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Input/output-to-state Stability of Switched Nonlinear Systems with an Improved Average Dwell Time Approach

Li-Xia Liu, Rong-Wei Guo*, and Shu-Ping Ma

Abstract: This paper investigates the input/output-to-state stable (IOSS) property of the switched systems under average dwell time (ADT) switching signals in two cases: 1) all of the subsystems are IOSS, 2) parts of the subsystems are IOSS, and proposes a number of new results on stability analysis. First, we present a new IOSS result for the switched nonlinear systems whose subsystems are IOSS with an improved ADT method. Second, extending the improved ADT method to unforced nominal switched nonlinear systems in which parts of subsystems are stable, we establish a new stability analysis result. IOSS property of switched nonlinear systems in which parts of subsystems are IOSS, we show that if the average dwell time is large enough and if the fraction of time where one of the non-IOSS system is active is not too big, then IOSS property of the switched system can be established. It should be pointed that the main results obtained in this paper have some advantages over the exiting ones. Finally, two illustrative examples with simulation verify the correctness and validity of our results.

Keywords: Average dwell time, input/output-to-state stable, switched nonlinear system, unstable subsystems.

1. INTRODUCTION

Switched systems arise in various fields of real life world, such as auto pilot design, machine manufacturing, communication networks, computer synchronization, automotive engine control, traffic control, hybrid electric automobile, and chemical processes, and so on. In the past two decades, increasing attention has been paid to the analysis and synthesis of switched systems due to their significance in both theory and applications, and many significant results have been obtained for the stability analysis and control design of switched systems, see [1–6] and references therein. For the stability analysis problem, there are two famous methods: Common Lyapunov Function (CLF) method [4, 5], and Multiple Lyapunov Functions (MLF) method [6]. For the CLF method, for a given switched system, it is very difficult to determine whether all the subsystems share a CLF or not, even for the switched linear systems. As for MLF method, it is well known that the switched system is globally asymptotically stable (GAS) for any switching signal if the time between consecutive switching (i.e., dwell time) is sufficiently large when all the subsystems are stable. From this, some results have appeared in recent works to compute lower bounds of the dwell time for ensuring the stability and obtained an average dwell time (ADT) method [9–13]. However, how to obtain the minimum dwell time for a given switched system has been no general method so far, even for the switched linear systems. As pointed out in [11], the ADT switching is a class of restricted switching signals which means that the times of switches in a finite interval is bounded and the average dwell time between consecutive switching is not less than a constant. It is well known that the ADT scheme characterizes a large class of stable switching signals than the dwell time scheme does, and its extreme case is the arbitrary switching. However, the ADT results mentioned above have two limitations: one is the obtained ADT has conservative property, the other is the conditions for the ADT method are too many to be satisfied in practice. In [14], we obtain an improved ADT method by modifying our previous results [15, 16], which not only simplifies the conditions of the above ADT methods, but also reduces the conservative property in determining the lower bound of ADT τ_a^* in some sense. Especially, for switched linear systems, we have given a rigorous proof that our improved ADT method is better than the existing ones.

When a control system is affected by an external input, the notion of input-to-state stability, introduced in [17], has been proved very useful for investigating stability

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Lixia Liu and Rongwei Guo are with the School of Science, Qilu University of Technology, Jinan 250353, P. R. China (e-mails: liulixiaqlu@163.com, rwguo@qlu.edu.cn). Shuping Ma is with the School of Mathematics, Shandong University, Jinan 250100, P. R. China (e-mail: mashup@sdu.edu.cn).
* Corresponding author.

properties of continuous-time nonlinear systems, even for switched nonlinear systems, see [18–21] and the references therein. The dual of this concept, output-to-state stability was discussed in [22], and the combination of the two, input/output-to-state stability, were established in [23, 24]. Although many works about the IOSS property of nonlinear systems and switched nonlinear systems have been done, the IOSS property problem of the switched nonlinear systems has not been solved completely so far. Therefore, investigating the IOSS property of the switched systems is not only important in theory, but also reasonable in practice.

