}$ in (5) is a stable matrix. It thus satisfies the following Lyapunov equation

$$\mathbf{F}^T\mathbf{P}+\mathbf{P}\mathbf{F}=-\mathbf{I}, \tag{7}$$

where $\mathbf{P}\in\Re^{2\times2}$ is a positive definite matrix, which has the following property.

Lemma 1: Let the vector $\mathbf{PG}=\begin{bmatrix}\alpha r\\ r\end{bmatrix}$ where $\mathbf{P}$ is from (7) and $\mathbf{G}$ is as in (6). Then both constants α and r are positive.

Proof: Let matrix $\mathbf{P}=\begin{bmatrix}p_{11} & p_{12}\\ p_{21} & p_{22}\end{bmatrix}$, then $\mathbf{PG}=\begin{bmatrix}p_{12}\\ p_{22}\end{bmatrix}=\begin{bmatrix}\alpha r\\ r\end{bmatrix}$. Solving the Lyapunov equation (7) directly, one obtains $p_{12}=\frac{1}{2a_0}>0$ and $p_{22}=\frac{1+\frac{1}{a_0}}{2a_1}>0$, hence, both r and $\alpha=\frac{a_1}{1+a_0}$ are positive. End of proof.

The following lemma is important for the Lyapunov stability analysis for the state space system (5).

Lemma 2: Define a function

$$L=(\dot{\sigma}+\alpha\sigma)\left(-\rho\frac{\sigma}{|\sigma|}+d\right), \tag{8}$$

where α is a positive constant as in (4), ρ satisfies the inequality (4), then $\int_0^t L(\tau)d\tau$ is upper-bound by a constant bound b.

Proof: Following the definition of L, one has

$$\begin{aligned}\int_0^t L(\tau)d\tau &= \int_0^t(\dot{\sigma}(\tau)+\alpha\sigma(\tau))(-\rho sign(\sigma(\tau))\\ &\quad +d(\tau))d\tau\\ &= \int_0^t \alpha\sigma(\tau)(-\rho sign(\sigma(\tau))+d(\tau))d\tau\end{aligned}$$

$$+\int_0^t \frac{d\sigma(\tau)}{d\tau} d(\tau) d\tau - \int_0^t \rho \frac{d\sigma(\tau)}{d\tau} sign(\sigma(\tau)) d\tau, \quad (9)$$

where $sign(\sigma) = \frac{\sigma}{|\sigma|}$ is the signum function. Integrating by parts of the second term in (9) gives

$$\int_0^t \frac{d\sigma(\tau)}{d\tau} d(\tau) d\tau = \sigma(\tau) d(\tau)|_0^t - \int_0^t \sigma(\tau)\dot{d}(\tau) d\tau \le |\sigma(t)||d(t)| + |\sigma(0)||d(0)| + \int_0^t |\sigma(\tau)| \left|\dot{d}(\tau)\right| d\tau. \quad (10)$$

Note further that

$$d|\sigma| = d(\sigma sign(\sigma)) = sign(\sigma) d\sigma + \sigma d sign(\sigma).$$

Hence, the last term in (9) can be written as

$$\int_0^t \rho \frac{d\sigma(\tau)}{d\tau} sign(\sigma(\tau)) d\tau = \int_0^t \rho \frac{d|\sigma(\tau)|}{d\tau} d\tau - \int_0^t \rho\sigma(\tau) \frac{d sign(\sigma(\tau))}{d\tau} d\tau = \rho|\sigma(\tau)||_0^t, \quad (11)$$

where it is proved in the Appendix that the second term in (11) is zero. Substituting (10) and (11) into (9), one obtains

$$\begin{aligned}\int_0^t \frac{d\sigma(\tau)}{d\tau} d(\tau) d\tau &\le \int_0^t \alpha|\sigma(\tau)| \left(|d(\tau)| + \frac{1}{\alpha}\dot{d}(\tau) - \rho\right) d\tau \\ &\quad + |\sigma(t)||d(t)| + |\sigma(0)||d(0)| - \rho|\sigma(\tau)||_0^t \\ &\le |\sigma(t)|(|d(t)| - \rho) + |\sigma(0)|(|d(0)| + \rho) \\ &\le |\sigma(0)|(|d(0)| + \rho) = b,\end{aligned}$$

where the second and third inequalities are due to (4). The proof is then completed.

