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Robust Decentralized Output Regulation of Heterogeneous Uncertain Linear Systems with Multiple Leaders via Distributed Adaptive Protocols

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Abstract: In the current paper we consider the robust decentralized output regulation of heterogeneous uncertain linear systems with multiple leaders. A novel class of distributed observers is proposed. The states of the distributed observers synchronize to the states of their leaders, respectively. In contrast to the existing results, we consider a more general class of systems and furthermore we utilize the adaptive protocols to estimate the coupling weights between neighboring agents online. Therefore the observers and internal model based control laws can be designed in a purely distributed way, i.e., without knowledge of the associated matrix of the network topology. Finally we apply the proposed methods to solve the synchronization problem of a group of RLC networks and the simulation results show the effectiveness of the methods.

Keywords: Adaptive protocol, internal model, linear uncertain systems, robust regulation.

1. INTRODUCTION

In this paper we consider a multi-agent system consisting of N heterogeneous uncertain linear systems or so called followers and $1 \le n \le N$ exosystems or so called leader systems. According to the n leaders, the N followers are divided into n subgroups represented by $\mathcal{N}_1,\cdots,\mathcal{N}_n$. Without loss of generality we suppose $\mathcal{N}_1 = \{1,\cdots,h_1\}$, $\mathcal{N}_2 = \{h_1+1,\cdots,h_1+h_2\},\cdots$, $\mathcal{N}_n = \{h_1+\cdots+h_{n-1}+1,\cdots,h_1+\cdots+h_{n-1}+h_n\}$. We assume $h_j \ge 1, j = 1,\cdots,n$ and $\mathcal{N}_i \cap \mathcal{N}_j = \emptyset$, $\sum_{i=1}^n h_i = N$. The j-th leader is as follows:

$$\dot{v}_j = S_j v_j, \quad j = 1,\cdots,n, \tag{1}$$

whose output is $y_{0j} = R_{0j} v_j$, where $R_{0j} \in \mathbb{R}^{p_i \times n_v}$ and $S_j \in \mathbb{R}^{n_v \times n_v}$, $j = 1,\cdots,n$. $v_j \in \mathbb{R}^{n_v}$ is the exogenous signal representing the reference input to be tracked or the disturbance to be rejected of the follower systems in $\mathcal{N}_j$, $j = 1,\cdots,n$. $\forall i \in \mathcal{N}_j$, i-th follower system takes the form:

$$\begin{aligned} \dot{x}_i &= \bar{A}_i x_i + \bar{B}_i u_i + \bar{E}_i v_j \\ y_i &= \bar{C}_i x_i + \bar{D}_i u_i + \bar{F}_i v_j, \end{aligned} \tag{2}$$

where $x_i \in \mathbb{R}^{n_i}$, $y_i \in \mathbb{R}^{p_i}$ and $u_i \in \mathbb{R}^{m_i}$ are the state, measurement output and control input of the i-th subsystem. $\bar{A}_i$, $\bar{B}_i$, $\bar{E}_i$, $\bar{C}_i$, $\bar{D}_i$, and $\bar{F}_i$ are unknown matrices and can be represented as

$$\begin{aligned} &\bar{A}_i = A_i + \Delta A_i, \bar{B}_i = B_i + \Delta B_i, \bar{E}_i = E_i + \Delta E_i, \\ &\bar{C}_i = C_i + \Delta C_i, \bar{D}_i = D_i + \Delta D_i, \bar{F}_i = F_i + \Delta F_i \end{aligned}$$

where A_i, B_i, E_i, C_i, D_i, F_i are known matrices and ΔA_i, ΔB_i, ΔE_i, ΔC_i, ΔD_i, ΔF_i are the unknown perturbations of the matrices. The control target is to design u_i such that $e_i = y_i - y_{0j} \to 0$, $i \in \mathcal{N}_j$; $j \in \{1,\cdots,n\}$, $i = 1,...,N$, when time t goes to infinity.

When only one leader exists we always call the aforementioned problem as cooperative output regulation or leader-following synchronization problem of linear system which has received a lot of attention in the last few years. In [1, 2] the cooperative output regulation problem of deterministic linear system has been solved through feedforward design. [1] assumed that all the linear subsystems are identical while [2] considered the heterogeneous linear systems. In [4–8], the robust cooperative output regulation problem of uncertain linear system were solved by introducing the internal model based control laws. In [6] the no-cycle assumption was made on the network topology and this was removed in [4]. [5] considered the switching network for some special form systems. In [7], the distributed observer was proposed and the cooperative output regulation of heterogeneous uncertain linear systems was achieved. In [8], the minimum phase system was considered by introducing the high gain feedback

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technique. The leaderless output synchronization or consensus problem of linear systems was investigated in [9–11]. However in the recent years there was an emerging trend to study the case that the group may incorporate or evolve into different subgroups which is so called clustering [3,12,13]. In addition to being cooperative, the systems belonging to different subgroups may be competitive or repulsive. Thus the problem is more challenging than the case that only one leader exists and can include the cooperative regulation as special case. It is worthy to note that the adaptive dynamic programming method is used to solve the consensus problem of multi-agent systems in [21].

In the current paper we study robust decentralized regulation of heterogeneous uncertain linear systems subject to multiple leaders. The followers are divided into several groups according to the leaders and the steady state manifolds of each subgroup are dependent on its leader only. But in addition to cooperation, the followers between different subgroups may have negative effect on each other, i.e., repulsion or competition. The main contributions of the paper are as follows:

- In this paper we deal with a more general linear system (2). To our best knowledge, all the existing literatures about robust output regulation of heterogeneous uncertain linear systems consider the special case of (2). That is $\bar{D}_i = 0$ in those literatures [3–8]. However this excludes an important class of real systems, such as RLC network in (30).
- The current distributed controller does not require the full knowledge of the associated matrix of network graph, i.e., the smallest eigenvalue of the associated matrix. This is actually a global information that is required in [1–9]. Thus the current result remarkably facilitates the local controller design which is now purely distributive.

To circumvent these two difficulties we propose novel distributed observers with adaptive coupling weights in the current paper. Therefore the states of leaders can be observed and moreover some coupling weights are estimated online such that the design of individual control law is independent of the associated matrix. Thus the control laws can be designed in a purely distributed way and the decentralized regulation of a more general class of systems as (2) can be handled.

The rest of this paper is organized as follows. In Section 2, we present some preliminaries and standard assumptions for the solvability of the problem. In Section 3, we propose our main results. In Section 4, we apply our approaches to solve a synchronization problem for a group of heterogeneous uncertain RLC systems subject to two leaders, the effectiveness of our approaches is illustrated by the simulations. Finally we close the paper with some conclusions. Throughout the paper $\otimes$ denotes the Kronecker product, $\mathbb{R}^n$ and $\mathbb{C}^n$ denote the n-dimensional real and complex vector space, respectively. x^T denotes the transpose of x and x^H denotes the conjugate transpose of x. $\mathrm{Re}(\cdot)$ means the real part of the argument.

2. PRELIMINARIES

2.1. Network topology

The N followers (2) exchange their information through a directed graph $\mathcal{G}$. Definitions and details of diagraph can be found in Appendix. Associated with the graph $\mathcal{G}$ there exist an adjacency matrix $A = [a_{ij}] \in \mathbb{R}^{N\times N}$, $a_{ij} = 1$ or -1 if there is coupling from j-th subsystem to i-th subsystem, otherwise $a_{ij} = 0$, and $a_{ii} = 0$. If $a_{ij} = 1$, a cooperative coupling is enforced, whereas if $a_{ij} = -1$, the coupling is competitive or repulsive. Associated with the graph $\mathcal{G}$, we define the Laplacian matrix $L = [l_{ij}] \in \mathbb{R}^{N\times N}$ and $L = D-A$ where $D = \mathrm{diag}(d_1,\cdots,d_N)$ where $d_i = \sum_{j=1,j\neq i}^{N} a_{ij}$. Moreover in each group $\mathcal{N}_j, j = 1,\cdots,n$, there exist at least one follower is coupled with the leader. To be consistent with matrix A, the coupling between the i-th follower and its leader is represented by a_{i0} and $a_{i0} = 1$ if they are coupled, otherwise $a_{i0} = 0, i \in \{1,\cdots,N\}$. Thus we can define a matrix $H = [h_{ij}] \in \mathbb{R}^{N\times N}$, where $h_{ii} = l_{ii} + a_{i0}$ and $h_{ij} = l_{ij}$ when $i \neq j$. Thus the matrix H and L can be written as

$$H = \begin{bmatrix} H_{11} & H_{12} & \cdots & H_{1n} \\ H_{21} & H_{22} & \cdots & H_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ H_{n1} & H_{n2} & \cdots & H_{nn} \end{bmatrix},$$

$$L = \begin{bmatrix} L_{11} & L_{12} & \cdots & L_{1n} \\ L_{21} & L_{22} & \cdots & L_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ L_{n1} & L_{n2} & \cdots & L_{nn} \end{bmatrix}.$$

We allow the competition or repulsion only happen between the followers that belong to different groups, i.e., L_{ii}, or $H_{ii}, 1 \le i \le n$ has non-positive off-diagonal elements. Associated with group $\mathcal{N}_i, 1 \le i \le n$, we can define a diagraph $\bar{\mathcal{G}}_i = \{\bar{\mathcal{V}}_i, \bar{\mathcal{E}}_i\}$, where $\bar{\mathcal{V}}_i = \{i_o, h_1 + \cdots, h_{i-1} + 1, \cdots, h_1 + \cdots, h_{i-1} + h_i\}$, i_o denotes the leader of group $\mathcal{N}_i$ and the edge set $\bar{\mathcal{E}}_i = \{(k,j), a_{jk} = 1, j,k \in \bar{\mathcal{V}}_i\}$. We make the following assumption on the graph $\bar{\mathcal{G}}_i$.

