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Discontinuous $\mathcal{H}_\infty$ Control of Underactuated Mechanical Systems with Friction and Backlash

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Abstract: Nonlinear $\mathcal{H}_\infty$-control is extended to discontinuous mechanical systems with degree of underactuation one, where nonlinear phenomena such as Coulomb friction and backlash are considered. The problem in question is to design a feedback controller via output measurements so as to obtain the closed-loop system in which all trajectories are locally ultimate bounded, and the underactuated link is regulated to a desired position while also attenuating the influence of external perturbations and nonlinear phenomena. It is considered that positions are the only measurements available for feedback in the system. Performance issues of the discontinuous $\mathcal{H}_\infty$-regulation controller are illustrated in an experimental study made for a rectilinear plant with friction modified to have a gap in the point of contact between bodies.

Keywords: Backlash, discontinuous $\mathcal{H}_\infty$ control, friction, mechanical systems.

1. INTRODUCTION

The design of a robust feedback control which asymptotically stabilizes a nominal plant while also attenuates the influence of parameter variations and external perturbations is a major problem in control engineering. This problem was intensely studied and research efforts have resulted in the development of a systematic design methodology for nonlinear systems. A survey of the methods, which are fundamental in this respect, is given in [1].

On the other hand, backlash is one of the most important non-linearities that limit the performance of speed and position control in industrial robotics, automotive, automation and other applications [2].

According to [3], backlash is clearance or lost motion in a mechanism caused by gaps between the parts. It can be defined as the maximum distance or angle through which any part of a mechanical system may be moved in one direction without applying appreciable force or motion to the next part in mechanical sequence.

Backlash is a common non-linearity in mechanical systems. Depending on the mechanical surrounding of the backlash, and the operating conditions, different mathematical models must be utilized to model the behavior. In the present document, it is used a dead zone model of backlash proposed in [4].

The dead zone model of backlash is non differentiable at the moment of impact, because of this, in our case of study: an underactuated rectilinear plant, it is proposed the usage of a monotonic approximation of a dead zone model in order to fit the model with the requirements of the nonlinear $\mathcal{H}_\infty$ control design.

Control of mechanical systems with backlash has been attempted with a wide range of methodologies such as switched control [5], predictive control [6], fuzzy control [7], and optimal control [8] among others.

The nonlinear $\mathcal{H}_\infty$ control was considered, which is capable of handling the above-mentioned factors, thereby yielding good performance in real systems. The main advantage of using $\mathcal{H}_\infty$-control technique is that it can minimize the external perturbations and model uncertainties of the plant, also, it can attenuate the influence of the error when the system measurements are corrupted. On the other side, a disadvantage of $\mathcal{H}_\infty$ technique include the need for a reasonably good model of the system to be controlled. A previous work of $\mathcal{H}_\infty$ control can be found in [9] where a controller design is developed for mechanical systems with friction, which was derived via the nonlinear $\mathcal{H}_\infty$ control approach coupled to a feedback linearization technique. Although the proposed controller is rather attractive due to their robustness and simple implementation, the backlash and discontinuous friction have not been considered. In [10], an output regulation problem for a servomechanism with backlash in the absence of friction was solved by applying nonlinear $\mathcal{H}_\infty$ control synthesis. In [11] a nonlinear $\mathcal{H}_\infty$-controller synthesis was developed

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for discontinuous time-varying systems via measurement feedback, without considering the nonlinear phenomena and addressing fully actuated mechanical systems.

An application of $\mathcal{H}_\infty$ control to a frictionless and unperturbed constrained system can be found in [12], where a simulation example was provided to validate the effectiveness of the proposed approach. Recently in [13], the problem of output-feedback $\mathcal{H}_\infty$ control for a class of active quarter-car suspension systems with control delay was addressed, considering also that the maximum actuator control force is constrained. As well in [14] is designed a robust linear control $\mathcal{H}_\infty$ and μ-synthesis for a linear model approximation of an active suspension on a quarter car test-rig.

A previous works of mechanical systems with backlash, considering viscous friction only, can be found in [15] and [16], where are proposed sliding mode controllers which included an $\mathcal{H}_\infty$ control on its sliding surface to reduce unmatched perturbations on the unactuated link of the mechanical system. Moreover, in [17] is addressed the regulation problem of the force exerted by a spring on a wall of a one degree-of-freedom mechanical system with a position constraint, affected by Coulomb friction and an external perturbation, using a sliding mode controller.

Furthermore, [9] extended the nonlinear $\mathcal{H}_\infty$-output feedback approach to a class of nonsmooth systems to account for nonsmooth dynamic friction models such as the Dahl [18] and LuGre models [19]. These dynamic models were brought into play to accurately describe observed frictional effects (the stiction behavior and the stribeck effect).

A drawback of the use of the afore-mentioned dynamic friction models for control purposes is in the need of augmenting the state vector dimension to account for the dynamics of the friction model, thus incrementing the controller computational cost. This motivates the use of a static friction model such as the Coulomb model so that the system dimension remains the same. The cost one should pay is to deal with discontinuous systems to be robustly controlled.

Recent works about $\mathcal{H}_\infty$-control encompass a wide range of applications: in [20] are addressed nonlinear systems with parameter uncertainty, in which a sensor fault detection observer is proposed. Moreover, $\mathcal{H}_\infty$-control for linear and nonlinear systems with unknown system model can be found in [21, 22], respectively.

Since dead zone model of backlash is non differentiable at the moment of impact, it is proposed the usage of a monotonic approximation of a dead zone model in order to fit the model with the requirements of the nonlinear $\mathcal{H}_\infty$ control design.