The final lemma introduced is well-known Baralat lemma.

Lemma 3: Consider a time function $f(t)$. If $f(t)$ satisfies $\dot{f}(t) \in L_\infty$ and $f(t) \in L_2$, then $f(t) \to 0$ as $t \to \infty$.

Proof: See [16].

One is now in a position to prove Theorem 2.

Proof of Theorem 2: The objective is to prove that the system state $x^T = \begin{bmatrix} \sigma & \dot{\sigma} \end{bmatrix}$ of the state space system (5) converges asymptotically to zero. Construct a Lyapunov function

$$V = \mathbf{x}^T\mathbf{P}\mathbf{x} + 2rb - \int_0^t 2r(\dot{\sigma} + \alpha\sigma)\left(-\rho\frac{\sigma}{|\sigma|} + d\right) d\tau, \quad (12)$$

where $\mathbf{P}$ is from the Lyapunov equation (7), r is as in Lemma 1, and b is as in Lemma 2. Note that V is always non-negative because of Lemma 2. Taking the time derivative of the Lyapunov function V along the trajectory of (5), one obtain

$$\begin{aligned}V &= -\mathbf{x}^T\mathbf{x} + \left(2\mathbf{G}^T\mathbf{P}\mathbf{x}\right)\left(-\rho\frac{\sigma}{|\sigma|} + d\right) \\ &\quad - 2r(\dot{\sigma} + \alpha\sigma)\left(-\rho\frac{\sigma}{|\sigma|} + d\right) \\ &\le -\mathbf{x}^T\mathbf{x},\end{aligned} \quad (13)$$

where the second equality is obtained by noting that $2\mathbf{G}^T\mathbf{P}\mathbf{x} = 2\begin{bmatrix} r\alpha & r \end{bmatrix}\begin{bmatrix} \sigma \\ \dot{\sigma} \end{bmatrix} = 2r(\dot{\sigma} + \alpha\sigma)$. From this, one obtains that $x \in L_\infty$. Integrating the equality (13) and noting that V is always non-negative further show that $x \in L_2$. One can then infer from the state equation (5) that $\dot{x} \in L_\infty$. Finally, one quotes Lemma 3 to conclude that $\lim_{t\to\infty} x^T(t) = \lim_{t\to\infty} \begin{bmatrix} \sigma(t) & \dot{\sigma}(t) \end{bmatrix} = 0$. End of proof.

Remark 1: Previous analysis of the differential equation (3) or other second-order sliding mode control proves its stability using a geometric view point [14]. This paper is the first literature that proves the stability of the second-order sliding mode control using rigorous Lyapunov stability theorems. It is important to obtain a stability proof via the Lyapunov function since the Lyapunov stability theorem or the converse Lyapunov theorem is very useful in extending the application domain. For example, in this paper, one constraints the disturbance d to be bounded and has bounded first order time derivative. Using the Lyapunov stability theorem, one can relax this constraint, and allow the disturbance to be potentially unbounded. This result cannot be obtained from a geometric view point. It is also mentioned that in [14], the second order differential equation (3) must satisfy $a_1^2 - 4a_0 > 0$. This paper relaxes this constraint.

