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Risk-sensitive Control of Markov Jump Linear Systems: Caveats and Difficulties

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Abstract: In this technical note, we revisit the risk-sensitive optimal control problem for Markov jump linear systems (MJLSs). We first demonstrate the inherent difficulty in solving the risk-sensitive optimal control problem even if the system is linear and the cost function is quadratic. This is due to the nonlinear nature of the coupled set of Hamilton-Jacobi-Bellman (HJB) equations, stemming from the presence of the jump process. It thus follows that the standard quadratic form of the value function with a set of coupled Riccati differential equations cannot be a candidate solution to the coupled HJB equations. We subsequently show that there is no equivalence relationship between the problems of risk-sensitive control and H^∞ control of MJLSs, which are shown to be equivalent in the absence of any jumps. Finally, we show that there does not exist a large deviation limit as well as a risk-neutral limit of the risk-sensitive optimal control problem due to the presence of a nonlinear coupling term in the HJB equations.

Keywords: Markov jump linear systems, risk-sensitive control, stochastic zero-sum differential games.

1. INTRODUCTION

Markov jump linear systems (MJLSs) are switching systems, where the switching process is determined according to a finite- or infinite-state Markov chain. They are also known as piecewise-deterministic or stochastic hybrid linear systems. One of the remarkable features of MJLSs is that they allow for modeling of a number of different system modes depending on a state space of the Markov chain, which enables capturing more general system behaviors that are subject to dynamic uncertainties and abrupt changes in modeling parameters. Hence, applications of MJLSs can be found in a wide variety of fields, such as solar power stations, flight systems, economic systems, power systems, and communication systems, among many others [1].

Optimal control of MJLSs driven by Brownian motion process is one of the fundamental stochastic control problems, and the linear-quadratic-Gaussian (LQG) and/or the H^∞ control settings have been of particular interest to many researchers [1–4] (detailed expositions can be found in the recent book [1] and/or the recent survey papers [5, 6], and the references therein). Specifically, it was shown that the state feedback mode-dependent (LQG or H^∞) optimal controller can be obtained by solving a set of coupled Riccati differential equations associated with the coupled Hamilton-Jacobi-Bellman (HJB) equations.

The problem of risk-sensitive control of MJLSs, on the other hand, has not been discussed as extensively due to its inherent difficulty. Specifically, in [1, page 82], it is noted that *"Although the avenues of research to a risk sensitivity approach for the optimal control of MJLS seem to be fascinating, this is a topic which has defied the researchers up to now."* The difficulty was discussed briefly in [7], but the main source of the underlying difficulty was not presented clearly. Moreover, the author in [7] did not discuss the large deviation limit as well as the risk-neutral limit, which are fundamental properties of risk-sensitive control as mentioned in [8]. It does not appear that there is any literature on risk-sensitive control for MJLSs except [7].

In this technical note, we revisit the risk-sensitive optimal control problem of MJLSs to address the issues mentioned above. Our main contributions of the note are as follows:

(i) We obtain a set of coupled HJB equations for risk-sensitive control of MJLSs, and show that it is *not* possible to solve it in closed form analytically even if the system is linear and the cost function is quadratic.

(ii) We show that for the case of MJLSs, the risk-sensitive control is *not* equivalent to the H^∞ control studied in [4, 9].

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(iii) We prove *nonexistence* of the large deviation limit as well as the risk-neutral limit.

In Section 2, we show that the main difficulty of the problem stems from the nonlinear coupling nature of the corresponding HJB equations by which the quadratic value, not admitting, for example, a quadratic structure for its solution, thus ruling out a Riccati differential equation based representation. In Sections 3 and 4, we discuss, respectively, (ii) and (iii), which lead to the conclusion that the equivalence relationship, and existence of the two limiting behaviors for the jump-free case[1] do not hold for the Markov jump case. These two results also follow from the nonlinear coupling nature of the HJB equations. In Section 5, we conclude by providing a recap of highlights of the note and also discussing some future research directions.