The nonlinear $\mathcal{H}_\infty$-regulator approach for nonsmooth systems was addressed for mechanical systems with degree of underactuation one, where is considered nonlinear phenomena such Coulomb friction and backlash. Due to the nature of the local approach, the resulting controller is additionally expected to yield desired robustness properties in spite of the discrepancy between the dead zone model of backlash and their monotonic approximation. Experiments confirmed the validity of the theoretical analysis. The proposed $\mathcal{H}_\infty$ synthesis procedure considering discontinuous friction, external perturbations and backlash in an underactuated mechanical system constitutes the main contribution of the present work, which to the best of our knowledge had never been addressed before as a whole.

The study is organized as follows. The $\mathcal{H}_\infty$-control problem for mechanical systems with degree of underactuation one considering friction and backlash is presented in Section 2. To facilitate exposition, the friction model chosen for treatment was confined to the discontinuous static Coulomb model augmented with viscous friction. In Section 3 is addressed the nonlinear $\mathcal{H}_\infty$-control synthesis, since the position was assumed to be the only available measurement in the system, the resulting nonlinear $\mathcal{H}_\infty$-controller design necessarily includes a filter to have access to the remaining states. Section 4 presents the $\mathcal{H}_\infty$ controller design in order to stabilize the experimental rectilinear plant configured to have backlash. Finally, Section 5 presents conclusions.

2. $\mathcal{H}_\infty$-REGULATION CONTROL OF UNDERACTUATED MECHANICAL MANIPULATORS WITH FRICTION AND BACKLASH

A mathematical model for such a manipulator is given by

$$M(q)\ddot{q}+C(q,\dot{q})\dot{q}+G(q)+F(\dot{q})=\tau+w_1+DT(q). \quad (1)$$

In the above equation, $q(t)\in\mathbb{R}^n$ is the joint position vector, $\tau(t)\in\mathbb{R}^n$ is the input torque, $\dot{q}$ and $\ddot{q}$ are the velocity and acceleration vectors, $w_1(t)\in\mathbb{R}^n$ is an external perturbation, $F(\dot{q})$, $G(q)$, $M(q)$, and $C(q,\dot{q})$ are matrix functions of appropriate dimensions. From the physical point of view, q is the vector of generalized coordinates, τ is the vector of external torques, $M(q)$ is the inertia matrix, symmetric and positive definite for all $q\in\mathbb{R}^n$, $C(q,\dot{q})\dot{q}$ is the vector of Coriolis and centrifugal torques, $G(q)$ is the vector of gravitational torques affecting only the actuated link, the components $F_j(\dot{q}_j)$, $j=1,\ldots,n$ of $F(\dot{q})$ are friction forces acting independently in each joint. Throughout, The functions $M(q)$, $C(q,\dot{q})$, $G(q)$ are twice continuously differentiable.

Moreover, $D\in\mathbb{R}^n$ is the matrix that maps $T(q)\in\mathbb{R}$ to the joint coordinates space, $T(q)$ is the vector of transmitted torque or force through a backlash, where the dead

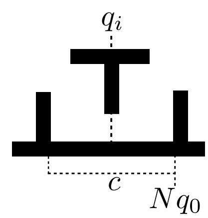

Fig. 1. Schematic of backlash.

zone model of backlash is modelled by

$$T(q) = \begin{cases} k(\Delta q - c/2), & \text{if } \Delta q \geq c/2; \\ 0, & \text{if } -c/2 < \Delta q < c/2; \\ k(\Delta q + c/2), & \text{if } \Delta q \leq -c/2; \end{cases} \tag{2}$$

where $\Delta q = q_i - Nq_o + c/2$. The expression (2) can be rewritten as

$$T(q) = \frac{k}{2}\left[2\Delta q + |\Delta q - c/2| - |-\Delta q - c/2|\right], \tag{3}$$

where $k \in \mathbb{R}$ is the stiffness of the spring, $c \in \mathbb{R}$ is the size of the clearance, q_i is the actuated link, q_0 is the unactuated link, and N is the reducer ratio. Since the action of two mating gears can be represented by the action of one pair of teeth, backlash is commonly represented by the schematic shown in Fig. 1.

In order to fulfill the requirements of control design for a local solution of the $\mathcal{H}_\infty$-position regulation problem is replaced the dead zone model of backlash (3), with its strictly monotonic approximation, this approximation ensures that (1) is at least twice continuously differentiable according to the aforementioned,

$$T(q) = k\Delta q + k\eta(\Delta q), \tag{4}$$

where

$$\eta = -c\frac{1 - e^{-(\Delta q/0.5c)}}{1 + e^{-(\Delta q/0.5c)}}. \tag{5}$$

Hereinafter this approximation of $T(q)$ will be used, the present backlash approximation is inspired from [23]. Coupled to the drive system (1) subject to position measurements of the links, it is subsequently shown to constitute a smooth approximation of the underlying mechanical system, operating under uncertainty $w_1(t)$ to be attenuated. As a matter of fact, this uncertainty involve discrepancies between the physical backlash model (3) and its approximation (4)–(5).

The friction model chosen for the treatment is the static Coulomb model augmented with viscous friction:

$$F_j = \sigma_{0j}\dot{q}_j + F_{cj}\,\mathrm{sign}(\dot{q}_j), \qquad j = 1,\ldots,n, \tag{6}$$

where $\sigma_{0j} > 0$ and $F_{cj} > 0$ are the viscous friction coefficient and the Coulomb friction level respectively, corresponding to the j-th manipulator joint. Moreover, the sign