Assumption 1: The diagraph $\bar{\mathcal{G}}_i$ contains a spanning tree with i_o as root and $\mathcal{G}$ is undirected.

Remark 1: From Lemma 4 in [19], eigenvalues of H_{ii} has positive real parts if and only if $\bar{\mathcal{G}}_i$ contains a spanning tree. Since $\mathcal{G}$ is undirected, $H_{ii}, 1 \le i \le n$ is positive definite. When only one leader exists, Assumption 1 is equivalent to Assumption 3.2 in [18].

Furthermore we introduce the following assumption on $H_{ij}, i \neq j, 1 \le i,j \le n$.

Assumption 2: The matrix $H_{ij}, i \neq j, 1 \le i,j \le n$ has zero row sum.

Remark 2: Assumption 2 means the effects between different groups can vanish when leader-following synchronization in each group is achieved. Thus this guarantees the synchronization of each group with different leaders can be realized.

2.2. Distributed observer and problem formulation

In the following we propose the distributed observer as

$$\begin{aligned}\dot{\hat{v}}_i =& S_j\hat{v}_i + \sum_{k\in\mathcal{N}_j} c_{ik}(t)a_{ik}(\hat{v}_k-\hat{v}_i) + \sum_{k\notin\mathcal{N}_j} a_{ik}(\hat{v}_k-\hat{v}_i)\\ &+c_{i0}(t)a_{i0}(v_j-\hat{v}_i),\\ \dot{c}_{ik} =& a_{ik}(\hat{v}_k-\hat{v}_i)^T\Gamma(\hat{v}_k-\hat{v}_i),\ \ i,k\in\mathcal{N}_j,\\ \dot{c}_{i0} =& a_{i0}(\hat{v}_i-v_j)^T\Gamma(\hat{v}_i-v_j), i\in\mathcal{N}_j,\ \ i=1,\cdots,N,\end{aligned} \tag{3}$$

where Γ is a positive definite matrix to be verified and the initial conditions of $c_{ik}(0)$ and $c_{ki}(0)$ are the same. (3) is assigned with each follower system to observe the states of exosystems, i.e., v_j when $i\in\mathcal{N}_j$. a_{ik} is the element of adjacency matrix A and $c_{ik}(t)$ and $c_{i0}(t)$ are the coupling weights that are adapted online. It can be inferred from (3) that only the couplings between the followers in the same subgroup are adapted. This is because the introduction of the adaptive protocol is aimed to enforce the cooperations among the same subgroup.

Due to the network, the regulation error $e_i = y_i - y_{0j}$ is not available for feedback for each subsystem, we use the following variable for feedback of i-th subsystem:

$$e_{vi} = y_i - R_{0j}\hat{v}_i.$$

It can be easily seen that $e_{vi} = e_i$ when $\hat{v}_i = v_j, \forall i\in\mathcal{N}_j$.

Thus the distributed control law will take the following form:

$$\begin{aligned}u_i =& k_i(z_i,e_{vi}),\\ \dot{z}_i =& g_i(z_i,e_{vi}),\quad i=1,\cdots,N,\end{aligned} \tag{4}$$

where k_i, g_i are linear functions of their arguments. For notation convenience we introduce the variable

$$w = \operatorname{vec}\left(\left[\begin{array}{c}\Delta A_1,\cdots,\Delta A_N,\Delta B_1,\cdots,\Delta B_N,\Delta E_1,\cdots,\Delta E_N\\ \Delta C_1,\cdots,\Delta C_N,\Delta D_1,\cdots,\Delta D_N,\Delta F_1,\cdots,\Delta F_N\end{array}\right]\right).$$

Thus we give the formulation of the problem we are dealing with as

Problem 1: Given the N followers (2) subject to n leader systems (1), with the network topology $\mathcal{G}$ satisfies Assumptions 1 and 2, find a control law of the form (4) such that :
1) when $w=0$ and $v_j=0, j=1,\cdots,n$, the closed-loop system matrix is Hurwitz.
2) there exist an open neighborhood W of $w=0$, for any $w\in W$ and any conditions $x_i(0), z_i(0), i=1,\cdots,N, v_j(0), j=1,\cdots,n$, for all $i\in\mathcal{N}_j$, the regulation error $\lim_{t\to\infty} e_i(t) = y_i - y_{0j} = 0, i=1,\cdots,N$.

Remark 3: The Problem 1, when only one leader exists ($n=1$), has been studied in [4,6] and [7] with somewhat different e_{vi} in (4) and network conditions. However the systems considered in these papers are the special cases of system (2), i.e., $\bar{D}_i = 0$.

2.3. Assumptions

Next we list some assumptions that are needed for the solvability of the Problem 1. The assumptions are standard ones for the solvability of the linear robust output regulation problem of single system [17], [16].

Assumption 3: $S_j, j=1,\cdots,n$ has no eigenvalues with negative real parts.

Assumption 4: The pair (A_i,B_i) is stabilizable, $i=1,\cdots,N$.

Assumption 5: The pair (C_i,A_i) is detectable, $i=1,\cdots,N$.

Assumption 6: For all $\lambda\in\sigma(S_j), j=1,\cdots,n$, where $\sigma(S_j)$ is the spectrum of S_j, rank $\begin{bmatrix} A_i-\lambda I & B_i\\ C_i & D_i\end{bmatrix} = n_i + p_i$, for all $i\in\mathcal{N}_j$.

3. MAIN RESULTS

In the beginning of the section we introduce two lemmas that are proposed in [20] and [14], respectively.

Lemma 1: [20] Let A and B be $N\times N$ Hermitian matrices, and let the eigenvalues $\sigma_i(A), \sigma_i(B), \sigma_i(A+B)$ be arranged in increasing order as $\sigma_1(\cdot)\le\sigma_2(\cdot)\le\cdots\le\sigma_N(\cdot)$. For each $k=1,2,\cdots,N$, we have

$$\sigma_k(A)+\sigma_1(B)\le\sigma_k(A+B)\le\sigma_k(A)+\sigma_N(B).$$

Lemma 2: [14] Given any stabilizable pair (A,B), $A\in\mathbb{R}^{n\times n}, B\in\mathbb{R}^{n\times m}$, the algebraic Riccati equation $A^TP+PA+I_n-PBB^TP=0$ admits a positive definite solution. Then for any $\rho\in\mathbb{C}$ with $\mathrm{Re}(\rho)\ge 1$, $A-\rho BB^TP$ is Hurwitz. Furthermore $(A-\rho BB^TP)^HP+P(A-\rho BB^TP) = -I_n-(1+2\varepsilon)PBB^TP$, where $\varepsilon = \mathrm{Re}(\rho)-1$.

Next we will analyze the stability of the distributed observer (3).

Lemma 3: Under Assumptions 1 and 2, the state $\hat{v}_i$ of the distributed observer (3) converge to the state v_j of its leader (1), with $\Gamma = I$, as time t goes to infinity for all $i\in\mathcal{N}_j, i=1,\cdots,N; j\in\{1,\cdots,n\}$.

Proof: Introduce the variable as

$$\tilde{v}_i = \hat{v}_i - v_j,\quad i\in\mathcal{N}_j.$$

Thus we have that

$$\begin{aligned}\dot{\tilde{v}}_i =& S_j\tilde{v}_i + \sum_{k\in\mathcal{N}_j} c_{ik}(t)a_{ik}(\tilde{v}_k-\tilde{v}_i)\\ &+\sum_{k\notin\mathcal{N}_j} a_{ik}(\hat{v}_k-\hat{v}_i) - c_{i0}(t)a_{i0}\tilde{v}_i.\end{aligned} \tag{5}$$

We define Lyapunov function as

$$V_{o1}=\sum_{i=1}^{N}\tilde{v}_i^T\tilde{v}_i+\sum_{i=1}^{N}\sum_{k\in\mathcal{N}_j}\frac{(c_{ik}-\mu)^2}{2}+\sum_{i=1}^{N}(c_{i0}-\mu)^2,$$

where μ is a positive constant to be verified.