2. RISK-SENSITIVE CONTROL OF MJLSS: DIFFICULTY

Consider a controlled stochastic differential equation (SDE) defined on a complete probability space $(\Omega,\mathcal{F},\mathbb{P})$:

$$dx(t) = A(\theta(t))x(t)dt + B(\theta(t))u(t)dt + \sqrt{\varepsilon}D(\theta(t))dB(t), \tag{1}$$

where $x(0)=x_0$, $x\in\mathbb{R}^n$ is the state, $u\in\mathbb{R}^q$ is the control, $\{B(t),t\geq 0\}$ is a p-dimensional standard Brownian motion, and $\varepsilon>0$ is a noise intensity parameter. In (1), $\{\theta(t),t\geq 0\}$ is a continuous-time Markov chain taking values in the finite state space $S=\{1,2,\ldots,s\}$ with infinitesimal generator $\Lambda=\{\lambda_{ij},i,j=1,\ldots,s\}$. In this formulation, the initial state x_0 is not random, and $\{B(t)\}$ and $\{\theta(t)\}$ are independent of each other. Moreover, for any $\theta(t)=i$, $i\in S$, $A_i:=A(\theta(t)=i)$, $B_i:=B(\theta(t)=i)$, and $D_i:=D(\theta(t)=i)$ are time-invariant matrices with appropriate dimensions. Under this setting, (1) is known as a Markov jump linear system (MJLS) [1]. We assume that $D_iD_i^T>0$ for all $i\in S$.

The control input u is generated by a Markov strategy $\mu:\mathbb{R}^n\times S\to\mathbb{R}^q$:

$$u(t)=\mu(x(t),\theta(t)), \tag{2}$$

where μ is measurable both in x and θ, and Lipschitz continuous in x. Let us denote the class of all such state-feedback Markov control policies by $\mathcal{U}$.

The performance index to be minimized is the risk-sensitive one, given by

$$J(\mu;t,x,i) = \delta\log\mathbb{E}\Big\{e^{\frac{1}{\delta}[\int_t^T \|x(\tau)\|^2_{Q(\theta(\tau))}+\|u(\tau)\|^2_{R(\theta(\tau))}d\tau+\|x(T)\|^2_{L(\theta(T))}]} \,\Big|\, x(t)=x,\ \theta(t)=i\Big\}, \tag{3}$$

where $\delta>0$ is the risk-sensitivity (risk-averse) parameter, $Q_i:=Q(\theta(t)=i)\geq 0$, $L_i:=L(\theta(T)=i)\geq 0$ and $R_i:=R(\theta(t)=i)>0$ for all $i\in S$. Note that the optimal control problem thus formulated may be thought of as being *simple*, due to the linearity of the system (1) and the quadratic nature of the exponent in the cost function. In the following, we show that the problem above is, on the contrary, not easy to solve even if the problem setting may seem to be simple.

Let $\phi(t,x,i)$, $i\in S$, be the value function associated with

$$\mathbb{E}\Big\{e^{\frac{1}{\delta}[\int_t^T \|x(\tau)\|^2_{Q(\theta(\tau))}+\|u(\tau)\|^2_{R(\theta(\tau))}d\tau+\|x(T)\|^2_{L(\theta(T))}]} \,\Big|\, x(t)=x,\ \theta(t)=i\Big\},$$

that is,

$$\phi(t,x,i) = \inf_{\mu\in\mathcal{U}}\mathbb{E}\Big\{e^{\frac{1}{\delta}[\int_t^T \|x(\tau)\|^2_{Q(\theta(\tau))}+\|u(\tau)\|^2_{R(\theta(\tau))}d\tau+\|x(T)\|^2_{L(\theta(T))}]} \,\Big|\, x(t)=x,\ \theta(t)=i\Big\}. \tag{4}$$

subject to (1). It is clear that for $i\in S$,

$$V(t,x,i) := \inf_{u\in\mathcal{U}} J(\mu;t,x,i) = \delta\log\phi(t,x,i). \tag{5}$$

From Lemma 1 in Appendix A, $\phi(t,x,i)$ can be differentiated in the Itô sense:

$$d\phi(t,x,i) = \Big[\phi_t(t,x,i)+\phi_x^T(t,x,i)(A_ix(t)+B_iu(t)) + \sum_{j=1}^s\lambda_{ij}\phi(t,x,j)+\frac{\varepsilon}{2}\mathrm{Tr}(D_iD_i^T\phi_{xx}(t,x,i))\Big]dt,$$

where ϕ_t and ϕ_x are the partial derivatives of ϕ with respect to t and x, respectively, and ϕ_{xx} is the second partial derivative with respect to x. Using the Itô-Dynkin formula, the dynamic optimization problem yields [11]

$$\inf_{\mu\in\mathcal{U}}\{d\phi(t,x,i)+\frac{1}{\delta}(\|x(t)\|^2_{Q_i}+\|u(t)\|^2_{R_i})\phi(t,x,i)dt\}=0.$$