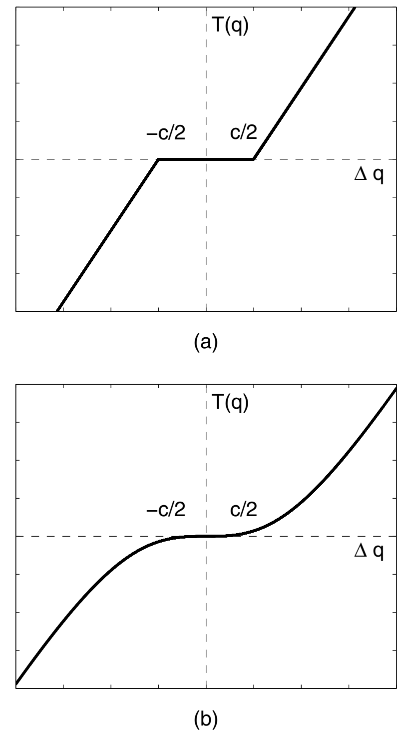

Fig. 2. (a) The dead zone model of backlash and (b) the monotonic approximation of the dead zone model.

is the signum function, defined by

$$\mathrm{sign}(\dot{q}) = \begin{cases} 1 & \dot{q} > 0 \\ 0 & \dot{q} = 0 \\ -1 & \dot{q} < 0. \end{cases} \tag{7}$$

The relation (6) can be rewritten in the vector form

$$F = \sigma_0\dot{q} + F_c\,\mathrm{sign}(\dot{q}), \tag{8}$$

where $F = \mathrm{col}\{F_j\}$, $\sigma_0 = \mathrm{diag}\{\sigma_{0j}\}$, $\dot{q} = \mathrm{col}\{\dot{q}_j\}$,$F_c = \mathrm{diag}\{F_{cj}\}$ and $\mathrm{sign}(\dot{q}) = \mathrm{col}\{\mathrm{sign}(\dot{q}_j)\}$, the notations diag and col are used to denote a diagonal matrix and a column vector, respectively.

Since the right hand side of the equation (1) has discontinuous terms due to Coulomb friction, the solutions of system (1) are understood in the Filippov sense (see [24]).

To manage the model discontinuities while controlling the plant, let us consider the dynamical model counterpart of the static Coulomb model, which is given by the Dahl model represented by the following equations:

$$F = \sigma_0\dot{q} + F_d, \tag{9}$$

$$\dot{F}_d = \sigma_1\dot{q} - \sigma_1\mathrm{diag}\{|\dot{q}_i|\}F_c^{-1}F_d, \tag{10}$$

where $F = \mathrm{col}\{F_j\}$, is the cumulative friction force, affecting the manipulator, $F_d = \mathrm{col}\{F_{dj}\}$ is the Dahl friction component, $\sigma_1 = \mathrm{diag}\{\sigma_{1j}\}$ is the stiffness, $\sigma_0 =$

$\mathrm{diag}\{\sigma_{0j}\}$ and $F_c = \mathrm{diag}\{F_{cj}\}$ are as before the viscous friction coefficient and the Coulomb friction level, respectively. The dynamical friction model (9) and (10) approximates the static Coulomb friction plus viscous friction as in (8) when the parameter σ_1 tends to infinity. In this case, friction model (9) degenerates into (8), which follows the fact that the Coulomb friction model is a limit case of the Dahl friction model when its internal dynamics approaches zero.

Hereafter the friction model (8) will be used in the rest of the document. An advantage of using the static friction model (8) instead of the dynamic friction model (9) is that reduces the number of states in the system model (1) and the parameter σ_1 does not need to be characterized. Now, let $q_d = \mathrm{col}\{q_{dj}\}$ be the desired position. Then if there were no initial and external perturbations the following feedback compensator τ_d will impose on the disturbance-free manipulator motion, desired stability properties around q_d enforced by the external torque/force

$$\tau_d = G(q) - B\theta\left(\int_{t_0}^{T}(q(t)-q_d)\,dt, q-q_d, \dot{q}\right), \quad (11)$$

where $\theta\left(\int_{t_0}^{T}(q(t)-q_d)\,dt, q-q_d, \dot{q}\right) \in \mathbb{R}^m$ being $m < n$ is a control input, subject to the initial condition $t_0 = 0 \in \mathbb{R}$ (the absence of initial and external perturbations means that $q(0) = 0$, $\dot{q}(0) = 0$, and $w_1 = 0$), and $B \in \mathbb{R}^{n\times m}$ is the input matrix that maps the control input θ of dimension m to the joint coordinates space of dimension n.

Since it is used only position measurements, the velocity term $\dot{q}$ used in the compensation, is estimated from the expression $\dot{q} = \xi_2$, where ξ_2 is the velocity estimated by the controller's filter (26), that will be presented later in this document.

Our objective is to design a controller of the form

$$\tau = \tau_d + u \quad (12)$$

that imposes on the perturbation-free manipulator motion desired stability properties around the desired position, otherwise it will locally attenuate the effect of the perturbations. Thus, the controller to be constructed consists of the regulation compensator (11) and a perturbation attenuator $u(t)$, internally stabilizing the closed-loop system around the desired position.

For certainty, let confine our investigation to the position regulation control problem were (i) the output to be controlled is given by

$$z = \rho\begin{bmatrix}\mathbf{0_{m\times 1}}\\ q_o - q_{do}\end{bmatrix} + \begin{bmatrix}\mathbf{I_m}\\ \mathbf{0_{1\times m}}\end{bmatrix}u \quad (13)$$

where q_{do} is the desired position of the underactuated link q_o, with a positive weight coefficient ρ, and (ii) the position measurements

$$y = q + w_0, \quad (14)$$

corrupted by the error vector $w_0(t) \in \mathbb{R}^n$, are only available.

3. NONLINEAR $\mathcal{H}_\infty$-CONTROL SYNTHESIS

The $\mathcal{H}_\infty$ control problem for position regulation in robot manipulators with discontinuous friction and backlash can formally be stated as follows. Given a mechanical system (1), (6)–(14) is thus to design a nonlinear $\mathcal{H}_\infty$ controller so as to obtain the closed-loop system in which all these trajectories are bounded and the output of the underactuated link $q_o(t)$ asymptotically decays to a desired position q_{do} as $t\to\infty$ while also attenuating the influence of the external perturbations $w_0(t)$ and $w_1(t)$.