The derivative of V_{o1} along (3) and (5) is

$$\begin{aligned}\dot{V}_{o1}=&\sum_{i=1}^{N}\tilde{v}_i^T(\bar{S}_i+\bar{S}_i^T)\tilde{v}_i+\sum_{i=1}^{N}\sum_{k\in\mathcal{N}_j}2c_{ik}(t)a_{ik}\tilde{v}_i^T(\tilde{v}_k-\tilde{v}_i)\\&+\sum_{i=1}^{N}\sum_{k\notin\mathcal{N}_j}2a_{ik}\tilde{v}_i^T(\hat{v}_k-\hat{v}_i)-\sum_{i=1}^{N}2c_{i0}(t)a_{i0}\tilde{v}_i^T\tilde{v}_i\\&+\sum_{i=1}^{N}\sum_{k\in\mathcal{N}_j}(c_{ik}-\mu)a_{ik}(\tilde{v}_k-\tilde{v}_i)^T\Gamma(\tilde{v}_k-\tilde{v}_i)\\&+\sum_{i=1}^{N}(c_{i0}-\mu)2a_{i0}\tilde{v}_i^T\Gamma\tilde{v}_i,\end{aligned}\tag{6}$$

where $\bar{S}_i=S_j$ when $i\in\mathcal{N}_j$. Due to the fact that

$$\begin{aligned}&\sum_{i=1}^{N}\sum_{k\in\mathcal{N}_j}(c_{ik}-\mu)a_{ik}(\tilde{v}_k-\tilde{v}_i)^T\Gamma(\tilde{v}_k-\tilde{v}_i)\\&=2\sum_{i=1}^{N}\sum_{k\in\mathcal{N}_j}(c_{ik}-\mu)a_{ik}\tilde{v}_i^T\Gamma(\tilde{v}_i-\tilde{v}_k)\end{aligned}$$

this equality holds because $a_{ik}=a_{ki}$ and $c_{ik}(t)=c_{ki}(t)$, $c_{ik}(t)$ are cancelled out from (6). And H_{ij} is zero-row-sum matrix,

$$\sum_{k\notin\mathcal{N}_j}a_{ik}=0,\ \ i=1,\cdots,N.$$

It follows that

$$\sum_{i=1}^{N}\sum_{k\notin\mathcal{N}_j}2a_{ik}\tilde{v}_i^T\Gamma(\hat{v}_k-\hat{v}_i)=\sum_{i=1}^{N}\sum_{k\notin\mathcal{N}_j}2a_{ik}\tilde{v}_i^T\Gamma(\tilde{v}_k-\tilde{v}_i)$$

and we choose $\Gamma=I$, it gives

$$\begin{aligned}\dot{V}_{o1}=&\sum_{i=1}^{N}\tilde{v}_i^T(\bar{S}_i+\bar{S}_i^T)\tilde{v}_i+\sum_{i=1}^{N}\sum_{k\in\mathcal{N}_j}2\mu a_{ik}\tilde{v}_i^T(\tilde{v}_k-\tilde{v}_i)\\&+\sum_{i=1}^{N}\sum_{k\notin\mathcal{N}_j}2a_{ik}\tilde{v}_i^T(\tilde{v}_k-\tilde{v}_i)-\sum_{i=1}^{N}2\mu a_{i0}\tilde{v}_i^T\tilde{v}_i\\=&\tilde{v}^T[D_S-D_H\otimes I_{n_v}-\check{D}\otimes I_{n_v}]\tilde{v},\end{aligned}\tag{7}$$

where $D_S=\mathrm{diag}\{\bar{S}_1+\bar{S}_1^T,\cdots,\bar{S}_N+\bar{S}_N^T\}$, $D_H=\mathrm{diag}\{2\mu H_{11},2\mu H_{22},\cdots,2\mu H_{nn}\}$ and

$$\check{D}=2\begin{bmatrix}0&H_{12}&\cdots&H_{1n}\\H_{21}&0&\cdots&H_{2n}\\\cdots&\cdots&\cdots&\cdots\\H_{n1}&H_{n2}&\cdots&0\end{bmatrix}.$$

We define

$$L_H=D_H+\check{D}.$$

Then, (7) changes to

$$\dot{V}_{o1}=\tilde{v}^T[D_S-L_H\otimes I_{n_v}]\tilde{v}.\tag{8}$$

According to Lemma 1 we choose μ sufficiently large such that L_H is positive definite and hence there exist an unitary matrix U such that $U^TL_HU=\Lambda=\mathrm{diag}(\lambda_1,\lambda_2,\cdots,\lambda_N)$, where $\lambda_i, i=1,\cdots,N$ is eigenvalue of L_H.

We introduce the new variable

$$\xi=(U\otimes I_{n_v})\tilde{v}$$

and thus (8) changes to

$$\begin{aligned}\dot{V}_{o1}=&\xi^T[D_{us}-\Lambda\otimes I_{n_v}]\xi=-\sum_{i=1}^{N}\lambda_i\|\xi_i\|^2+\xi^TD_{us}\xi\\&\le-\underline{\lambda}\|\xi\|^2+\xi^TD_{us}\xi,\end{aligned}\tag{9}$$

where $D_{us}=(U\otimes I_{n_v})D_S(U^T\otimes I_{n_v})$ and $\underline{\lambda}=\min_{i=1,\cdots,N}\lambda_i$. Therefore according to Lemma 1, $\underline{\lambda}$ can be sufficiently large by choosing μ sufficiently large. Here we choose μ sufficiently large such that $\underline{\lambda}=\|D_{us}\|+1$. From (9) it follows that $\dot{V}_{o1}\le-\|\xi\|^2=-\|\tilde{v}\|^2$. Hence by Lasalle's Invariance Principle, $\tilde{v}\to 0$. □

Remark 4: From the preceding proof we know that $\lim_{t\to\infty}\tilde{v}=0$ and in (3)

$$\begin{aligned}\dot{c}_{ik}=&a_{ik}(\hat{v}_k-\hat{v}_i)^T\Gamma(\hat{v}_k-\hat{v}_i), i,k\in\mathcal{N}_j\\=&a_{ik}[(\hat{v}_k-v_j)-(\hat{v}_i-v_j)]^T\Gamma[(\hat{v}_k-v_j)\\&-(\hat{v}_i-v_j)], i,k\in\mathcal{N}_j\\=&a_{ik}[\tilde{v}_k-\tilde{v}_i]^T\Gamma[\tilde{v}_k-\tilde{v}_i], i,k\in\mathcal{N}_j\end{aligned}$$

Thus $\lim_{t\to\infty}\dot{c}_{ik}(t)=0$ and hence c_{ik} converges to constant.

Remark 5: It can be seen that the stability analysis of the distributed observer (3) is independent of control law and thus it has potential to be used in other cases. If we assume $S_1=S_2=\cdots=S_n=S_0$, $R_{01}=R_{02}=\cdots=R_{0n}=R_0$ and the pair (R_0,S_0) is detectable, i.e., there exists a matrix Q such that S_0-QR_0 is Hurwitz, the observer (3) can be modified as

$$\begin{aligned}\dot{\hat{v}}_i=&S_0\hat{v}_i+\sum_{k\in\mathcal{N}_j}c_{ik}(t)a_{ik}QR_0(\hat{v}_k-\hat{v}_i)\\&+\sum_{k\notin\mathcal{N}_j}a_{ik}QR_0(\hat{v}_k-\hat{v}_i)+c_{i0}(t)a_{i0}QR_0(v_j-\hat{v}_i),\\\dot{c}_{ik}=&a_{ik}(\hat{v}_k-\hat{v}_i)^TR_0^T\Gamma R_0(\hat{v}_k-\hat{v}_i),\ \ i,k\in\mathcal{N}_j,\\\dot{c}_{i0}=&a_{i0}(\hat{v}_i-v_j)^TR_0^T\Gamma R_0(\hat{v}_i-v_j),\ \ i\in\mathcal{N}_j, i=1,\cdots,N.\end{aligned}\tag{10}$$

Following steps in the proof of Lemma 3, we have that

$$\begin{aligned}\dot{\tilde{v}}_i=&S_0\tilde{v}_i+\sum_{k\in\mathcal{N}_j}c_{ik}(t)a_{ik}QR_0(\tilde{v}_k-\tilde{v}_i)\\&+\sum_{k\notin\mathcal{N}_j}a_{ik}QR_0(\hat{v}_k-\hat{v}_i)-c_{i0}(t)a_{i0}QR_0\tilde{v}_i.\end{aligned}\tag{11}$$

We define Lyapunov function as

$$V_{o2} = \sum_{i=1}^{N} \tilde{v}_i^T P^{-1} \tilde{v}_i + \sum_{i=1}^{N} \sum_{k\in\mathcal{N}_j} \frac{(c_{ik}-\mu)^2}{2} + \sum_{i=1}^{N} (c_{i0}-\mu)^2,$$

where P is a positive definite matrix, μ is a positive constant to be verified.