Thus, one obtains

$$\phi_t(t,x,i)+\frac{\varepsilon}{2}\mathrm{Tr}(D_iD_i^T\phi_{xx}(t,x,i))+\sum_{j=1}^s\lambda_{ij}\phi(t,x,j) + \inf_{\mu\in\mathcal{U}}\{\phi_x^T(t,x,i)(A_ix(t)+B_iu(t)) + \frac{1}{\delta}(\|x(t)\|^2_{Q_i}+\|u(t)\|^2_{R_i})\phi(t,x,i)\}=0, \tag{6}$$

where $\phi(T,x,i)=e^{(1/\delta)x^TL_ix}$, $i\in S$.

To establish the connection between (4) and (5), in view of (5), we use the value function transformation:

$$\phi_t=\frac{1}{\delta}V_t\phi,\ \ \phi_x=\frac{1}{\delta}V_x\phi,\ \ \phi_{xx}=\frac{1}{\delta}V_{xx}\phi+\frac{1}{\delta^2}V_xV_x^T\phi.$$

[1] It was shown in [8, 10] that for the jump-free case, there exist two limiting behaviors in risk-sensitive control, and risk-sensitive optimal control is equivalent to H^∞ control.

Substituting these in (6), and dividing throughout by ϕ/δ, leads to the set of coupled Hamilton-Jacobi-Bellman (HJB) equations for the risk-sensitive optimal control problem of MJLSs:

$$\begin{aligned} &-V_t(t,x,i)\\ &=\frac{\varepsilon}{2}\operatorname{Tr}(D_iD_i^TV_{xx}(t,x,i))+\frac{\varepsilon}{2\delta}\|D_i^TV_x(t,x,i)\|^2\\ &\quad+\delta\frac{\sum_{j=1}^s\lambda_{ij}e^{(1/\delta)V(t,x,j)}}{e^{(1/\delta)V(t,x,i)}} \qquad (7)\\ &\quad+\inf_{\mu\in\mathcal{U}}\{V_x^T(t,x,i)(A_ix(t)+B_iu(t))\\ &\qquad+\|x(t)\|_{Q_i}^2+\|u(t)\|_{R_i}^2\}, \end{aligned}$$

where the boundary condition is $V(T,x,i)=x^TL_ix,\ \ i\in S$.

Note that the above optimization problem has a unique solution, which can be written as

$$u^*(t)=-\frac{1}{2}R_i^{-1}B_i^TV_x(t,x,i)\ ,i\in S. \qquad (8)$$

Substituting the above optimal solution in the HJB equations yields

$$\begin{aligned} &-V_t(t,x,i)\\ &=\frac{\varepsilon}{2}\operatorname{Tr}(D_iD_i^TV_{xx}(t,x,i))+x^T(t)Q_ix(t)\\ &\quad+\delta\frac{\sum_{j=1}^s\lambda_{ij}e^{(1/\delta)V(t,x,j)}}{e^{(1/\delta)V(t,x,i)}}\\ &\quad+V_x^T(t,x,i)A_ix(t)-\frac{1}{4}V_x^T(t,x,i)B_iR_i^{-1}B_i^TV_x(t,x,i)\\ &\quad+\frac{\varepsilon}{2\delta}V_x^T(t,x,i)D_iD_i^TV_x(t,x,i). \end{aligned}$$

Now, from the verification theorem [11, Theorem 4.1], if there is a value function $V(t,x,i)$ that is a solution to the above set of HJB equations for all $i\in S$, then (8) is the optimal controller for the MJLS in (1) that minimizes the risk-sensitive performance index in (3). Note that since the set of HJB equations is uniformly parabolic for all $i\in S$ ($D_iD_i^T>0$, $\forall i\in S$), it admits a unique bounded positive solution. Unfortunately, unlike the cases of LQG and H^∞ control of MJLSs, it does not seem to be possible to find an explicit expression for the value function, $V(t,x,i)$, for all $i\in S$, analytically, due to the nonlinear coupling term (7). Specifically, the usual quadratic value function with the coupled Riccati differential equations cannot be a candidate solution to the set of HJB equations due to the presence of (7). This shows that the nonlinear coupling term (7) is the main source of the difficulty that defies attempts to find closed-form analytical solutions to the risk-sensitive optimal control problem for MJLSs.