Remark 2: In Theorem 2, the sliding surface σ is a scalar function. In fact, Theorem 2 can be established for vector $\sigma \in \Re^m$, $m \ge 1$. Thus, Theorem 2 can be applied to the sliding mode control design for multivariable systems

4. CHATTERING-FREE SMC

Consider the sliding mode control design for a multivariable system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}(\mathbf{u}(t) + \mathbf{d}(t)), \quad (14)$$

where $\mathbf{x} \in \Re^n$ is the accessible system state, $\mathbf{u} \in \Re^m$ is the control input, and $\mathbf{d} \in \Re^m$ is an unknown disturbance. In this paper, it is assumed that the disturbance satisfies a smoothness assumption that d, $\dot{d}$, and $\ddot{d}$ are all uniformly bounded. Without loss of generality, the system (14) can be decomposed a

$$\begin{bmatrix} \dot{\mathbf{x}}_1(t) \\ \dot{\mathbf{x}}_2(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}\begin{bmatrix} \mathbf{x}_1(t) \\ \mathbf{x}_2(t) \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{B}_2 \end{bmatrix}(\mathbf{u}(t) + \mathbf{d}(t)), \quad (15)$$

where $\mathbf{x}_1 \in \Re^{n-m}$ and $\mathbf{x}_2 \in \Re^m$, and the system matrices are partitioned accordingly. The system is assumed to be controllable in the sense that $\mathbf{B}_2 \in \Re^{m\times m}$ is invertible and $(\mathbf{A}_{11}, \mathbf{A}_{12})$ is a controllable pair. The control objective is to design a sliding-mode control that can suppress the unknown disturbance d and drive the system state $\mathbf{x}$ to zero. Furthermore, the control signal $\mathbf{u}$ of the sliding mode design should be chattering free. At the first stage of design, one needs to choose a sliding surface σ for the system. The sliding surface should have the invariance property that on the sliding mode ($\sigma = 0$), the system is not subject to the interference of the disturbance, and the system state will automatically converge to zero. Furthermore, the relative degree from the control input u to σ should be one. To meet these requirements, one chooses the sliding variable

$$\sigma = \begin{bmatrix} \mathbf{K} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} = Cx, \quad C = \begin{bmatrix} \mathbf{K} & \mathbf{I} \end{bmatrix}. \quad (16)$$

To verify that such a choice makes σ satisfy the invariance property on the sliding surface, note that when $\sigma = 0$, one obtains $\mathbf{x}_2 = -\mathbf{K}\mathbf{x}_1$. Substituting this into the system equation (15), one has $\dot{\mathbf{x}}_1 = (\mathbf{A}_{11} - \mathbf{K}\mathbf{A}_{12})\mathbf{x}_1$. Since $(\mathbf{A}_{11}, \mathbf{A}_{12})$ is controllable, one can always choose $\mathbf{K}$ such that $\mathbf{A}_{11} - \mathbf{K}\mathbf{A}_{12}$ is a stable matrix; hence, $\mathbf{x}_1$ converges to zero exponentially after $\sigma = 0$, and so does $\mathbf{x}_2\ (= -\mathbf{K}\mathbf{x}_1)$. Note also that $\mathbf{CB} = \begin{bmatrix} \mathbf{K} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ \mathbf{B}_2 \end{bmatrix} = \mathbf{B}_2$, which is invertible by assumption. At the second stage of design, one constructs the control u to drive the system state to reach, from any initial condition, the sliding surface. At this stage, one will present two designs of u; one design is based on the first-order sliding mode control (1), and the other based on the second-order sliding mode control (3), respectively

Conventional SMC Design: Take the time derivative of the sliding surface$\sigma = \mathbf{Cx}$ along (14); one obtain

$$\dot{\sigma} = \mathbf{CAx} + \mathbf{CB}(\mathbf{u} + \mathbf{d}). \quad (17)$$