In this paper, we present several new sufficient conditions under which a switched nonlinear system with an improved ADT switching signal is IOSS, also examining the case where parts of the subsystems are not IOSS. First, we introduce an improved ADT method, and by which present a new sufficient condition for the switched nonlinear system whose subsystems are IOSS. Even for the unforced nominal switched nonlinear system, we also give some novel results, namely, on asymptotic stability for such switched systems parts of the subsystems are GAS. Then, we obtain some new IOSS results for the switched nonlinear systems that parts of the subsystems are IOSS. Finally, two illustrative examples with simulation are studied by using the results obtained in this paper. The study of examples shows that our analysis methods work very well in analyzing the IOSS property for some classes of switched nonlinear systems.

The rest of this paper is structured as follows. Section 2 introduces the notation and basic definitions used throughout this paper. Section 3 presents the main results of this paper. In Section 4, two illustrative examples are given to support our new results, which is followed by the conclusion in Section 5.

2. PRELIMINARIES

Consider a family of systems

$$\dot{x} = f_i(x,u), \tag{1}$$
$$y = h_i(x), \quad i \in \mathcal{I}, \tag{2}$$

where the state $x \in \mathbb{R}^n$, the input $u \in \mathbb{R}^m$, the output $y \in \mathbb{R}^l$ and $\mathcal{I}$ is an index set. For every $i \in \mathcal{I}$, $f_i(.,.)$ is locally Lipschitz, $h_i(.)$ is continuous, $f_i(0,0) = 0$ and $h_i(0) = 0$. A switched nonlinear system

$$\dot{x} = f_{\sigma(t)}(x,u), \tag{3}$$
$$y = h_{\sigma(t)}(x), \tag{4}$$

is generated by the family of systems (1) and (2), an initial condition $x_0 = x(t_0)$ with initial time $t_0 \geq 0$, and a switching signal $\sigma(t)$, where $\sigma(t)$: $[t_0,\infty) \to \mathcal{I} = \{1,2,3,\cdots,N\}$ is a piecewise right-continuous function, and $\sigma(t) = i$ means that the i-th subsystem is active. Admissible input signals $u(.)$ applied to the switched system (3) and (4) are measurable and locally bounded. In order to simplify notation, in the following we assume that the solution of the switched system (3) and (4) exists for all times. If this is not the case, but the solution is only defined on some finite interval $[t_0,t_{\max})$, all subsequent results are still valid for this interval.

For an arbitrary switching path $\sigma(t) = i_m \in \mathcal{I}$ $(t \in [t_m,t_{m+1}), m = 0,1,2,3,\cdots,)$, $\{t_m\}_{m=0}^{+\infty}$ is called the switching time sequence, which is assumed to satisfy

$$t_0 < t_1 < t_2 < \cdots < t_m < \cdots < +\infty.$$

Let $\tau_k = t_k - t_{k-1}$ denote the dwell time, $k = 1,2,3,\cdots$.

Especially, when $u \equiv 0$, the switched nonlinear system (3) becomes

$$\dot{x} = f_{\sigma(t)}(x), \quad x \in \mathbb{R}^n. \tag{5}$$

For the development of this paper, we introduce two definitions and a lemma first.

Definition 1 [9]**:** A switching signal $\sigma(t)$ has an average dwell time τ_a if there are numbers N_0, τ_a such that

$$N_\sigma(\tau,t) \leq N_0 + \frac{t-\tau}{\tau_a}, \ \forall\, t \geq \tau \geq 0, \tag{6}$$

where $N_\sigma(\tau,t)$ denotes the number of switching of $\sigma(t)$ over the interval $[\tau,t)$, and N_0 denotes the chatter bound .