Remark 1: From [8, Lemma 9.1, Chapter VI], we can show that the nonlinear coupling term in (7) can be written as

$$\begin{aligned} &\delta\frac{\sum_{j=1}^s\lambda_{ij}e^{(1/\delta)V(t,x,j)}}{e^{(1/\delta)V(t,x,i)}}\\ &=\delta\sup_{k_i>0}\Big\{\sum_{j=1}^s\frac{\bar\lambda_{ij}}{\delta}V(t,x,j)+\frac{\sum_{j=1}^s\lambda_{ij}k_j}{k_i}-\sum_{j=1}^s\bar\lambda_{ij}\log k_j\Big\}, \end{aligned}$$

where $\bar\lambda_{ij}=\lambda_{ij}\frac{k_j}{k_i}$ when $i\neq j$ and $\bar\lambda_{ii}=-\sum_{j\neq i}\bar\lambda_{ij}$. Moreover, the supremum can be achieved by $k_i^*=e^{\frac{1}{\delta}V(t,x,i)}$ for all $i\in S$. With this transformation, the HJB equation becomes

$$\begin{aligned} &-V_t(t,x,i)\\ &=\frac{\varepsilon}{2}\operatorname{Tr}(D_iD_i^TV_{xx}(t,x,i))+\frac{\varepsilon}{2\delta}\|D_i^TV_x(t,x,i)\|^2\\ &\quad+\delta\sup_{k_i>0}\Big\{\sum_{j=1}^s\frac{\bar\lambda_{ij}}{\delta}V(t,x,j)+\frac{\sum_{j=1}^s\lambda_{ij}k_j}{k_i}-\sum_{j=1}^s\bar\lambda_{ij}\log k_j\Big\}\\ &\quad+\inf_{\mu\in\mathcal{U}}\{V_x^T(t,x,i)(A_ix(t)+B_iu(t))\\ &\quad+\|x(t)\|_{Q_i}^2+\|u(t)\|_{R_i}^2\}. \end{aligned}$$

Then it is easy to see that the underlying difficulty of the problem stems from the additional optimization term induced by the nonlinear coupling term. Note that the above HJB equation cannot be solved with a quadratic value function. □

It should be mentioned that a similar difficulty was identified via the large deviation theory in [7]. The main source of the difficulty (that is the nonlinear coupling term in (7)), however, was not presented in a transparent way, and the corresponding HJB equation was not provided. Specifically, in [7], it was claimed that the risk-sensitive optimal control problem of MJLSs is difficult because the assumptions given in [7] are hard to check (see [7, Theorem 4]), but the author did not provide any sufficient conditions for such assumptions, and did not mention the nonlinear coupling nature of the HJB equation as the main source of the difficulty. In this technical note, we do not make any critical assumptions on the problem, and provide a clear reason as to why it is hard to obtain the closed-form analytical optimal solution. The following example further demonstrates that point, for the simplest possible scenario.

Example 1: Suppose that $A_1=-0.2$, $A_2=0.4$, $S=2$, $L=0$, and $\lambda_{12}=\lambda_{21}=B=D=Q=R=T=\varepsilon=1$. Then the associated pair of HJB equations can be written as

$$\begin{aligned} &-V_t(t,x,i)\\ &=\frac{1}{2}V_{xx}(t,x,i)+\frac{1}{2\delta}V_x^2(t,x,i)\\ &\quad+\frac{\delta(-e^{(1/\delta)V(t,x,i)}+e^{(1/\delta)V(t,x,j)})}{e^{(1/\delta)V(t,x,i)}}\\ &\quad+\inf_{\mu\in\mathcal{U}}\{V_x(t,x,i)(A_ix(t)+u(t))+x^2(t)+u^2(t)\}, \end{aligned}$$

where $V(1,x,1)=V(1,x,2)=0$. It should be apparent from the above that it is not possible to have a closed-form solution with a quadratic value function because of the nonlinear exponential coupling term. □

Remark 2: As can seen from Example 1, even for the simplest possible case, one has to resort to numerical techniques to solve the coupled partial differential equations. A detailed discussion on numerical techniques is provided in Section 5. □

Before concluding this section, we note that if a solution to the coupled HJB equations, $V(t,x,i)$, exists, for all $i \in S$, then as to be expected, the optimal controller in (8) is depends on the transition rate $\{\lambda_{ij}\}$, since $V(t,x,i)$ depends on $\{\lambda_{ij}\}$ in view of (7).