If the control u is chosen as

$$\mathbf{u} = (\mathbf{CB})^{-1}(-a\mathbf{Cx} - \mathbf{CAx} - \rho sign(\sigma)), \quad (18)$$

where a is any positive constant, and $\rho > 0$ is the switching control gain satisfying the bound $\rho > |\mathbf{CB}d|$, one can verify that the sliding surface satisfies

$$\dot{\sigma} + a\sigma = -\rho sign(\sigma) + d. \quad (19)$$

Quoting Theorem 1 yields the desired result that the sliding surface σ will converge to zero exponentially; thus, achieving the control objective of driving $\boldsymbol{x}$ to zero asymptotically by quoting the invariance property of the sliding surface. Note that the first-order sliding mode control u in (18) is a switching control, and hence, will exhibit the chattering phenomenon after the sliding surface σ is driven to zero. One common solution to this control chattering problem is to use the boundary layer design, in which the discontinuous switching function $sign(\sigma) = \frac{\sigma}{|\sigma|}$ is replaced by a continuous approximation function $\frac{\sigma}{|\sigma|+\varepsilon}$, where ε is a small positive constant. However, the boundary layer design has two weaknesses. First the control accuracy is sacrificed because the system state no longer converges to the origin of the state space but a residual set around the origin. Second, the control chattering is still unavoidable in noisy environments even if a boundary layer is used [5]. In the sequel, the design procedure for the first-order sliding mode control is summarized below.

Step 1: Design a state feedback gain $\mathbf{K}$ such that $\mathbf{A}_{11} - \mathbf{K}\mathbf{A}_{12}$has desired stable eigenvalues.

Step 2: Set the sliding surface $\sigma = \mathbf{Cx}$, where $\mathbf{C} = \begin{bmatrix} \mathbf{K} & \mathbf{I} \end{bmatrix}$.

Step 3: Choose the first-order sliding mode control design parameters in (19) as follows. Let a be any positive number, and the switching control gain $\rho > 0$ be such that $\rho > |\mathbf{CB}d|$.

Step 4: Set the control law as in (18).

New SMC Design: Take the second-order time derivative of the sliding surface σ to obtain

$$\ddot{\sigma} = \mathbf{CA}^2 x + \mathbf{CAB}(u + d) + \mathbf{CB}(\dot{u} + \dot{d}). \quad (20)$$

Propose the chattering-free second-order sliding mode control law as

$$b_1\dot{u} + b_0 u = -\left(a_0\mathbf{C} + a_1\mathbf{CA} + \mathbf{CA}^2\right)x - \rho sign(\sigma), \quad (21)$$

where $b_1 = \mathbf{CB}$ and $b_0 = \mathbf{CAB} + a_1\mathbf{CB}$ with $a_1 > 0$ chosen such that b_0 has the same sign as b_1. Notice that this can always be achieved if a_1 is large enough so that $a_1\mathbf{I}_m > (\mathbf{CB})^{-1}\mathbf{CAB}$. In (21), ρ is the switching control gain satisfying the bound

$$\rho > |D| + \frac{1}{\alpha}\left|\dot{D}\right|, \quad |D| = b_1\dot{d} + b_0 d, \quad (22)$$

where $\alpha = \frac{a_1}{1+a_0}$. Using (17), (20), and (21), one can show that the control law (21) drives the sliding surface σ such that it satisfies the second-order differential equation

$$\ddot{\sigma} + a_1\dot{\sigma} + a_0\sigma = -\rho sign(\sigma) + D. \quad (23)$$

Quoting Theorem 2 yields the result that the sliding surface σ will converge to zero asymptotically; thus, achieving the control objective of driving $\boldsymbol{x}$ to zero asymptotically.

Note that the second-order sliding mode control u in (21) is the output of a stable low-pass filter

$$u = \frac{1}{b_1 s + b_0}\left(-\rho sign(\sigma) - \left(a_0\mathbf{C} + a_1\mathbf{CA} + \mathbf{CA}^2\right)\mathbf{x}\right).$$

It is important to note that the high frequency chattering of $\rho sign(\sigma)$ will be filtered out by the low-pass filter; hence, the control variable u will be chattering-free. In the sequel, the design procedure for the second-order chattering free sliding mode control is summarized below.

Step 1: Design a state feedback gain $\mathbf{K}$ such that $\mathbf{A}_{11}-\mathbf{KA}_{12}$ has desired stable eigenvalues.