To begin with, let us introduce the state deviation vector $x = (x_1, x_2)^T$ where $x_1(t) = q(t) - q_d$ is the position deviation from the desired position, and $x_2(t) = \dot{q}(t)$ is the velocity. After that, let us rewrite the state equations (1), (6)–(14) in terms of the state vector x

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= M^{-1}(x_1+q_d)[-C(x_1+q_d,x_2)x_2\\ &\quad -\sigma_0 x_2 - F_c\mathrm{sign}(x_2) + DT(\Delta x) - B\theta + u + w_1]\end{aligned} \quad (15)$$

considering $\Delta x = x_i + q_{di} - N(x_0 + q_{do}) + c/2$ and $q_{di} = Nq_{do} - c/2$, then the expression of Δx can be simplified to $\Delta x = x_i - x_0$. It should be pointed out, that the $\mathcal{H}_\infty$ regulation problem for the discontinuous mechanical system can be specified in a similar manner as in [9] which is given as follows:

$$\begin{aligned}\dot{x} &= f_1(x) + f_2(x) + g_1(x)w + g_2(x)u\\ z &= h_1(x) + k_{12}(x)u\\ y &= h_2(x) + k_{21}(x)w\end{aligned} \quad (16)$$

when equations are specified with

$$f_1(x) = \begin{bmatrix} x_2\\ M^{-1}(x_1+q_d)\,[-C(x_1+q_d,x_2)x_2\\ -\sigma_0 x_2 + DT(\Delta x) - B\theta]\end{bmatrix}, \quad (17)$$

$$f_2(x) = \begin{bmatrix}\mathbf{0_{n\times 1}}\\ -M^{-1}(x_1+q_d)F_c\mathrm{sign}(x_2)\end{bmatrix}, \quad (18)$$

$$g_1(x) = \begin{bmatrix}\mathbf{0_{n\times n}} & \mathbf{0_{n\times n}}\\ \mathbf{0_{n\times n}} & M^{-1}(x_1+q_d)\end{bmatrix}, \quad (19)$$

$$g_2(x) = \begin{bmatrix}0_{n\times n}\\ M^{-1}(x_1+q_d)\end{bmatrix},$$

$$h_1(x) = \rho\begin{bmatrix}\mathbf{0_{m\times 1}}\\ \mathbf{x_{1o}}\end{bmatrix}, \quad h_2(x) = x_1 + q_d,$$

$$k_{12}(x) = \begin{bmatrix}\mathbf{I_m}\\ \mathbf{0_{1\times m}}\end{bmatrix}, \; k_{21}(x) = \begin{bmatrix}\mathbf{I_n} & \mathbf{0_{n\times n}}\end{bmatrix}, \quad (20)$$

where $x_{1o} \in \mathbb{R}$ is the position error of the underactuated link and the perturbations $w = [w_0, w_1]^T \in \mathbb{R}^{2n}$. From system (15) let us derive a local solution of the $\mathcal{H}_\infty$ regulation problem. Therefore the local solution of the $\mathcal{H}_\infty$-position regulation problem subject to (17)–(20) has the following output feedback

$$\begin{aligned}\dot{x} &= Ax + B_1 w + B_2 u,\\ z &= C_1 x + D_{12} u,\\ y &= C_2 x + D_{21} w,\end{aligned} \tag{21}$$

where

$$\begin{aligned}&A = \frac{\partial f_1}{\partial x}(0),\ \ B_1 = g_1(0),\ \ B_2 = g_2(0),\\ &C_1 = \frac{\partial h_1}{\partial x}(0),\ \ D_{12} = k_{12}(0),\ \ C_2 = \frac{\partial h_2}{\partial x}(0),\\ &D_{21} = k_{21}(0)\end{aligned} \tag{22}$$

with matrices A, B_1, C_1, C_2, D_{12}, D_{21} of appropriate dimensions. The general state-space representation with nonzero feedthrough terms D_{11} and D_{22} can be treated as in [25] by constructing an equivalent problem with $D_{11} = 0$ and $D_{22} = 0$. In addition, the simplifying assumptions

(A_1) (A, B_1) is stabilizable and (C_1, A) is detectable,

(A_2) (A, B_2) is stabilizable and (C_2, A) is detectable,

(A_3) $D_{12}^T C_1 = 0$ and $D_{12}^T D_{12} = I$,

(A_4) $B_1 D_{21}^T = 0$ y $D_{21} D_{21}^T = I$,

presented in [26], are made throughout.

A local solution is then derived by means of the perturbed Riccati equations (see [9]), given by

$$\begin{aligned}0 &= P_\varepsilon A + A^T P_\varepsilon + C_1^T C_1\\ &\quad + P_\varepsilon \left[\frac{1}{\gamma^2} B_1 B_1^T - B_2 B_2^T\right] P_\varepsilon + \varepsilon I,\end{aligned} \tag{23}$$

$$\begin{aligned}0 &= A_\varepsilon Z_\varepsilon + Z_\varepsilon A_\varepsilon^T + B_1 B_1^T\\ &\quad + Z_\varepsilon \left[\frac{1}{\gamma^2} P_\varepsilon B_2 B_2^T P_\varepsilon - C_2^T C_2\right] Z_\varepsilon + \varepsilon I.\end{aligned} \tag{24}$$

There exists a positive constant ε_0 such that the system of the perturbed algebraic Riccati equations has a unique positive definite symmetric solution $(P_\varepsilon, Z_\varepsilon)$ for each $\varepsilon \in (0, \varepsilon_0)$ where $A_\varepsilon = A + (1/\gamma^2) B_1 B_1^T P_\varepsilon$. Equations (23) and (24) are utilized to derive a local solution of the $\mathcal{H}_\infty$-control problem for a mechanical system with friction and backlash (17)-(20).