Definition 2 [17]**:** The switched system (3) and (4) is input/output-to-state stable (IOSS), if there exist functions $\gamma_1, \gamma_2 \in \mathcal{K}_\infty$, and $\beta \in \mathcal{KL}$ such that for each $t_0 \geq 0$, each $x_0 \in \mathbb{R}^n$ and each input u, the corresponding solution satisfies

$$|x(t)| \leq \beta(|x_0|,\, t-t_0) + \gamma_1(\|u\|_{[0,\,t)}) + \gamma_2(\|y\|_{[0,\,t)}), \quad \forall\, t \geq 0, \tag{7}$$

where $|.|$ denotes the Euclidean norm, and $\|.\|_{\mathscr{F}}$ is the supremum norm of a signal over an interval $\mathscr{F} \subseteq [0,\,\infty)$.

Remark 1: If no outputs are considered and equation (7) holds for $\gamma_2 \equiv 0$, then this system is said to be input-to-state stable (ISS). If also no inputs are present, then the system (3) and (4) reduces to be globally asymptotically stable (GAS).

Next, we introduce an improved ADT method from [14].

Lemma 1: Consider the switched nonlinear system (3) and (4), if there exist $\mathcal{C}^1$ functions $V_i:\ \mathbb{R}^n \to [0,\,\infty)$ and functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that

$$\alpha_1(|x|) \leq V_i(x) \leq \alpha_2(|x|), \tag{8}$$

and

$$\dot{V}_i(x)\big|_{(i)} = \frac{\partial^T V_i(x)}{\partial x} f_i(x) \leq -\lambda_i V_i(x), \tag{9}$$

where $\lambda_i > 0,\ i \in \mathcal{I}$, then the switched system (5) is GAS for any switching signal with ADT

$$\tau_a > \tau_a^* = \frac{a}{\lambda_s}, \tag{10}$$

where

$$a = \ln\mu,\ \ \mu = \sup_{x\neq 0}\frac{\alpha_2(|x|)}{\alpha_1(|x|)},\ \ \lambda_{\min} = \min_{i\in\mathcal{I}}\lambda_i. \tag{11}$$

Remark 2: In general, $\mu \geq 1$. Especially, if $\mu = 1$, which implies that $V_i(x) \equiv V(x), i \in \mathcal{I}$, i.e., $V(x)$ is a CLF for the switched system (5), and thus this system is GAS under arbitrary switching. It is also noted that the ADT method proposed in [9,10,12,13] needs the conditions (8)-(9) and an additional condition as "$V_i(x) \leq \mu V_j(x),\ \mu \geq 1,\ i \neq j,\ i,j \in \mathcal{I}$". Comparing Lemma 1 with the results in [9, 10, 12, 13], Lemma 1 needs fewer conditions, and thus can be applied to a wider range of systems.

Moreover, it is noted that the above α_1 and α_2 in (8) should have the same order, and which can ensure that μ exist. Furthermore, if $V_i(x) = x^T P_i x,\ P_i > 0$, then inequality (8) becomes

$$\alpha_{1i}|x|^2 \leq V_i(x) \leq \alpha_{2i}|x|^2 \tag{12}$$

and μ is given as

$$\mu = \max_{i\in\mathcal{I}}\frac{\alpha_{2i}}{\alpha_{1i}}. \tag{13}$$

For this case, if we use the ADT method in [9, 10, 12, 13], we can get

$$\mu' = \max_{i,j\in\mathcal{I}}\frac{\alpha_{2i}}{\alpha_{1j}}. \tag{14}$$

Obviously, $\mu \leq \mu'$.

Lemma 2: Consider the switched linear system $\dot{x} = A_i x,\ i \in \mathcal{I}$. If there exist a family of matrices $P_i > 0,\ i \in \mathcal{I}$ such that

$$A_i^T P_i + P_i A_i + \lambda_i P_i \leq 0, \tag{15}$$

where $\lambda_i > 0,\ i \in \mathcal{I}$, then the switched system $\dot{x} = A_i x$ is GAS for any switching signal with ADT

$$\tau_a > \tau_a^* = \frac{a}{\lambda_{\min}}, \tag{16}$$

where

$$a = \ln\mu,\ \ \mu = \max_{i\in\mathcal{I}}\frac{\lambda_{\max}(P_i)}{\lambda_{\min}(P_i)},\ \ \lambda_{\min} = \min_{i\in\mathcal{I}}\lambda_i. \tag{17}$$