The derivative of V_{o2} along (10) and (11) is

$$\begin{aligned}\dot{V}_{o2} =& \sum_{i=1}^{N} 2\tilde{v}_i^T P^{-1} S_0 \tilde{v}_i + \sum_{i=1}^{N} \sum_{k\notin\mathcal{N}_j} 2a_{ik}\tilde{v}_i^T P^{-1} QR_0(\hat{v}_k - \hat{v}_i) \\ &+ \sum_{i=1}^{N} \sum_{k\in\mathcal{N}_j} 2c_{ik}a_{ik}\tilde{v}_i^T P^{-1} QR_0(\tilde{v}_k - \tilde{v}_i) \\ &- \sum_{i=1}^{N} 2c_{i0}a_{i0}\tilde{v}_i^T P^{-1} QR_0 \tilde{v}_i \\ &+ \sum_{i=1}^{N} \sum_{k\in\mathcal{N}_j} (c_{ik}-\mu)a_{ik}(\tilde{v}_k-\tilde{v}_i)^T R_0^T \Gamma R_0(\tilde{v}_k-\tilde{v}_i) \\ &+ \sum_{i=1}^{N} (c_{i0}-\mu)2a_{i0}\tilde{v}_i^T R_0^T \Gamma R_0 \tilde{v}_i. \end{aligned} \tag{12}$$

Due to the fact that

$$\begin{aligned}&\sum_{i=1}^{N} \sum_{k\in\mathcal{N}_j} (c_{ik}-\mu)a_{ik}(\tilde{v}_k-\tilde{v}_i)^T R_0^T \Gamma R_0(\tilde{v}_k-\tilde{v}_i) \\ &= 2\sum_{i=1}^{N} \sum_{k\in\mathcal{N}_j} (c_{ik}-\mu)a_{ik}\tilde{v}_i^T R_0^T \Gamma R_0(\tilde{v}_i-\tilde{v}_k)\end{aligned}$$

and

$$\begin{aligned}&\sum_{i=1}^{N} \sum_{k\notin\mathcal{N}_j} 2a_{ik}\tilde{v}_i^T R_0^T \Gamma R_0(\hat{v}_k-\hat{v}_i) \\ &= \sum_{i=1}^{N} \sum_{k\notin\mathcal{N}_j} 2a_{ik}\tilde{v}_i^T R_0^T \Gamma R_0(\tilde{v}_k-\tilde{v}_i)\end{aligned}$$

and if we can choose Γ such that $R_0^T \Gamma R_0 = P^{-1} QR_0$, it will give

$$\begin{aligned}\dot{V}_{o2} =& \sum_{i=1}^{N} 2\tilde{v}_i^T P^{-1} S_0 \tilde{v}_i + \sum_{i=1}^{N} \sum_{k\notin\mathcal{N}_j} 2a_{ik}\tilde{v}_i^T P^{-1} QR_0(\tilde{v}_k - \tilde{v}_i) \\ &+ \sum_{i=1}^{N} \sum_{k\in\mathcal{N}_j} 2\mu a_{ik}\tilde{v}_i^T P^{-1} QR_0(\tilde{v}_k - \tilde{v}_i) \\ &- \sum_{i=1}^{N} 2\mu a_{i0}\tilde{v}_i^T P^{-1} QR_0 \tilde{v}_i \end{aligned} \tag{13}$$

We introduce new variable $\check{v}_i = P^{-1}\tilde{v}_i$ and hence (13) changes to

$$\begin{aligned}\dot{V}_{o2} =& \sum_{i=1}^{N} 2\check{v}_i^T S_0 P \check{v}_i + \sum_{i=1}^{N} \sum_{k\in\mathcal{N}_j} 2\mu a_{ik}\check{v}_i^T QR_0 P(\check{v}_k - \check{v}_i) \\ &+ \sum_{i=1}^{N} \sum_{k\notin\mathcal{N}_j} 2a_{ik}\check{v}_i^T QR_0 P(\check{v}_k - \check{v}_i) - \sum_{i=1}^{N} 2\mu a_{i0}\check{v}_i^T QR_0 P \check{v}_i \\ =& \check{v}^T [I_N \otimes 2S_0 P - L_H \otimes QR_0 P]\check{v}, \end{aligned} \tag{14}$$

where $\check{v} = [\check{v}_1^T, \cdots, \check{v}_N^T]^T$, and we define $\zeta = (U \otimes I_{n_v})\check{v}$,

$$\begin{aligned}\dot{V}_{o2} &= \zeta^T [I_N \otimes 2S_0 P - \Lambda \otimes QR_0 P]\zeta \\ &= \sum_{i=1}^{N} \zeta_i^T [(S_0 - \frac{\lambda_i}{2} QR_0)P + P(S_0 - \frac{\lambda_i}{2} QR_0)^T]\zeta_i.\end{aligned} \tag{15}$$

Due to the fact that (S_0^T, R_0^T) is stabilizable, according to Lemma 2, there exists a positive definite matrix P_0 such that

$$S_0 P_0 + P_0 S_0^T + I_{n_v} - P_0 R_0^T R_0 P_0 = 0. \tag{16}$$

And hence, for any $\rho \in \mathbb{C}$ with $\mathrm{Re}(\rho) \geq 1$, $S_0^T - \rho R_0^T R_0 P_0$ is Hurwitz and P_0 satisfies $(S_0 - \rho P_0 R_0^T R_0)P_0 + P_0(S_0 - \rho P_0 R_0^T R_0)^T = -I_n - (1+2\varepsilon)P_0 R_0^T R_0 P_0$ where $\varepsilon = \mathrm{Re}(\rho) - 1$. Thus we can choose $P = P_0$, $Q = P_0 R_0^T$ and hence $S_0 - \frac{\lambda_i}{2} QR_0$ is Hurwitz if we choose μ sufficiently large such that $\underline{\lambda} \geq 2$, it gives

$$\begin{aligned}\dot{V}_{o2} &= \sum_{i=1}^{N} \zeta_i^T [(S_0 - \frac{\lambda_i}{2} QR_0)P + P(S_0 - \frac{\lambda_i}{2} QR_0)^T]\zeta_i \\ &= \sum_{i=1}^{N} -\zeta_i^T [I_{n_v} + (1+2\varepsilon_i)P_0 R_0^T R_0 P_0]\zeta_i \leq -\|\zeta\|^2,\end{aligned} \tag{17}$$

where $\varepsilon_i = \frac{\lambda_i}{2} - 1 \geq 0$. Therefore by Lasalle's Invariance Principle, $\zeta \to 0$ and hence $\tilde{v} \to 0$. Thus we have the following lemma.

Lemma 4: Under Assumptions 1 and 2, $S_1 = S_2 = \cdots = S_n = S_0$ and $R_{01} = R_{02} = \cdots = R_{0n} = R_0$, the pair (R_0, S_0) is detectable, the state $\hat{v}_i$ of the distributed observer (10) converges to the state v_j of its leader (1) for all $i \in \mathcal{N}_j$, $i = 1, \cdots, N; j \in \{1, \cdots, n\}$, with $\Gamma = I$ and $Q = P_0 R_0^T$ where P_0 is the solution of the Ricatti equation (16).

Remark 6: Roughly speaking, observer (3) is based on state feedback while (10) is based on output feedback. However Lemma 4 requires more restrictions on the dynamics of the exosystem, i.e., $S_1 = S_2 = \cdots = S_n = S_0$ and $R_{01} = R_{02} = \cdots = R_{0n} = R_0$, the pair (R_0, S_0) should be detectable.

Remark 7: If we further assume that the associated matrix H is known to all subsytems, i.e., μ can be implemented in the distributed observers design, the observer (10) can be modified as

$$\begin{aligned}\dot{\hat{v}}_i =& S_0 \hat{v}_i + \mu \sum_{k\in\mathcal{N}_j} a_{ik} QR_0(\hat{v}_k - \hat{v}_i) + \sum_{k\notin\mathcal{N}_j} a_{ik} QR_0(\hat{v}_k - \hat{v}_i) \\ &+ \mu a_{i0} QR_0(v_j - \hat{v}_i).\end{aligned} \tag{18}$$

And thus we have that

$$\dot{\tilde{v}} = (I_N \otimes S_0 - \frac{L_H}{2} \otimes QR_0)\tilde{v} \tag{19}$$

and with $\xi = (U \otimes I_m)\tilde{v}$,

$$\dot{\xi} =(I_N \otimes S_0 - \frac{\Lambda}{2} \otimes QR_0)\xi. \tag{20}$$

Based on the arguments in Remark 5 and μ is chosen such that $\underline{\lambda} = 2$ and $Q = P_0R_0^T$ where P_0 is the solution of (16), it follows that

$$I_N \otimes S_0 - \frac{\Lambda}{2} \otimes QR_0$$

is Hurwitz. Thus the states of the observer (18) exponentially converge to states of their leaders, respectively. Then we have the following lemma.

Lemma 5: Under Assumptions 1 and 2, and $S_1 = S_2 = \cdots = S_n = S_0$ and $R_{01} = R_{02} = \cdots = R_{0n} = R_0$, the pair (R_0, S_0) is detectable, the state $\hat{v}_i$ of the distributed observer (18) converge to the state v_j of its leader (1) exponentially for all $i \in \mathcal{N}_j$, $i = 1, \cdots, N$; $j \in \{1, \cdots, n\}$, with a sufficiently large μ and $Q = P_0R_0^T$ where P_0 is the solution of the Ricatti equation (16).

Next we will give internal model based distributed dynamic state feedback control law and output feedback control law. In the following we will give the definition of p-copy internal model [16, 17].