Step 2: Set the sliding surface $\sigma = \mathbf{Cx}$, where $\mathbf{C}=\left[\begin{array}{cc}\mathbf{K} & \mathbf{I}\end{array}\right]$.

Step 3: Choose the second-order sliding mode control design parameters in (21) as follows. Let a_0 be any positive number, $a_1>0$ be such that $a_1\mathbf{I} > (\mathbf{CB})^{-1}\mathbf{CAB}$, and the switching control gain $\rho>0$ be such that $\rho > |D| + \frac{1}{\alpha}\left|\dot{D}\right|$, in which $\alpha = \frac{a_1}{1+a_0}$, $b_1 = CB$, and $b_0 = CAB + a_1 CB$.

Step 4: Set the control law as in (21).

In this paper, the proposed second-order sliding mode control is based on state feedback. It is mentioned that when the system state is not accessible, and only the system output is measured, one can use the robust observer proposed in [17] for accurate state estimation, and obtain an observer-based state feedback second-order sliding mode control. To confirm the effectiveness of the proposed chattering-free second-order sliding mode control design, two simulation examples are presented and compared. The first example presents the first-order sliding mode control plus the boundary layer design. The second example presents the second-order sliding mode control design.

Example 1 (Boundary layer control design): Consider the system (14) with system matrices

$$\mathbf{A}=\begin{bmatrix} 1.38 & -0.2077 & 6.715 & -5.676 \\ -0.5814 & -4.29 & 0 & 0.675 \\ 1.067 & 4.273 & -6.654 & 5.893 \\ 0.048 & 4.273 & 1.343 & -2.104 \end{bmatrix},$$

$$\mathbf{B}=\begin{bmatrix} 0 & 0 \\ 5.679 & 0 \\ 1.136 & -3.146 \\ 1.136 & 0 \end{bmatrix}, \mathbf{d}=\begin{bmatrix} 1.5\sin(0.5t) \\ 1\cos(\pi t) \end{bmatrix},$$

and the initial condition $x(0)=\left[\begin{array}{cccc}3 & -1 & 2 & 2\end{array}\right]^T$. The control design parameters are such that $\mathbf{K}$ in (16) places the eigenvalues of $\mathbf{A}_{11}-\mathbf{KA}_{12}$ to $\{-2,-3\}$, $a=3$ in (18), the switching control gain $\rho=2$, and the boundary layer width $\varepsilon = 0.05$. The control result is shown in Figure 2. It is seen from the upper plot that the system states are finally constrained in the bounded region but cannot asymptotically converge to zero. The lower plot depicts the time history of the control input.

Example 2 (Second-order sliding mode control design): Consider the same system as in Example 1. The control design parameters are such that $\mathbf{K}$ in (16) places the eigenvalues of $\mathbf{A}_{11}-\mathbf{KA}_{12}$ to $\{-2,-3\}$, $a_0=1$, $a_1=5$ in (21). The control result is shown in Figure 3; in which it is seen from the upper plot that the system state asymptotically converges to zero in about 2 seconds. The lower plot depicts the time history of the control input, which is chattering-free and sufficiently smooth. Note that no boundary layer is introduced in this sliding mode control design.

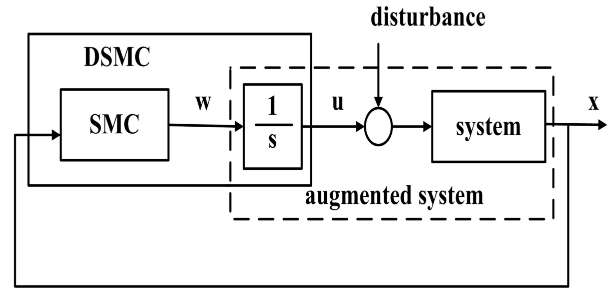

Fig. 1. Structure of dynamic sliding mode control.