Let $(P_\varepsilon, Z_\varepsilon)$ be a positive definite solution of (23), (24) under some $\varepsilon > 0$. Then the output feedback is given by

$$u = -g_2^T(\xi) P_\varepsilon \xi, \tag{25}$$

which is a local solution of the $\mathcal{H}_\infty$-control problem. A filter to have access to the remaining states is

$$\begin{aligned}\dot{\xi} =& f_1(\xi) + f_2(\xi)\\ &+ \left[\frac{1}{\gamma^2} g_1(\xi) g_1^T(\xi) - g_2(\xi) g_2^T(\xi)\right] P_\varepsilon \xi\\ &+ Z_\varepsilon C_2^T [y - h_2(\xi)].\end{aligned} \tag{26}$$

The purpose of the control is twofold: to achieve closed-loop stability and to attenuate the influence of the external input w on the penalty variable z. A controller which locally asymptotically stabilizes the equilibrium $(x, \xi) = (0, 0)$ of the closed-loop system is said to be an admissible controller. The perturbation attenuation depends on the specific class of external signals to be considered and/or the performance criteria chosen to evaluate the penalty variable. Given a real number $\gamma > 0$, it is said that system (16), (25), (26) has $\mathcal{L}_2$ gain less than γ if the response z, resulting from w for initial state $x(0) = 0$, $\xi(0) = 0$, satisfies

$$\int_0^T z^T(t) z(t) dt \le \gamma^2 \int_0^T w^T(t) w(t) dt \tag{27}$$

for all $T > 0$ and all piecewise continuous functions $w(t)$, for which the corresponding state trajectory of the closed-loop system, initialized at the origin, remains in some neighborhood of this point.

4. CASE OF STUDY: RECTILINEAR PLANT WITH BACKLASH

Let us now apply the $\mathcal{H}_\infty$ control design developed in the previous Sections to a rectilinear plant with backlash (see Fig. 3). The mathematical model of the laboratory prototype of the rectilinear plant, in the joint coordinates space, is given by

$$\begin{aligned}m_1 \ddot{q}_i + F_1(\dot{q}_i) + T(q) &= \tau + w_{1,1},\\ m_2 \ddot{q}_o + F_2(\dot{q}_o) &= T(q) + w_{1,2}.\end{aligned} \tag{28}$$

In the above equations, $q_i(t)$, $\dot{q}_i(t)$, $\ddot{q}_i(t) \in \mathbb{R}$ represent the displacement, velocity, and acceleration of the actuated mass $m_1 \in \mathbb{R}$, respectively; and $q_o(t)$, $\dot{q}_o(t)$, $\ddot{q}_o(t) \in \mathbb{R}$ represent the displacement, velocity, and acceleration of the underactuated mass $m_2 \in \mathbb{R}$, respectively; $\tau \in \mathbb{R}$ is the input torque, and $w_1 \in \mathbb{R}^2$ is an external perturbations, a,

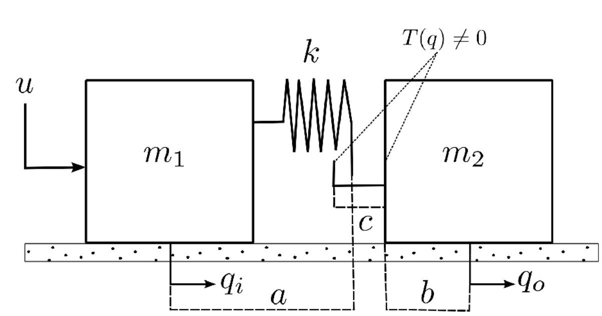

Fig. 3. Mechanical system with backlash.

b, and $c \in \mathbb{R}$ are distances greater than zero. The friction forces $F_j(\dot{q})$, $j = 1,2$ are specified as in (6). Finally, $T(q)$ is the dead zone model of backlash between the masses is modelled as in (2) and (3) but considering now $\Delta q = q_i + a + b - q_o + c/2$.

Provided the actuated mass position $q_i(t)$ and underactuated mass position $q_o(t)$ are the only available measurements on the system. The above unforced system (28) possesses a set of equilibria (q_i, q_o) with $q_i \in [\zeta - a - b - c, \zeta - a - b]$ where ζ is any constant and $q_o = \zeta$.

In order to fulfill the requirements of control design for a local solution of the $\mathcal{H}_\infty$-position regulation problem is replaced the dead zone model of backlash (3), with its strictly monotonic approximation (4)-(5).

Let us propose the following regulator

$$\tau_d = -\lambda_1 \int_0^T (q_o - q_{do})dt - \lambda_2(q_o - q_{do}) - \lambda_3 \dot{q}_o \tag{29}$$

that imposes on the perturbation-free system motion desired stability properties around $q_d \in \mathbb{R}$.

Throughout, the output to be controlled is given by

$$z = \rho \begin{bmatrix} 0 \\ q_o - q_{do} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \tag{30}$$

with a positive weight coefficient ρ, and the position measurements

$$y = \begin{bmatrix} q_i \\ q_o \end{bmatrix} + \begin{bmatrix} w_{0,1} \\ w_{0,2} \end{bmatrix}, \tag{31}$$

corrupted by the error vectors $w_{0,1}(t), w_{0,2}(t) \in \mathbb{R}$, are only available.