Remark 3: For this switched linear systems $\dot{x} = A_i x$, not only an additional condition ($P_i \leq \mu P_j,\ i,\ j \in \mathcal{I}$ in the existing result [9]) is not needed, but also the lower bound of ADT τ_a^* obtained by Lemma 2 is smaller than the lower bound of ADT τ_a' obtained in [9]. In fact,

$$\tau_a^* = \max_{i\in\mathcal{I}}\left\{\frac{\lambda_{\max}(P_i)}{\lambda_{\min}(P_i)\lambda_0}\right\} \leq \max_{i,j\in\mathcal{I}}\left\{\frac{\lambda_{\max}(P_j)}{\lambda_{\min}(P_i)\lambda_0}\right\} = \tau_a'. \tag{18}$$

3. MAIN RESULTS

3.1. All subsystems are IOSS

In this section, we first investigate the IOSS of the switched nonlinear system (3) and (4) in the case where all of the constituent subsystems are IOSS. According to Lemma 1, we obtain the following result.

Theorem 1: Considering the switched nonlinear system (3) and (4). Suppose there exist functions $\alpha_1, \alpha_2, \varphi_1, \varphi_2 \in \mathcal{K}_\infty$, $\mathcal{C}^1$ functions $V_i:\ \mathbb{R}^n \to [0,\ \infty)$, constants $\lambda_i > 0, i \in \mathcal{I}$ such that (8) for all $x \in \mathbb{R}^n$, and moreover, the following inequalities holds:

$$\begin{aligned} &|x| \geq \varphi_1(\|u\|) + \varphi_2(\|h_i(x)\|) \\ &\Rightarrow \frac{\partial^T V_i}{\partial x} f_i(x,u) \leq -\lambda_i V_i(x),\ \ i \in \mathcal{I}. \end{aligned} \tag{19}$$

Then, the switched system (3) and (4) is IOSS for any switching signal with ADT

$$\tau_a > \tau_a^* = \frac{a}{\lambda_s}, \tag{20}$$

where a and λ_s are given as (11).

Proof: Let $t_0 \geq 0$ be arbitrary. For $t \geq t_0$, define $\nu(t) := \varphi_1(\|u\|_{[t_0,t]}) + \varphi_2(\|y\|_{[t_0,t]})$ and $\xi(t) := \alpha_1^{-1}(\mu^{N_0}\alpha_1(\nu(t)))$, where N_0 comes from (6). Moreover, define the balls around the origin $B_\nu(t) := \{x:\ |x| \leq \nu(t)\}$, $B_\xi(t) := \{x:\ |x| \leq \xi(t)\}$. Note that ν, and thus also ξ, are non-decreasing functions of time, and thus the ball B_ν and B_ξ has non-decreasing volume.

If $|x(t)| \geq \nu(t) \geq \varphi_1(\|u(t)\|) + \varphi_2(\|y\|_{[t_0,t]})$ during some time interval $t \in [t', t'']$, then $x(t)$ can be bounded above by

$$\begin{aligned} |x(t)| &\leq \alpha_1^{-1}(\mu^{N_0}e^{-\lambda_s(t-t')}\alpha_1(|x(t')|)) \\ &:= \beta(|x(t')|, t-t'). \end{aligned} \tag{21}$$

In fact, on any interval $[\tau_i, \tau_{i+1})\bigcap[t',\ t'']$, noting that the inequality (19) and according to the proof of Lemma 1, we arrive at

$$\alpha_1(|x(t)|) \leq e^{aN_\sigma(t',\ t)-\lambda_s(t-t')}\alpha_1(|x(t')|). \tag{22}$$

Then, according to (6), we conclude from (22)

$$\begin{aligned} \alpha_1(|x(t)|) &\leq e^{aN_0}e^{(\frac{a}{\tau_a}-\lambda_s)(t-t')}\alpha_1(|x(t')|) \\ &= \mu^{N_0}e^{(\frac{a}{\tau_a}-\lambda_s)(t-t')}\alpha_1(|x(t')|) \end{aligned} \tag{23}$$

Thus, if $\tau_a > \tau_a^*$, we get (21).