Definition 1: A pair of matrices (G_1, G_2) is said to incorporate the minimum p-copy internal model of the matrix S if

$$G_1 = \text{blockdiag}\underbrace{[\beta, \cdots, \beta]}_{p-tuple}, \quad G_2 = \text{blockdiag}\underbrace{[\sigma, \cdots, \sigma]}_{p-tuple},$$

where β is a constant square matrix whose characteristic polynomial equals the minimal polynomial of S, and σ is a constant column vector such that (β, σ) is controllable.

Thus our distributed state feedback control law will be chosen as

$$\begin{aligned} u_i &= K_{1i}x_i + K_{2i}z_i, \\ \dot{z}_i &= G_{1i}z_i + G_{2i}e_{vi} \quad i = 1, \cdots, N, \end{aligned} \tag{21}$$

where $z_i \in \mathbb{R}^{n_{z_i}}$ and K_{1i}, K_{2i} are matrices with appropriate dimensions and (G_{1i}, G_{2i}) incorporate the minimum p-copy internal model of the matrix $S_j, \forall i \in \mathcal{N}_j, j \in \{1, \cdots, n\}$. The distributed output feedback control law is chosen as

$$\begin{aligned} u_i &= [\begin{array}{cc} K_{1i} & K_{2i} \end{array}]z_i \triangleq \mathcal{K}_i z_i, \\ \dot{z}_i &= \begin{bmatrix} A_i + B_iK_{1i} - L_i(C_i + D_iK_{1i}) & (B_i - L_iD_i)K_{2i} \\ 0 & G_{1i} \end{bmatrix} z_i \\ &\quad + \begin{bmatrix} L_i \\ G_{2i} \end{bmatrix} e_{vi}, \\ &\triangleq \mathcal{G}_{1i}z_i + \mathcal{G}_{2i}e_{vi}, \quad i = 1, \cdots, N, \end{aligned} \tag{22}$$

where L_i is matrix with appropriate dimension such that $A_i - L_iC_i$ is Hurwitz. Then the closed-loop system consisting of the system of (2) and controller (21) or (22) is

$$\begin{aligned} \begin{bmatrix} x_i \\ z_i \end{bmatrix}' &= \begin{bmatrix} \bar{A}_i + \bar{B}_iK_{1i} & \bar{B}_iK_{2i} \\ G_{2i}(\bar{C}_i + \bar{D}_iK_{1i}) & G_{1i} + G_{2i}\bar{D}_iK_{2i} \end{bmatrix} \begin{bmatrix} x_i \\ z_i \end{bmatrix} \\ &\quad + \begin{bmatrix} \bar{E}_i \\ G_{2i}(\bar{F}_i - R_{0j}) \end{bmatrix} v_j + \begin{bmatrix} 0 \\ -G_{2i}R_{0j} \end{bmatrix} \tilde{v}_i \\ &\triangleq A_{wi}^s \begin{bmatrix} x_i \\ z_i \end{bmatrix} + E_{1i}^s v_j + E_{2i}^s \tilde{v}_i \end{aligned} \tag{23}$$

and

$$\begin{aligned} \begin{bmatrix} x_i \\ z_i \end{bmatrix}' &= \begin{bmatrix} \bar{A}_i & \bar{B}_i\mathcal{K}_i \\ \mathcal{G}_2\bar{C}_i & \mathcal{G}_1 + \mathcal{G}_2\bar{D}_i\mathcal{K}_i \end{bmatrix} \begin{bmatrix} x_i \\ z_i \end{bmatrix} \\ &\quad + \begin{bmatrix} \bar{E}_i \\ \mathcal{G}_2(\bar{F}_i - R_{0j}) \end{bmatrix} v_j + \begin{bmatrix} 0 \\ \mathcal{G}_2R_{0j} \end{bmatrix} \tilde{v}_i \quad (24) \\ &\triangleq A_{wi}^o \begin{bmatrix} x_i \\ z_i \end{bmatrix} + E_{1i}^o v_j + E_{2i}^o \tilde{v}_i. \end{aligned}$$

The nominal parts A_{0i}^s and A_{0i}^o of A_{wi}^s and A_{wi}^o are

$$A_{0i}^s = \begin{bmatrix} A_i + B_iK_{1i} & B_iK_{2i} \\ G_{2i}(C_i + D_iK_{1i}) & G_{1i} + G_{2i}D_iK_{2i} \end{bmatrix},$$

$$A_{0i}^o = \begin{bmatrix} A_i & B_iK_{1i} & B_iK_{2i} \\ L_iC_i & A_i + B_iK_{1i} - L_iC_i & B_iK_{2i} \\ G_{2i}C_i & G_{2i}D_iK_{1i} & G_{1i} + G_{2i}D_iK_{2i} \end{bmatrix}.$$

Due to Assumptions 3, 4, 6 and Lemma 1.26 in [16], there exist K_{1i}, K_{2i} such that A_{0i}^s is Hurwitz. We define transformation matrix T as

$$T = \begin{bmatrix} I_{n_i} & 0 & 0 \\ 0 & 0 & I_{n_i} \\ -I_{n_i} & I_{n_i} & 0 \end{bmatrix}.$$

And hence we have

$$TA_{0i}^oT^{-1} = \begin{bmatrix} A_i + B_iK_{1i} & B_iK_{2i} & B_iK_{1i} \\ G_{2i}(C_i + D_iK_{1i}) & G_{1i} + G_{2i}D_iK_{2i} & G_{2i}D_iK_{1i} \\ 0 & 0 & A_i - L_iC_i \end{bmatrix}.$$

Thus the spectrum of A_{0i}^o is given by those of $A_i - L_iC_i$ and A_{0i}^s. Then there exist a neighborhood W of $w = 0$ such that the matrix A_{wi}^s or A_{wi}^o is Hurwitz.

Without generality, the equations (23) and (24) can be written as an uniform equation as

$$\dot{\eta}_i = A_{wi}\eta_i + E_{1i}v_j + E_{2i}\tilde{v}_i,$$

where $\eta_i = \text{col}(x_i, z_i)$, $A_{wi} = A_{wi}^s$ or A_{wi}^o, $E_{1i} = E_{1i}^s$ or E_{1i}^o, $E_{2i} = E_{2i}^s$ or E_{2i}^o, respectively. Due to Assumptions 3, 6 and Lemma 1.27 in [16], there exist $X_i(w), Z_i(w)$ such that Francis's equations

$$\begin{bmatrix} X_i(w) \\ Z_i(w) \end{bmatrix} S_j = A_{wi} \begin{bmatrix} X_i(w) \\ Z_i(w) \end{bmatrix} + E_{1i} \tag{25}$$

hold when $w \in W$ and $i \in \mathcal{N}_j$. And moreover, $\forall i \in \mathcal{N}_j$, when $A_{wi} = A^s_{wi}$,

$$0 = (\bar{C}_i + \bar{D}_i K_{1i}) X_i(w) + \bar{D}_i K_{2i} Z_i(w) + \bar{F}_i - R_{0j} \quad (26)$$

and when $A_{wi} = A^o_{wi}$,

$$0 = \bar{C}_i X_i(w) + \bar{D}_i \mathcal{K}_i Z_i(w) + \bar{F}_i - R_{0j}. \quad (27)$$

It can be inferred from (26) and (27) that when the states of system (2) and controller (21) or (22) are $X_i(w)v_j$ and $Z_i(w)v_j$, respectively, the regulation errors $e_i, i = 1, \cdots, N$ is zero.

We define new variables as $\bar{\eta}_i = \mathrm{col}(X_i(w)v_j, Z_i(w)v_j)$ and

$$\tilde{\eta}_i = \eta_i - \bar{\eta}_i.$$

Thus

$$\dot{\tilde{\eta}}_i = A_{wi}\tilde{\eta}_i + E_{2i}\tilde{v}_i. \quad (28)$$

We define the Lyapunov function as

$$V_e = \sum_{i=1}^{N} \tilde{\eta}_i^T P_i \tilde{\eta}_i,$$

where $P_i A_{wi} + A_{wi} P_i = -I$, whose derivative along (28) satisfies

$$\dot{V}_e = -\|\tilde{\eta}\|^2 + \sum_{i=1}^{N} 2\tilde{\eta}_i^T P_i E_{2i}\tilde{v}_i \leq -\frac{3}{4}\|\tilde{\eta}\|^2 + \delta_1\|\tilde{v}\|^2, \quad (29)$$

where $\delta_1 = \max_{1\leq i\leq N} \|2P_i E_{2i}\|^2$. Thus we have the following theorem.

Theorem 1: Under conditions of Lemma 3 and Assumptions 3, 4, 5, 6 hold, the robust decentralized output regulation problem of (2) subject to the multiple exosystems (1) can be solved by the distributed control law consisting of (3) and (21) or (22).

Proof: We define the Lyapunov function for the whole closed-loop system as

$$V_1 = (\delta_1 + 1)V_{o1} + V_e,$$

whose derivative along the closed-loop system is

$$\dot{V}_1 \leq -(\delta_1+1)\|\tilde{v}\|^2 - \frac{3}{4}\|\tilde{\eta}\|^2 + \delta_1\|\tilde{v}\|^2 = -\|\tilde{v}\|^2 - \frac{3}{4}\|\tilde{\eta}\|^2.$$

Thus by Lasalle's Invariance Principle we have that $\tilde{\eta} \to 0, \tilde{v} \to 0$. $e_i \to 0$ as time t goes to infinity. □

Theorem 2: Under conditions of Lemma 4 and Assumptions 3,4,5,6 hold, the robust decentralized output regulation problem of (2) subject to the multiple exosystems (1) can be solved by the distributed control law consisting of (10) and (21) or (22).