4.1. Control objective

The objective of the $\mathcal{H}_\infty$-output regulation of the nonlinear drive system (28) with friction (6) and backlash model (4)-(5) is thus to design a nonlinear $\mathcal{H}_\infty$ controller so as to obtain the closed-loop system in which all these trajectories are bounded and the output $q_o(t)$ asymptotically decays to a desired position q_{do} as $t \to \infty$ in the perturbation-free case $w = 0$ otherwise the nonlinear $\mathcal{H}_\infty$ controller is going to attenuate the influence of the external perturbations $w = [w_{0,1}, w_{0,2}, w_{1,1}, w_{1,2}]^T$. Now let us shift the equilibrium point of (28) to the origin by introducing the state transformation based on the position error including also the integral value of the position error introduced for control purposes,

$$\begin{aligned} x_1 &= q_i - q_{di}, \qquad x_2 = \dot{q}_i, \\ x_3 &= \int_0^T x_4(t)dt, \quad x_4 = q_o - q_{do}, \quad x_5 = \dot{q}_o, \end{aligned} \tag{32}$$

where $q_{di} = q_{do} - a - b - c/2$. After that let us rewrite the state equations (28)–(31) in terms of the state vector x

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= m_1^{-1}(-\sigma_{0,1}x_2 - F_{c1}\operatorname{sign}(x_2) \\ &\quad - T(x) + \tau + w_{1,1}), \\ \dot{x}_3 &= x_4, \\ \dot{x}_4 &= x_5, \\ \dot{x}_5 &= m_2^{-1}(-\sigma_{0,2}x_5 - F_{c2}\operatorname{sign}(x_5) + T(x) + w_{1,2}), \end{aligned} \tag{33}$$

where

$$T(x) = k\Delta x + k\eta(\Delta x) \tag{34}$$

with $\Delta x = x_1 + q_{di} - x_4 - q_{do} + a + b + c/2$ which can be simplified to $\Delta x = x_1 - x_4$ where

$$\eta = -c\frac{1 - e^{-(\Delta x/0.5c)}}{1 + e^{-(\Delta x/0.5c)}}. \tag{35}$$

and

$$\tau_d = -\lambda_1 x_3 - \lambda_2 x_4 - \lambda_3 x_5 \tag{36}$$

that imposes on the perturbation-free system motion desired stability properties around $x_4 = 0$.

4.2. $\mathcal{H}_\infty$ control design

The control objective is to determine a feedback controller to solve the following regulation problem, such that the closed-loop response satisfies in the absence of perturbations

$$\lim_{t\to\infty} \|x_4\| = 0. \tag{37}$$

The representation of (33) applying (36) according to (16) is as follows

$$f_1(x) = \begin{bmatrix} x_2 \\ m_1^{-1}(-\sigma_{0,1}x_2 - T(x) - \lambda_1 x_3 - \lambda_2 x_4 - \lambda_3 x_5) \\ x_4 \\ x_5 \\ m_2^{-1}(-\sigma_{0,2}x_5 + T(x)) \end{bmatrix}, \tag{38}$$

$$f_2(x) = \begin{bmatrix} 0 \\ -m_1^{-1}F_{c1}\operatorname{sign}(x_2) \\ 0 \\ 0 \\ -m_2^{-1}F_{c2}\operatorname{sign}(x_5) \end{bmatrix}, \tag{39}$$

$$g_1(x) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & m_1^{-1} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & m_2^{-1} \end{bmatrix}, \tag{40}$$

$$g_2(x) = \begin{bmatrix} 0_{1\times1} \\ m_1^{-1} \\ 0_{3\times1} \end{bmatrix}, \tag{41}$$

$$h_1(x) = \rho \begin{bmatrix} 0 \\ x_4 \end{bmatrix}, \quad h_2(x) = \begin{bmatrix} x_1 + q_d^* \\ x_4 + q_d \end{bmatrix}, \tag{42}$$

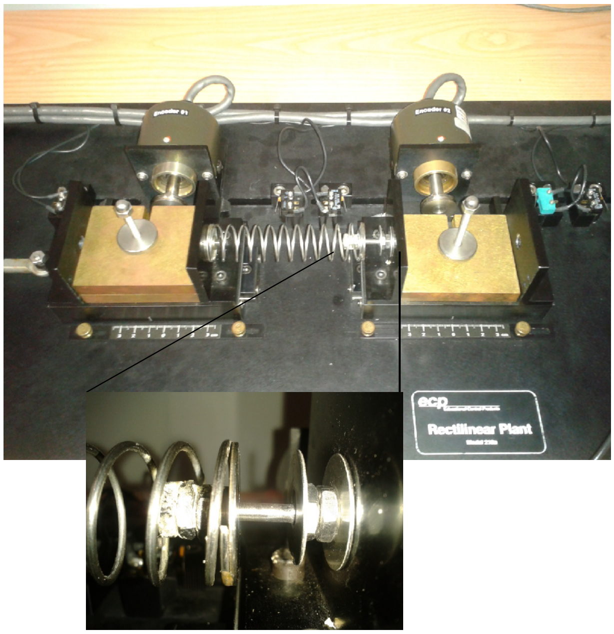

Fig. 4. Experimental platform ECP-210 configured to have gear play.

$$k_{12}(x) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad k_{21}(x) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}. \tag{43}$$

Then the output feedback (25)-(26) subject to (38)-(43) is a local solution of the H_∞-position regulation problem for the system with backlash (33). Thus, the controller τ to be constructed consists of the regulation compensator τ_d (36) and a perturbation attenuator u (25), internally stabilizing the closed-loop system (33) around the desired position, as follows

$$\tau = -\lambda_1 x_3 - \lambda_2 x_4 - \lambda_3 x_5 - g_2^T(\xi) P_\varepsilon \xi. \tag{44}$$

4.3. Experimental Study

Performance issues and robustness properties of the proposed compensator (36) and the perturbation attenuator $u(t)$ in (25) are tested in experiments. Since only state x_1 and x_4 measurements are available, the $\mathcal{H}_\infty$ filter (26) is applied to have access to the remaining states.

In the experiments performed using the platform ECP-210 configured to have gear play as in Fig. 4, the parameters of the mechanical system were considered as in Table 1. The desired position with respect to m_1 is given by $q_{do} = 2.0$ cm, being the desired position with respect to m_2 $q_{di} = 3.0$ cm. The controller feedback gains were set $\lambda_1 = 0.01$ kg/s^3, $\lambda_2 = 0.1$ kg/s^2 and $\lambda_3 = 0.01$ kg/s. It is worth mentioning that friction terms were calculated in accordance with the methodologies presented in [27].