Denote the first time when $x(t) \in B_\nu(t)$ by $\check{t}_1$, i.e., $\check{t}_1 := \inf\{t \geq t_0:\ |x(t)| \leq \nu(t)\}$. For $t_0 \leq t \leq \check{t}_1$, according to (21), we obtain

$$|x(t)| \leq \beta(|x_0|, t-t_0). \tag{24}$$

If $\check{t}_1 = \infty$, which only can happen if $\nu(t) \equiv 0$, i.e., both the input u as well as the output y are equivalent to zero for all times, then the switched system (3) and (4) is IOSS. Hence in the following we assume that $\check{t}_1 < \infty$.

For $t > \check{t}_1$, $x(t)$ can be bounded above in terms of $\nu(t)$. Namely, let $\hat{t}_1 := \inf\{t > \check{t}_1 : |x(t)| > \nu(t)\}$. If this is an empty set, let $\hat{t}_1 := \infty$. Clearly, for all $t \in [\check{t}_1, \check{t}_2)$, it holds that $|x(t)| \le \nu(t) \le \xi(t)$. For the case that $\hat{t}_1 < \infty$, due to continuity of $x(.)$ and monotonicity for $\nu(t)$ it holds that $|x(\hat{t}_1)| = \nu(\hat{t}_1)$. Furthermore, for all $\tau > \hat{t}_1$, if $|x(\tau)| > \nu(\tau)$ define

$$\hat{t} := \sup\{t < \tau : |x(t)| \le \nu(t)\} \tag{25}$$

which can be interpreted as the previous exit time of the trajectory $x(t)$ from the ball B_ν. Again, due to the same argument as above, one obtains that $|x(\hat{t}_1)| = \nu(\hat{t}_1)$. But then, according to (21), it holds that

$$\begin{aligned} |x(\tau)| &\le \beta(\nu(\hat{t}), \tau - \hat{t}) = \alpha_1^{-1}(\mu^{N_0} e^{-\lambda_s(\tau-\hat{t})} \alpha_1(\nu(\hat{t}))) \\ &\le \alpha_1^{-1}(\mu^{N_0}\alpha_1(\nu(\hat{t})) = \xi(\hat{t}) \le \xi(\tau) \end{aligned} \tag{26}$$

where the last inequality follows from the monotonicity of $\xi(.)$.

Summarizing the above, for all $t \ge \check{t}_1$, it holds that

$$\begin{aligned} x(t)| &\le \xi(t) = \alpha_1^{-1}(\mu^{N_0}\alpha_1(\varphi_1(\|u\|_{[t_0,t]}) + \varphi_2(\|y\|_{[t_0,t]}))) \\ &\le \alpha_1^{-1}(\mu^{N_0}\alpha_1(2\varphi_1(\|u\|_{[t_0,t]})) \\ &\quad + \alpha_1^{-1}(\mu^{N_0}\alpha_1(2\varphi_1(\|y\|_{[t_0,t]})) \\ &:= \gamma_1(\|u\|_{[t_0,t]}) + \gamma_2(\|y\|_{[t_0,t]}). \end{aligned} \tag{27}$$

Combining (24) and (27) we obtain that

$$|x(t)| \le \beta(|x_0|, t - t_0) + \gamma_1(\|u\|_{[t_0,t]}) + \gamma_2(\|y\|_{[t_0,t]}) \tag{28}$$

for all $t \ge t_0$. Since $t_0 \ge 0$ was arbitrary, this means the switched system (3) and (4) is IOSS, which completes the proof. □

Remark 4: Comparing Theorem 1 with Theorem 1 in [24], Theorem 1 needs fewer conditions, i.e., the corresponding result proposed in [24] needs the conditions (8), (19) and an additional condition as

$$V_i(x) \le \mu V_j(x), \mu \ge 1, \ i \ne j, \ i, j \in \mathcal{I}.$$

Therefore, Theorem 1 is really an improvement of the corresponding existing result.