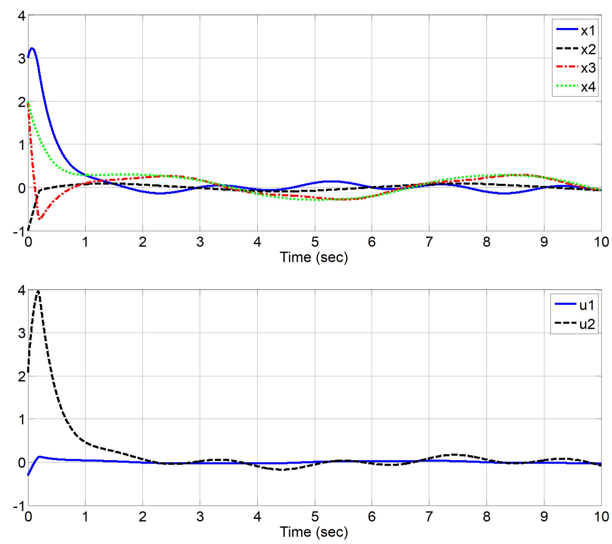

Fig. 2. Conventional sliding mode control design.

5. CONCLUSION

This paper proposes a new sliding mode control design with reduced control chattering. The proposed new design inherits the design concept from dynamic sliding mode control, in which the first-order time derivative of the control input is treated as the control variable for a chattering control design. Previous dynamic sliding mode control designs require an extra uncertainty observer or uncertainty estimator to construct the sliding variable. This paper is able to waive such observer or estimator.

APPENDIX A

The goal of this appendix is to prove that $\int_0^t \rho\sigma(\tau)\frac{dsign(\sigma(\tau))}{d\tau}d\tau = 0$ in equation (11). Note that $\frac{dsign(\sigma(\tau))}{d\tau} = 0$ is equal to zero when $\sigma \neq 0$, but equal to infinity when $\sigma = 0$. Hence, this integration term cannot be

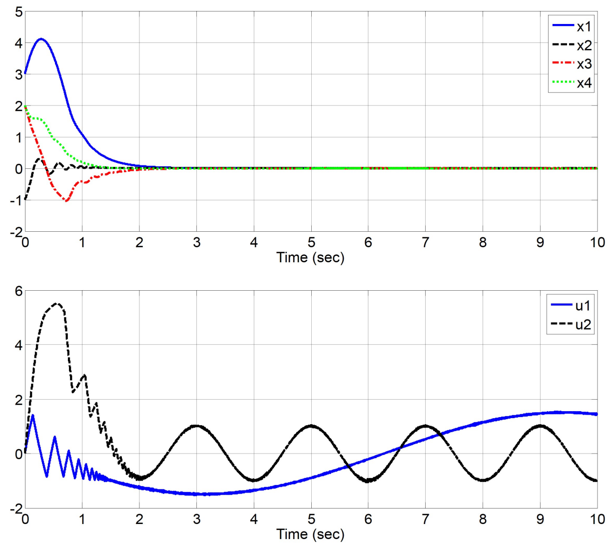

Fig. 3. New sliding mode control design.

neglected directly because one must pay special attention to its behavior at $\sigma = 0$. One will prove that this integration is equal to zero by utilizing the Heaviside step function H(?). Note that the signum function can be expressed by the Heaviside function as $sign(\sigma) = 2H(\sigma) - 1$. Hence, $\frac{dsign(\sigma(\tau))}{d\tau} = 2\frac{dH(\sigma)}{d\sigma}\frac{d\sigma}{d\tau} = 2\delta(\sigma)\frac{d\sigma}{d\tau}$, where one has used the fact that the derivative of the Heaviside function H(?) is the Dirac delta function $\delta(\cdot)$ [18]. Thus, the integration becomes

$$\int_0^t \rho\sigma(\tau)\frac{dsign(\sigma(\tau))}{d\tau}d\tau = \int_0^t 2\rho\delta(\sigma)d\sigma$$
$$= 2\rho\sigma|_{\sigma=0} = 0,$$

where one has used the formula $\int f(x)\delta(x)dx = f(x)|_{x=0} = f(0)$. This completes the proof.

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