3. NON-EQUIVALENCE BETWEEN RISK-SENSITIVE CONTROL AND STOCHASTIC ZERO-SUM DIFFERENTIAL GAMES FOR MJLSS

In this section, we show that the problem in Section 2 is *not* equivalent to the problem of H^∞ optimal control for MJLSs studied in [4, 9]. Toward that end, we compare the set of HJB equations in Section 2 to that of stochastic zero-sum differential games (more precisely, set of Hamilton-Jacobi-Isaacs (HJI) equations) of MJLSs.

Consider the following SDE:

$$\begin{aligned} dx(t) &= A(\theta(t))x(t)dt + B(\theta(t))u(t)dt \\ &\quad + D(\theta(t))v(t)dt + \sqrt{\varepsilon}D(\theta(t))dB(t), \end{aligned} \tag{9}$$

where $v \in \mathbb{R}^p$ is the disturbance that is generated by a state-feedback strategy ν as in (2). Let us denote the class of all admissible disturbance strategies by $\mathcal{V}$.

The (risk-neutral) cost function for this differential game is given by

$$\begin{aligned} &\bar{J}(\mu,\nu;t,x,i) \qquad (10)\\ &= \mathbb{E}\Big\{\int_t^T \|x(\tau)\|^2_{Q(\theta(\tau))} + \|u(\tau)\|^2_{R(\theta(\tau))} - \gamma^2\|v(\tau)\|^2 d\tau \\ &\quad + \|x(T)\|^2_{L(\theta(T))} | x(t) = x, \theta(t) = i\Big\}, \end{aligned}$$

where $\gamma > 0$ is the disturbance attenuation parameter.

Let $W(t,x,i)$, $i \in S$, be the value function associated with (10), that is,

$$\begin{aligned} &W(t,x,i) \\ &= \inf_{\mu\in\mathcal{U}} \sup_{\nu\in\mathcal{V}} \mathbb{E}\Big\{\int_t^T \|x(\tau)\|^2_{Q(\theta(\tau))} + \|u(\tau)\|^2_{R(\theta(\tau))} \\ &\quad - \gamma^2\|v(\tau)\|^2 d\tau + \|x(T)\|^2_{L(\theta(T))} | x(t) = x, \theta(t) = i\Big\} \\ &= \sup_{\nu\in\mathcal{V}} \inf_{\mu\in\mathcal{U}} \mathbb{E}\Big\{\int_t^T \|x(\tau)\|^2_{Q(\theta(\tau))} + \|u(\tau)\|^2_{R(\theta(\tau))} \\ &\quad - \gamma^2\|v(\tau)\|^2 d\tau + \|x(T)\|^2_{L(\theta(T))} | x(t) = x, \theta(t) = i\Big\}, \end{aligned}$$

subject to (9).

Note that the Brownian motion, $\{B(t), t \geq 0\}$, is independent of $\{\theta(t), t \geq 0\}$. Then from [4], and by employing the Itô-Dynkin formula, we obtain the corresponding set of Hamilton-Jacobi-Isaacs (HJI) equations for all $i \in S$ (after carrying out the maximization with respect to v)

$$\begin{aligned} &- W_t(t,x,i) \\ &= \frac{\varepsilon}{2}\mathrm{Tr}(D_i D_i^T W_{xx}(t,x,i)) + \frac{1}{4\gamma^2}\|D_i^T W_x(t,x,i)\|^2 \\ &\quad + \sum_{j=1}^s \lambda_{ij} W(t,x,j) \\ &\quad + \inf_{\mu\in\mathcal{U}}\{W_x^T(t,x,i)(A_i x(t) + B_i u(t)) \\ &\qquad + \|x(t)\|^2_{Q_i} + \|u(t)\|^2_{R_i}\}, \end{aligned} \tag{11}$$

where the boundary condition is $W(T,x,i) = x^T L_i x,\ i \in S$.

We can easily see that the set of HJI equations above is not identical to the set of HJB equations in Section 2 due to the difference between the linear coupling term in (11) and the nonlinear coupling term in (7) (or the additional optimization problem in Remark 1); hence, the corresponding value functions, $V(t,x,i)$ and $W(t,x,i)$ will generally not be the same. This shows that in contrast to the standard case (with no jump parameter process), the equivalence relationship between risk-sensitive and H^∞ control does not hold for the MJLS case.