3.2. Some subsystems are not IOSS

In this section, we will investigate the IOSS of the switched nonlinear systems in the case of where some subsystems are not IOSS. Before this, we analyze the stability of switched nonlinear systems (5) in which both stable and unstable subsystems coexist. For the switching signal $\sigma(t)$ and any $t > \tau$, we let $T^u(\tau,t)$ (resp., $T^s(\tau,t)$) denote the total activation time of the unstable subsystems (resp., the stable subsystems) on interval $[\tau,t)$. Then, we let $\mathcal{I} = \mathcal{I}_s \cup \mathcal{I}_u$ such that $\mathcal{I}_s \cap \mathcal{I}_u = \emptyset$, and introduce a switching law from [10]

S1: Determine the $\sigma(t)$ satisfying $\frac{T^s(t_0,t)}{T^u(t_0,t)} \ge \frac{\lambda_u + \lambda^*}{\lambda_s - \lambda^*}$ holds for any $t > t_0$, where $\lambda^* \in (0, \lambda_s)$, λ_u and λ_s are given as (32).

Next, we give the main results in the following.

Theorem 2: Consider the switched nonlinear system (5), if there exist $\mathcal{C}^1$ functions $V_i(x)$: $\mathbb{R}^n \to [0,\infty)$, and functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that (8), and

$$\dot{V}_i(x)\big|_{(i)} = \frac{\partial^T V_i(x)}{\partial x} f_i(x) \le \lambda_i V_i(x), \quad i \in \mathcal{I}_u \tag{29}$$

$$\dot{V}_i(x)\big|_{(i)} = \frac{\partial^T V_i(x)}{\partial x} f_i(x) \le -\lambda_i V_i(x), \quad i \in \mathcal{I}_s \tag{30}$$

where $\lambda_i > 0$, $i \in \mathcal{I}$, then under the switching law **S1** the switched system (5) is GAS for any switching signal with ADT

$$\tau_a > \tau_a^* = \frac{a}{\lambda^*}, \tag{31}$$

where a is given as (11), $\lambda^* \in (0, \lambda_s)$ is an arbitrarily chosen number, and

$$\lambda_s = \min_{i \in \mathcal{I}_s} \lambda_i, \ \ \lambda_u = \max_{i \in \mathcal{I}_u} \lambda_i. \tag{32}$$

Proof: Let $t_1, t_2, ..,$ denote the time points at which switching occurs, and write p_j for the value of $\sigma(t)$ on $[t_{j-1}, t_j)$. Integrating the inequality (29) or (30) over the interval $[t_{j-1}, t_j)$, we obtain that

$$\ln V_{p_j}(x_j) - \ln V_{p_j}(x_{j-1}) \le \mathrm{sign}(p_j)\lambda_{p_j}\tau_j$$

and then

$$V_{p_j}(x_j) \le e^{\mathrm{sign}(p_j)\lambda_{p_j}\tau_j} V_{p_j}(x_{j-1}), \tag{33}$$

where $\mathrm{sign}(p_j) = 1$, if $p_j \in \mathcal{I}_u$, $\mathrm{sign}(p_j) = -1$, if $p_j \le \mathcal{I}_s$. Thus,

$$\begin{aligned} \alpha_1(|x_j|) &\le V_{p_j}(x_j) \le e^{\mathrm{sign}(p_j)\lambda_{p_j}\tau_j} V_{p_j}(x_{j-1}) \\ &\le e^{\mathrm{sign}(p_j)\lambda_{p_j}\tau_j} \alpha_2(|x_{j-1}|) \\ &\le e^{\mathrm{sign}(p_j)\lambda_{p_j}\tau_j} \frac{\alpha_2(|x_{j-1}|)}{\alpha_1(|x_{j-1}|)} \alpha_1(|x_{j-1}|) \\ &\le e^{\mathrm{sign}(p_j)\lambda_{p_j}\tau_j} \mu\alpha_1(|x_{j-1}|), \end{aligned}$$

where μ is given as (11).