Additionally it is applied an external but bounded force perturbation governed by

$$w_{1,1} = 0.2\sin(10t)N. \tag{45}$$

Table 1. Nominal parameters.

Description	Notation	Value
First mass	m_1	1.06 kg
Second mass	m_2	0.61 kg
Distance	a	15.0 cm
Distance	b	5.1 cm
Clearance	c	1.0 cm
Viscous friction	σ_{01}	7.695 kg/s
Viscous friction	σ_{02}	2.1141 kg/s
Coulomb friction	F_{c1}	0.1 N
Coulomb friction	F_{c2}	0.1 N
Spring stiffness	k	375.42 N/m

For the selected $\gamma = 0.99$, $\rho = 1$ and $\varepsilon = 10^-5$, the corresponding perturbed Riccati equations (23) and (24) have the positive definite solutions:

$$P_\varepsilon = \begin{bmatrix} 13.8716 & 1.8564 & 0.0459 & -7.1751 & -1.7726 \\ 1.8564 & 0.2485 & 0.0062 & -0.9421 & -0.2325 \\ 0.0459 & 0.0062 & 0.0036 & -0.0055 & -0.0012 \\ -7.1751 & -0.9421 & -0.0055 & 7.8312 & 1.9559 \\ -1.7726 & -0.2325 & -0.0012 & 1.9559 & 0.4971 \end{bmatrix},$$

$$Z_\varepsilon = \begin{bmatrix} 8.7285 & 1.2005 & 1.5362 & -0.6914 & -0.1862 \\ 1.2005 & 0.1708 & 0.2104 & -0.0870 & -0.0232 \\ 1.5362 & 0.2104 & 0.4800 & 0.4172 & 0.0954 \\ -0.6914 & -0.0870 & 0.4172 & 1.8367 & 0.4743 \\ -0.1862 & -0.0232 & 0.0954 & 0.4743 & 0.1331 \end{bmatrix},$$

which have been numerically found using MATLAB. The initial values of the integral of position error, position errors and velocities were set to $x_1(0) = -0.03$ m, $x_2(0) = 0$ m/s, $x_3(0) = 0$ m.s, $x_4(0) = -0.02$ m and $x_5(0) = 0$ m/s, respectively. Initial conditions for $\mathcal{H}_\infty$ filter were set to $\xi(0) = 0$. As follows from Fig. 5, the monotonic backlash model (34)-(35) yields an appropriate approximation of the transmitted force. In order to illustrate the proposed control approach, a comparison is made using the plant under friction and backlash (as in Fig. 4), firstly, it is developed the regulation compensator + $\mathcal{H}_\infty$ control considering friction and backlash and then is developed and tested the regulation compensator + $\mathcal{H}_\infty$ control without the consideration of friction and backlash in its design. From Fig. 6–7, can be outline good performance and desired robustness properties of the mechanical system shown in Fig. 4 under the proposed regulation compensator + $\mathcal{H}_\infty$ control law (44). In Fig. 6, the unperturbed case, can be seen that the position errors x_1 and x_4, corresponding to the actuated and underactuated mass, respec-

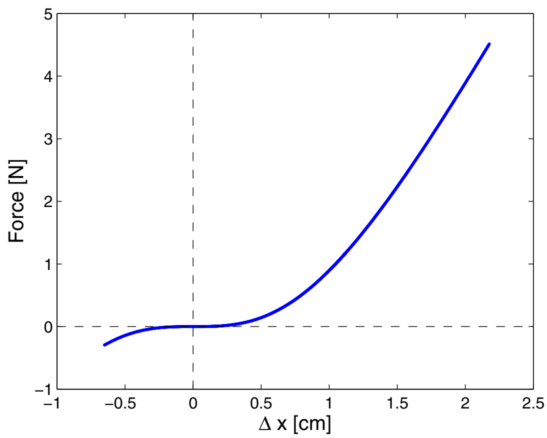

Fig. 5. Transmitted force, computed from the experiment.

tively, tend to zero in approximately 3.1 seconds in the case when friction and backlash were considered, the consequence of not considering the friction and backlash in the control design is that the convergence time was increased to 6.2 seconds approximately and the transient signal presents oscillations. The estimated velocities of the masses ξ_2 and ξ_5 corresponding to the first and second mass were obtained from the filter (26), and they were used as feedback in the controller, moreover, in the control signal in the case when it is considered friction and backlash, a peak of 4.9 N approximately can be observed, in contrast when it is not considered friction and backlash the peak is of 3.0 N approximately.

In Fig. 7, where the system is affected by a non vanishing perturbation besides of nonlinear phenomena such discontinuous friction and backlash, the positions errors x_1 and x_4 tend to its nominal value at 5 seconds approximately in the case when friction and backlash were considered, and when the friction and backlash are not considered it is reached the nominal value at 7 seconds approximately, being remarkable in both cases that the effect of perturbations on the plant are diminished considerably.

4.4. PI controller comparison

In order to appreciate the attenuation of external perturbations using the $\mathcal{H}_\infty$ controller, it is made a comparison against a PI control scheme, also it was considered a non vanishing perturbation as in (45). The gains of PI controller were set in $P = 0.5kg/s^2$ and $I = 0.01kg/s^3$, the derivative term was not considered because we only have position measurements and the $\mathcal{H}_\infty$ filter as in (26) was not used for the PI controller. The results can be seen in Fig. 8 where the $\mathcal{H}_\infty$ controller renders an outstanding performance over the PI controller.