The above set of HJI equations admits as solution a quadratic value function, $W(t,x,i) = x^T P_i(t) x$, where $P_i(t)$ is the solution to the following coupled generalized Riccati differential equations for all $i \in S$:

$$\begin{aligned} -\frac{dP_i(t)}{dt} &= A_i^T P_i(t) + P_i(t)A_i + Q_i \\ &\quad - P_i(t)(B_i R_i^{-1} B_i^T - \frac{1}{\gamma^2} D_i D_i^T)P_i(t) \\ &\quad + \sum_{j=1}^s \lambda_{ij} P_j(t), \end{aligned}$$

where $P_i(T) = L_i$. In this case, the optimal controller is

$$u^*(t) = -\frac{1}{2}R_i^{-1}B_i^T W_x(t,x,i)\ , i \in S, \tag{12}$$

where $W_x(t,x,i) = 2P_i(t)x$. Note that (12) is not identical to (8) because the corresponding value functions, W and V, cannot be identical.

Example 2: Given the parameters in Example 1, the corresponding pair of HJI equations can be solved in terms of the following pair of (linearly) coupled Riccati differential equations:

$$\begin{aligned} -\frac{dP_i(t)}{dt} &= 2A_i P_i(t) + 1 - P_i^2(t)\left(1 - \frac{1}{\gamma^2}\right) \\ &\quad - P_i(t) + P_j(t), \end{aligned}$$

where $P_1(1) = P_2(1) = 0$. This is a system of ordinary differential equations, which can be solved easily by conventional numerical methods [12]. □

4. NON-EXISTENCE OF LIMITING BEHAVIORS

4.1. Large deviation limit

We first discuss the large deviation limit (small noise limit), in which case we take $\delta = \varepsilon$ and let $\varepsilon \to 0$. Moreover, it is necessary to have the modified infinitesimal generator, $\Lambda = \Lambda/\varepsilon$, since we need to observe a rare event of the jump process. In this case, the corresponding set of HJB equations becomes similar to that in Section 2, which can be written as (for all $i \in S$)

$$\begin{aligned}
&-V_t(t,x,i)\\
&= \frac{\varepsilon}{2}\operatorname{Tr}(D_iD_i^TV_{xx}(t,x,i)) + \frac{1}{2}\|D_i^TV_x(t,x,i)\|^2 \qquad (13)\\
&\quad + \frac{\sum_{j=1}^{s}\lambda_{ij}e^{(1/\varepsilon)V(t,x,j)}}{e^{(1/\varepsilon)V(t,x,i)}}\\
&\quad + \inf_{\mu\in\mathcal{U}}\{V_x^T(t,x,i)(A_ix(t)+B_iu(t))\\
&\qquad + \|x(t)\|_{Q_i}^2 + \|u(t)\|_{R_i}^2\}.
\end{aligned}$$

Note that as $\varepsilon \to 0$, the first term in (13) becomes zero and the second term remains the same.

To see the limiting behavior of the third term in (13) as $\varepsilon \to 0$, consider the case when $S = \{1,2\}$, and expand the corresponding nonlinear coupling term for $i = 1$

$$\begin{aligned}
&\frac{\lambda_{11}e^{(1/\varepsilon)V(t,x,1)} + \lambda_{12}e^{(1/\varepsilon)V(t,x,2)}}{e^{(1/\varepsilon)V(t,x,1)}}\\
&= \lambda_{11} + \lambda_{12}e^{(1/\varepsilon)(V(t,x,2)-V(t,x,1))}, \qquad (14)
\end{aligned}$$

and for $i = 2$

$$\begin{aligned}
&\frac{\lambda_{21}e^{(1/\varepsilon)V(t,x,1)} + \lambda_{22}e^{(1/\varepsilon)V(t,x,2)}}{e^{(1/\varepsilon)V(t,x,2)}}\\
&= \lambda_{22} + \lambda_{21}e^{(1/\varepsilon)(V(t,x,1)-V(t,x,2))}, \qquad (15)
\end{aligned}$$

where $\lambda_{12} > 0$ and $\lambda_{21} > 0$.