Proof: We define the Lyapunov function for the whole closed-loop system as

$$V_2 = \varpi V_{o2} + V_e,$$

where ϖ is a positive constant to be verified, the derivative of V_2 along the closed-loop system is

$$\begin{aligned}\dot{V}_2 &\leq -\varpi\|\zeta\|^2 - \frac{3}{4}\|\tilde{\eta}\|^2 + \delta_1\|\tilde{v}\|^2 \\ &= -\varpi\tilde{v}^T(I\otimes(P^{-1})^2)\tilde{v} - \frac{3}{4}\|\tilde{\eta}\|^2 + \delta_1\|\tilde{v}\|^2 \\ &\leq -(\varpi\delta_2 - \delta_1)\|\tilde{v}\|^2 - \frac{3}{4}\|\tilde{\eta}\|^2,\end{aligned}$$

where δ_2 is the smallest eigenvalue of $I\otimes(P^{-1})^2$. By choosing $\varpi = (\delta_1+1)/\delta_2$, we can achieve that

$$\dot{V}_2 \leq -\|\tilde{v}\|^2 - \frac{3}{4}\|\tilde{\eta}\|^2.$$

Thus by Lasalle's Invariance Principle we have that $\tilde{\eta} \to 0, \tilde{v} \to 0$. Thus $e_i \to 0$ as time t goes to infinity. □

It can be seen that if the distributed observer (18) is implemented, the observer error $\tilde{v}_i$ in (28) converges to origin exponentially. Thus the regulation errors $e_i, i = 1, \cdots, n$ converges to origin exponentially. Therefore we have the following theorem.

Theorem 3: Under conditions of Lemma 5 and Assumptions 3,4,5,6 hold, the robust decentralized output regulation problem of (2) subject to the multiple exosystems (1) can be solved by the distributed control law consisting of (18) and (21) or (22).

Remark 8: The work in [3] is closest to the current paper, i.e., it considers the decentralized output regulation of heterogenous uncertain linear systems under multiple leaders or so called reference signals in [3]. There are, however, three differences. First, the systems considered in [3] are the special forms of (2), i.e., $\bar{E}_i = 0, \bar{D}_i = 0, \bar{F}_i = 0$ in [3]. Moreover, $S_j, j = 1, \cdots, n$ of the multiple leaders (1) can be different matrices with equal dimensions in Theorem 1. While in [3], the matrices of the multiple leaders should be totally same. Second, the distributed control laws proposed in Theorem 1 or Theorem 2 are independent of the associated matrix of the network diagraph due to the introduction of adaptive protocol. While this is not the case in [3], i.e., the design parameter $c_j, j = 1, \cdots, n$ in (20) of [3] are dependent on the associated matrix. Therefore the current design is in a purely decentralized way. Third, even the matrix H is known to all subsystems, i.e., (18) can be implemented, the order of (18) is still lower than the order of observer (20) proposed in [3]. There are also two differences in terms of the proved technical properties of the control laws. First, the uncertainty range W in [3] can be any given arbitrarily large compact set and the control laws in [3] rely on the high gain feedback, thus the systems considered in [3] should be minimum phase. While in the current paper, the control laws are constructed such that the matrix of closed-loop system is Hurwitz and thus the uncertainty range W is a neighborhood of origin but the systems (2) can be non-minimum phase. Second due to the introduction of adaptive protocols, the proposed distributed observers (3) and (10) are essentially nonlinear while the observer proposed in [3] is linear.

4. EXAMPLES

In this section we apply the proposed methods to a group of RLC networks [15]. In the first we consider the group consisting of six RLC networks and two leaders as in (1) are

$$\dot{v}_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} v_1,\ \ \dot{v}_2 = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} v_2$$

and output matrices $R_{01} = I, R_{02} = [1 \quad 0]$. Thus the whole group is divided into two groups. The electronic circuit diagrams in two subgroups are given in Fig. 1. The state-space equation of the network in first group is

$$\begin{aligned}
\begin{bmatrix} x_{i1} \\ x_{i2} \\ x_{i3} \end{bmatrix}' &= \begin{bmatrix} -\frac{1}{R_1C_1} & 0 & -\frac{1}{C_1} \\ 0 & 0 & \frac{1}{C_2} \\ \frac{1}{L} & -\frac{1}{L} & -\frac{R_2}{L} \end{bmatrix} \begin{bmatrix} x_{i1} \\ x_{i2} \\ x_{i3} \end{bmatrix} \\
&\quad + \begin{bmatrix} \frac{1}{C_1R_1} & \frac{1}{C_1} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_{i1} \\ u_{i2} \end{bmatrix}, \\
\begin{bmatrix} y_{i1} \\ y_{i2} \end{bmatrix} &= \begin{bmatrix} 1 & -1 & -R_2 \\ -\frac{1}{R_1} & 0 & 0 \end{bmatrix} \begin{bmatrix} x_{i1} \\ x_{i2} \\ x_{i3} \end{bmatrix} \\
&\quad + \begin{bmatrix} 0 & 0 \\ \frac{1}{R_1} & 0 \end{bmatrix} \begin{bmatrix} u_{i1} \\ u_{i2} \end{bmatrix},
\end{aligned} \tag{30}$$

$i = 1, 2, 3, 4$, and x_{i1}, x_{i2}, x_{i3} represent the voltages across capacitors C_1, C_2 and current through inductor L, respectively. Control inputs u_{i1} and u_{i2} are the voltage source and current source, respectively. Outputs y_{i1} and y_{i2} are the voltage across inductor L and current through resistor R_1, respectively. The state-space equation of the followers in second group is

$$\begin{aligned}
\begin{bmatrix} x_{i1} \\ x_{i2} \\ x_{i3} \end{bmatrix}' &= \begin{bmatrix} -\frac{1}{R_1C_1} & 0 & -\frac{1}{C_1} \\ 0 & 0 & \frac{1}{C_2} \\ \frac{1}{L} & -\frac{1}{L} & 0 \end{bmatrix} \begin{bmatrix} x_{i1} \\ x_{i2} \\ x_{i3} \end{bmatrix} \\
&\quad + \begin{bmatrix} \frac{1}{C_1R_1} \\ 0 \\ 0 \end{bmatrix} u_i \\
y_i &= \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_{i1} \\ x_{i2} \\ x_{i3} \end{bmatrix}
\end{aligned} \tag{31}$$

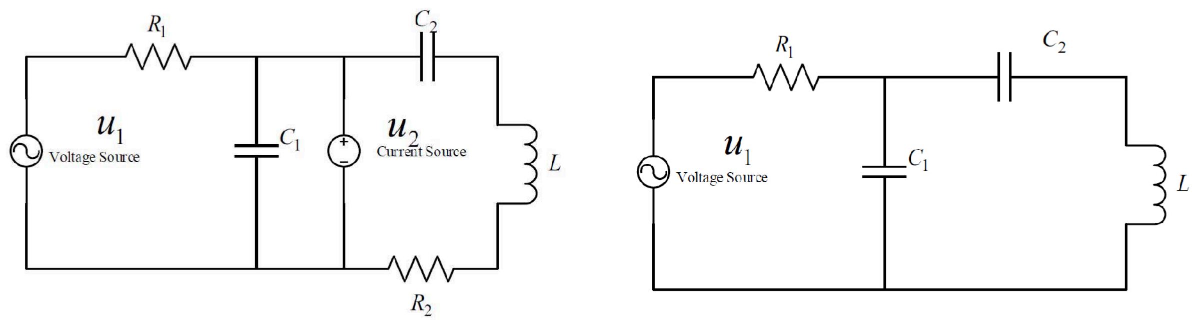
$i = 5, 6$, and x_{i1}, x_{i2}, x_{i3} represent the voltages across capacitors C_1, C_2 and current through inductor L, respectively. Control input u_i is the voltage source. Output y_i is the voltage across inductor L. Here the distributed observer (3) is implemneted since $S_1 \neq S_2$, and we design $u_i, i = 1,2,3,4$, such that $y_i \to v_1$ and $u_i, i = 5,6$, such that $y_i \to v_{21}$. The network diagraph is showed in Fig. 2. The solid lines in Fig. 2 between followers mean cooperation and the dot lines mean repulsion or competetion. Thus the marix H is defined as

$$H = \left(\begin{array}{c|c} H_{11} & H_{12} \\ \hline H_{21} & H_{22} \end{array}\right) = \left(\begin{array}{cccc|cc} 3 & -1 & 0 & -1 & 1 & -1 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ -1 & 0 & -1 & 2 & 0 & 0 \\ \hline 1 & 0 & -1 & 0 & 2 & -1 \\ -1 & 0 & 1 & 0 & -1 & 1 \end{array}\right)$$

The nominal values of R_1, R_2, L, C_1, C_2 are 2Ω, 1Ω, $2H$, $3F$ and $1F$, respectively. Hence Assumptions 3 $\sim$ 6 are satisfied. Due to the temperature effect, the resistors are undergoing one percent perturbations. In Fig. 3, we can see that the observation errors converge to origin. In Fig. 4, we can see that the regulation errors converge to origin. In Fig. 5, we can see that the adaptive coupling weights $c_{ik}(t)$ converge to constants and with identical initial conditions of $c_{ik}(t)$ and $c_{ki}(t)$, they converge to same constants.