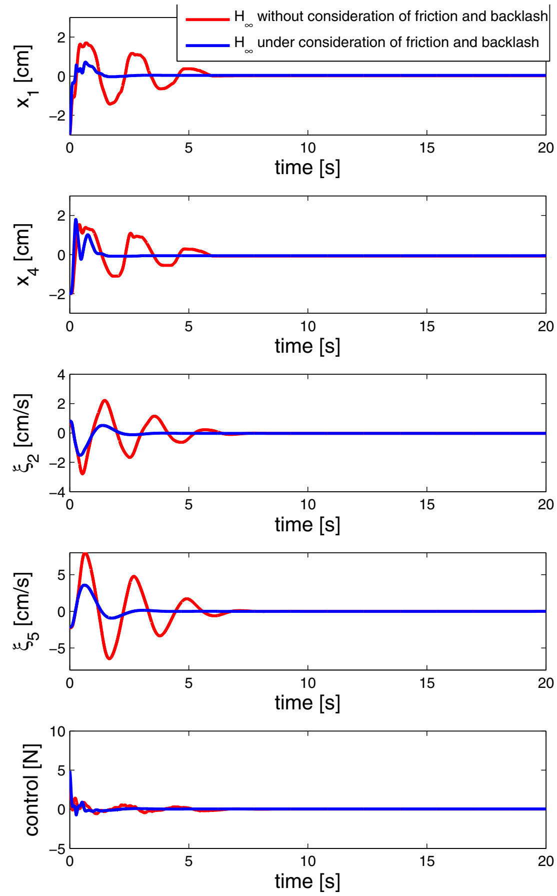

Fig. 6. Experimental results for $\mathcal{H}_\infty$ regulator considering only mass position measurements, and their velocities ξ_2 and ξ_5 are estimated by a filter.

5. CONCLUSIONS

A nonlinear $\mathcal{H}_\infty$ synthesis of discontinuous control systems is applied. The afore mentioned design procedure has been shown to be eminently suited for solving a position regulation problem for mechanical systems with degree of underactuation one under discontinuous friction and backlash. The friction model chosen for treatment has been confined to the static discontinuous Coulomb friction model augmented with viscous friction. In the case of study, made for a rectilinear plant while under the mass position measurements, the system is not twice differentiable due to backlash phenomenon and discontinuous friction, a monotonic approximation of well-known dead zone model of backlash is used to be replaced in the original system, moreover the discontinuous friction is viewed as a particular case of the Dahl friction model when parameter σ_1 tends to infinity, making it possible by this way to design an $\mathcal{H}_\infty$ control for mechanical systems. The nonlinear $\mathcal{H}_\infty$ output regulation synthesis proposed is shown to be eminently suited to locally solve the stabilization problem around a desired position while also

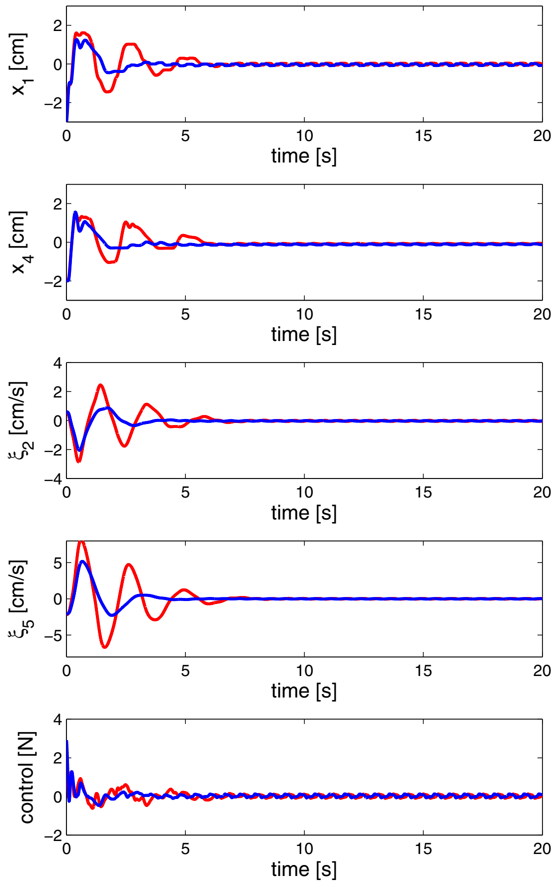

Fig. 7. Experimental results for $\mathcal{H}_\infty$ regulator, where mass position measurements and estimated velocities are used as feedback: Perturbed case.

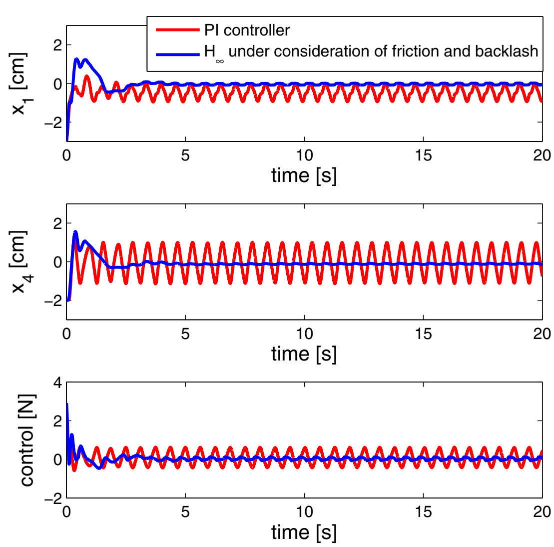

Fig. 8. Experimental comparison between the proposed $\mathcal{H}_\infty$ regulator and a PI controller: Perturbed case.

attenuating the dead zone model of backlash discrepancies and external perturbation. Effectiveness of the design procedure has been supported by experiments made for an underactuated two degrees-of-freedom mechanical system with backlash, where the platform ECP-210 was modified to present a slackness between their masses.

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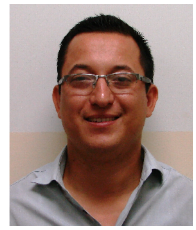

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