Now, since the value function is a mapping from $\mathbb{R} \times \mathbb{R}^n \times S$ to $\mathbb{R}_{\geq 0}$, we must have

$$V(t,x,2) - V(t,x,1) \geq 0 \text{ or } V(t,x,2) - V(t,x,1) \leq 0.$$

If $V(t,x,2) - V(t,x,1) > 0$, then (14) goes to infinity as $\varepsilon \to 0$. On the other hand, if $V(t,x,2) - V(t,x,1) < 0$, then (15) goes to infinity as $\varepsilon \to 0$. Therefore, (14) and (15) have a limit if and only if $V(t,x,2) - V(t,x,1) = 0$, which is not possible, since the system and/or cost parameters are not the same for $i = 1$ and $i = 2$.

This shows that the large deviation limit does not exist when $S = \{1,2\}$. From this, we can easily deduce that the large deviation limit does not exist for $S = \{1,2,\ldots,s\}$, and thereby the risk-sensitive control problem for MJLSs does not have any connection with the H^∞ control of MJLSs in [4, 9]. We should recall that for the jump-free case, the large deviation limit of risk-sensitive optimal control is equivalent to deterministic H^∞ control [8, 10].

4.2. Risk-neutral limit

The second limit we study is the risk-neutral limit, that is, we let $\delta \to \infty$ for a fixed ε. In this case, as can be seen from the set of HJB equations in Section 2, the second term in (13) approaches zero, but (7) goes to infinity. This implies that the risk-sensitive optimal control problem becomes not solvable; hence it is not equivalent to the standard LQG control of MJLSs in [2]. Let us again recall that for the jump-free case, under the risk-neutral limit, the risk-sensitive control and the LQG control become identical in the sense that the corresponding optimal controllers with the value functions are the same [8, 10].

5. CONCLUSIONS

In this technical note, we have revisited the risk-sensitive optimal control problem for Markov jump linear systems. We have shown that although the problem setting may seem to be simple, it is not possible to obtain closed-form analytical solutions to the associated set of HJB equations due to the presence of a nonlinear coupling term. We have also shown that the problem has no connection with that of H^∞ optimal control of MJLSs, and does not have a large deviation limit as well as a risk-neutral limit. These two results also follow from the nonlinear coupling nature of the HJB equations.

One immediate future research direction would be developing numerical techniques to solve the set of coupled HJB equations in Section 2 that leads to the optimal risk-sensitive controller for MJLSs. Toward that end, it would be possible to use a numerical approximation technique proposed in [13], where the algorithm constructs a discrete-time, finite-state, MJLS and the associated backward dynamic programming equation to approximate the corresponding SDE and the HJB equation, respectively. It was shown in [13] that under some conditions, when the approximation step size tends to zero, the value function of the (approximated) backward dynamic programming equation converges weakly to the solution of the HJB equation. The results in [13] are for the risk-neutral case, which can easily be extended to the risk-sensitive case. This extension is currently under study.

Finally, another important extension would be to the partial observation problem, which also does not admit a closed-form solution as can be expected from the result of this technical note, and hence constitutes a challenging research topic. Specifically, unlike the H^∞ control problem for MJLSs studied in [4, 9], it is not possible to construct the closed-form output feedback optimal controller for the partial observation case due to the presence of the nonlinear coupling term in the set of coupled HJB equations.

APPENDIX A

Lemma 1 (Martingale representation of a continuous–time Markov chain and its differentiation in the Itô sense) [1]**:** Suppose that $\mathcal{F}_t$ is the right-continuous complete filtration generated by $\sigma(\theta(s), s \leq t)$, where $\sigma(\cdot)$ is the σ-algebra generated by its argument. Set, for $i \in S$,

$$M_i(t) := \mathbb{1}_{\{\theta(t)=i\}} - \mathbb{1}_{\{\theta(0)=i\}} - \int_0^t \sum_{l=1}^{s} \lambda_{li} \mathbb{1}_{\{\theta(s-)=l\}} ds,$$

where the notation $\mathbb{1}_{\{\theta(s-)=l\}}$ represents the left-hand limit of $\mathbb{1}_{\{\theta(s)=l\}}$. Then $\{M_i(t); t \geq 0\}$ is a Martingale with respect to $\{\mathcal{F}_t; t \geq 0\}$ for any $i \in S$, where $M_i(0) = 0$ almost surely. Moreover, for any $i \in S$ with a differentiable function $f(t,i)$ taking values in $\mathbb{R}^n$, its differential form can be obtained in the Itô sense:

$$df(t,i) = \Big[f_t(t,i) + \sum_{l=1}^{s} \lambda_{il} f(t,l) \Big] dt + f(t,i) dM_i(t).$$

□

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