Next we will consider the case that the two leaders take the same forms as $\dot{v} = S_0 v$ where $S_0 = [0 \quad 1; -1 \quad 0]$ and the output matrix $R_0 = [1 \quad 3; 4 \quad 7]$. The six followers take the same form as (30). The diagraph is portrayed in Fig. 2 and the nominal values of the parameters in the first case are still used in the current case. The resistors are undergoing one percent perturbations. Thus the observer (10) can be implemented with $Q = [-0.4735 \quad 0.0412; 0.7486 \quad 1.1017]$. It can be seen from Fig. 6, 7, 8 that the observer errors converge to origin and the adapted coupling weights $c_{ik}(t)$ converge to constants.

Finally we will compare the current method with the static coupling gain method in [3]. However since $D \neq 0$ in (30), the system matrices of the two leaders are not same and (31) is not minimum phase, the design method proposed in [3] cannot be directly applied to the two systems. However we can adopt the static coupling gain method in [3] and we set c_{i0}, c_{ik} and c_{ki} as constant c in (3). The constant c should be large enough. Based on the proof of Lemma 3, c should be chosen such that $\underline{\lambda} \geq \|D_s\| + 1$ and here we choose $c = 20$. It is noted that the current D_H is the D_H with μ being replaced by c. It can be seen from Fig. 9 that the regulation errors converge to zero. However we need the exact knowledge of D_s and H that is actually a global information.



(a) RLC networks in group 1.

(b) RLC networks in group 2.

Fig. 1. RLC networks of two subgroups.

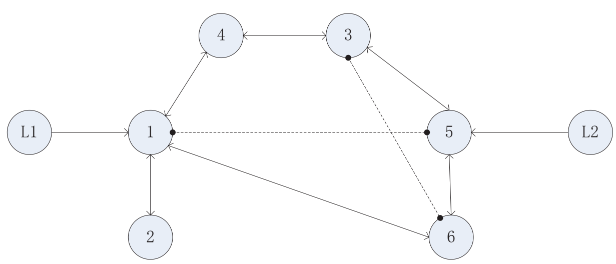


Fig. 2. The communication diagraph.

5. CONCLUSIONS

In the current paper we consider the robust decentralized output regulation of heterogeneous uncertain linear systems subject to multiple leaders. The distributed observers are proposed with adaptive protocols such that the design of the decentralized internal model based control laws can be independent of associated matrix, thus the purely decentralized design is achieved. The future work will focus on extending the current distributed observers with adaptive protocols to nonlinear multi-agent systems.

APPENDIX A

A.1 Diagraph

The diagraph $\bar{\mathcal{G}} = \{\bar{\mathcal{V}}, \bar{\mathcal{E}}\}$ where $\bar{\mathcal{V}} = \{1, 2, \cdots, n\}$ is the vertex set and $\bar{\mathcal{E}}$ is the edge set $\bar{\mathcal{E}} \subset \{(i, j), i, j \in \bar{\mathcal{V}}, i \neq j\}$. An edge of $\bar{\mathcal{E}}$ from node i to node j is denoted by (i, j), where the nodes i and j are called the parent node and the child node of each other. If the diagraph $\bar{\mathcal{E}}$ contains a sequence of edges of the form $(i_1, i_2), (i_2, i_3), \cdots, (i_k, i_{k+1})$, then the set $\{(i_1, i_2), (i_2, i_3), \cdots, (i_k, i_{k+1})\}$ is called a path of $\bar{\mathcal{E}}$ from node i_1 to node i_{k+1} and node i_{k+1} is said to be reachable from node i_1. A directed tree is a diagraph in which every node has exactly one parent except for one node, called the root, which has no parent and from which every other node is reachable. A diagraph $\bar{\mathcal{E}}_s = \{\bar{\mathcal{V}}_s, \bar{\mathcal{E}}_s\}$ is a subgraph of $\bar{\mathcal{G}} := \{\bar{\mathcal{V}}, \bar{\mathcal{E}}\}$ in which $\bar{\mathcal{V}}_s \subseteq \bar{\mathcal{V}}$ and $\bar{\mathcal{E}}_s \subseteq \bar{\mathcal{E}} \cap (\bar{\mathcal{V}}_s \times \bar{\mathcal{V}}_s)$. A subgraph $\bar{\mathcal{E}}_s = \{\bar{\mathcal{V}}_s, \bar{\mathcal{E}}_s\}$ of the graph $\bar{\mathcal{G}} := \{\bar{\mathcal{V}}, \bar{\mathcal{E}}\}$ is called a directed spanning tree of $\bar{\mathcal{G}}$ if $\bar{\mathcal{G}}_s$ is a directed spanning tree and $\bar{\mathcal{V}}_s = \bar{\mathcal{V}}$. The diagraph $\bar{\mathcal{G}} := \{\bar{\mathcal{V}}, \bar{\mathcal{E}}\}$ contains a directed spanning tree if a directed spanning tree is a subgraph of $\bar{\mathcal{G}}$. The diagraph $\bar{\mathcal{G}}$ is undirected if $(i, j) \in \bar{\mathcal{E}} \Leftrightarrow (j, i) \in \bar{\mathcal{E}}$.

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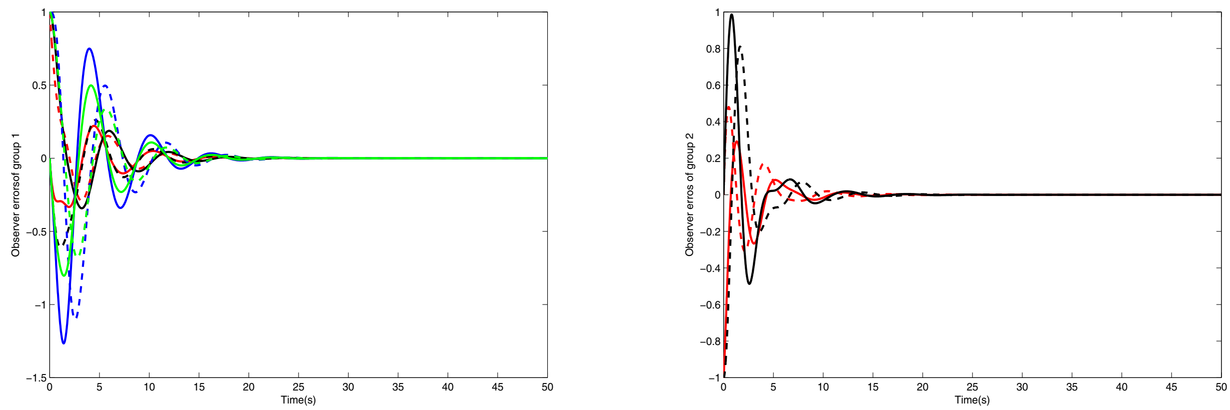


(a) Observer errors $\tilde{v}_{i1}, \tilde{v}_{i2}$, $i = 1, ..., 4$.

(b) Observer errors $\tilde{v}_{i1}, \tilde{v}_{i2}$, $i = 5, 6$.

Fig. 3. The observer errors $\tilde{v}_i$ of two groups.

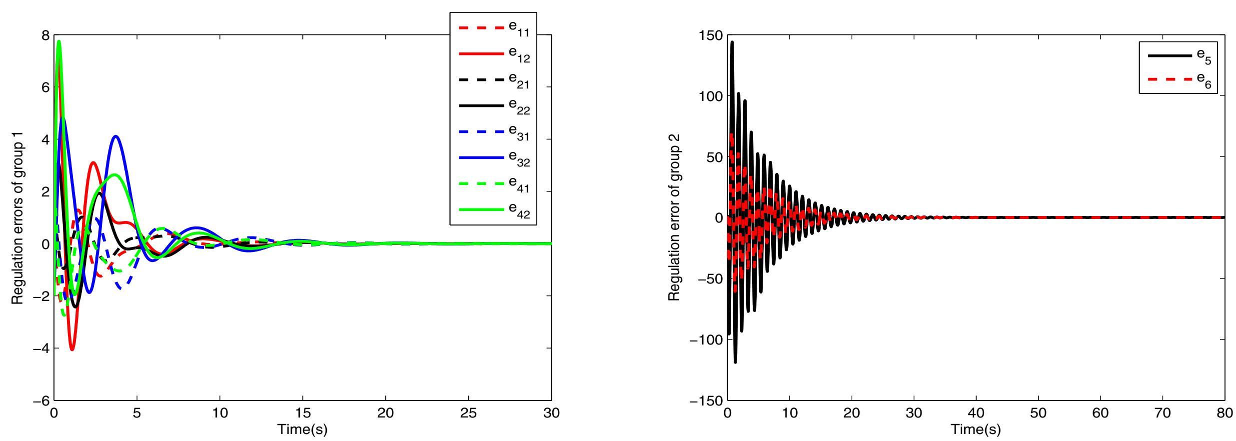


Fig. 4. The regulation errors e_i of two groups.

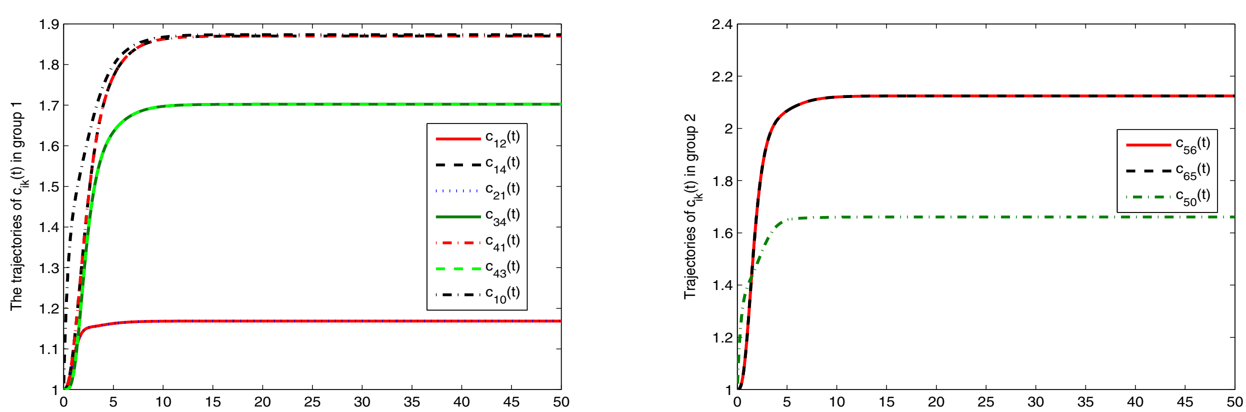
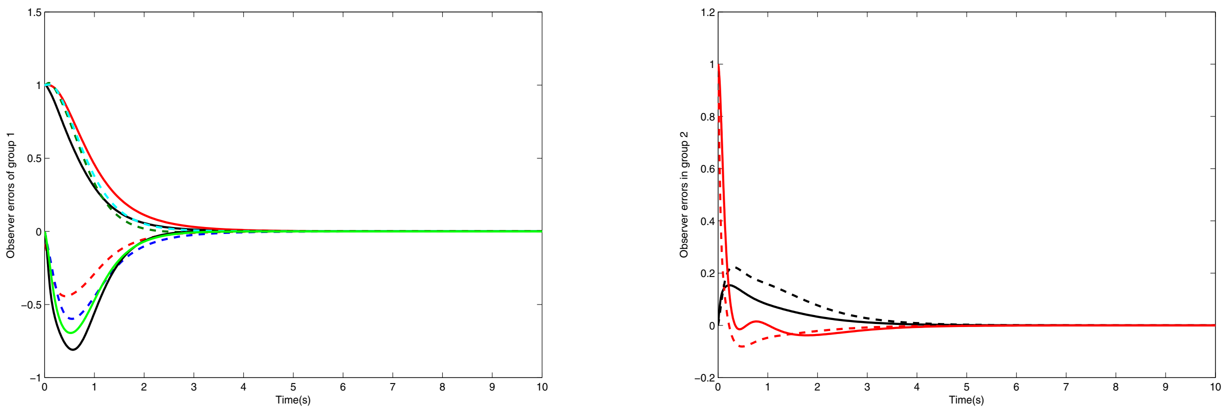


Fig. 5. The trajectories of $c_{ik}(t)$ of two groups.



(a) Observer errors $\tilde{v}_{i1}, \tilde{v}_{i2}$, $i = 1,..., 4$. (b) Observer errors $\tilde{v}_{i1}, \tilde{v}_{i2}$, $i = 5, 6$.

Fig. 6. The observer errors $\tilde{v}_i$ of two groups.

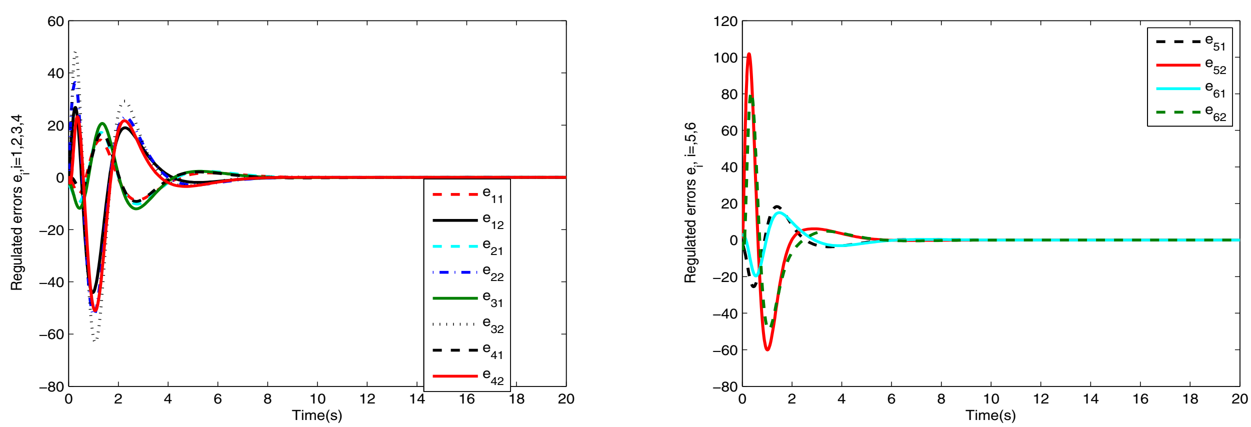


Fig. 7. The regulation errors e_i of two groups.

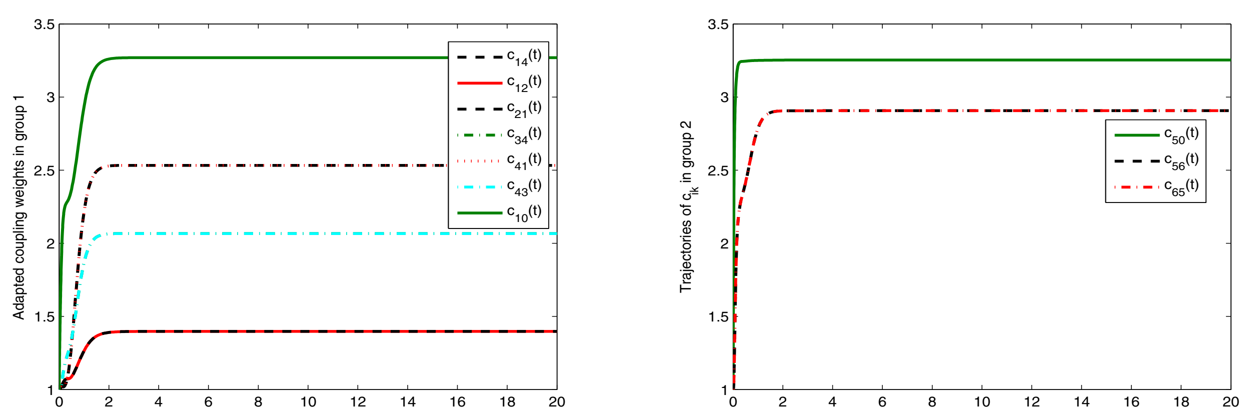


Fig. 8. The trajectories of $c_{ik}(t)$ of two groups.

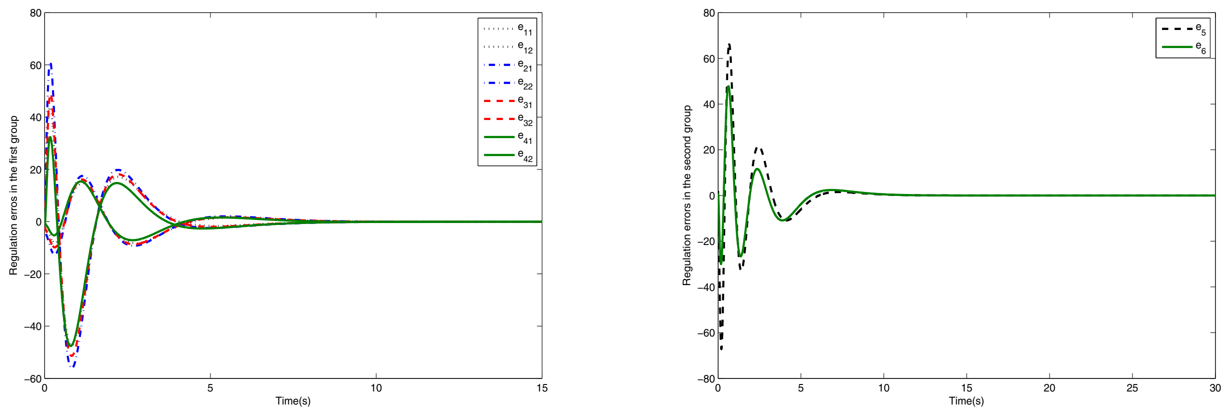


Fig. 9. The regulation errors when static coupling gain is used.

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