Then, for any t satisfying $t_0 < t_1 < ... < t_i \le t < t_{i+1}$, we obtain

$$\begin{aligned} \alpha_1(|x_t|) &\le V_{p_{i+1}}(x_t) \le e^{\mathrm{sign}(p_{i+1})\lambda_{p_{i+1}}(t-t_i)} V_{p_{i+1}}(x_{t_i}) \\ &\le e^{\mathrm{sign}(p_{i+1})\lambda_{p_{i+1}}(t-t_0)} \alpha_2(|x_{t_i}|) \\ &\le \mu e^{\mathrm{sign}(p_{i+1})\lambda_{p_{i+1}}(t-t_i)} \alpha_1(|x_{t_i}|) \cdots \\ &\le \mu^{i+1} e^{\lambda_u T^u(t_0,t) - \lambda_s T^s(t_0,t)} \alpha_1(|x_0|) \\ &= e^{(i+1)a + \lambda^+ T^u(t_0,t) - \lambda_s T^s(t_0,t)} \alpha_1(|x_0|) \\ &= c e^{a N_\sigma(t_0,t) + \lambda_u T^u(t_0,t) - \lambda_s T^s(t_0,t)} \alpha_1(|x_0|), \end{aligned}$$

where $c = e^a$.

According to the switching law **S1**, i.e.,

$$\begin{aligned}\lambda_u T^u(t_0,t) - \lambda_s T^s(t_0,t) &\leq -\lambda^*(T^u(t_0,t)+T^s(t_0,t)) \\ &= -\lambda^*(t-t_0) \qquad (34)\end{aligned}$$

we obtain from (34) that

$$\alpha_1(|x_t|) \leq c e^{aN_\sigma(t_0,t)-\lambda^*(t-t_0)}\alpha_1(|x_0|). \qquad (35)$$

When $a = 0$, i.e, $\mu = 1$, we can obtain from (35) that

$$\alpha_1(|x_t|) \leq e^{-\lambda^*(t-t_0)}\alpha_1(|x_0|), \qquad (36)$$

which implies that the switched system (5) is GAS for arbitrary switching paths.

When $a > 0$, according to (6), we arrive at

$$aN_\sigma(t_0,t) - \lambda^*(t-t_0) \leq aN_0 + (\frac{a}{\tau_a} - \lambda^*)(t-t_0) \quad (37)$$

and then

$$\alpha_1(|x_t|) \leq c e^{aN_0} e^{(\frac{a}{\tau_a}-\lambda^*)(t-t_0)}\alpha_1(|x_0|). \qquad (38)$$

If $\tau_a > \frac{a}{\lambda^*}$, then under the switching law **S1** the switched system (5) is GAS. □

In the next, we consider the switched nonlinear systems (3) and (4) in which both IOSS and not IOSS subsystems coexist. Similarly, for the switching signal $\sigma(t)$ and any $t > \tau$, we let $T^u(\tau,t)$ (resp., $T^s(\tau,t)$) denote the total activation time of the not IOSS subsystems (resp., the IOSS subsystems) on interval $[\tau,t)$.

By extending the previous analysis results to investigate the IOSS of the switched nonlinear system (3) and (4), we obtain the following conclusion.

Theorem 3: Considering the switched nonlinear system (3) and (4). Suppose there exist functions $\alpha_1, \alpha_2, \varphi_1, \varphi_2 \in \mathcal{K}_\infty$, $\mathcal{C}^1$ functions $V_i :\ \mathbb{R}^n \to [0,\ \infty)$, $i \in \mathcal{I}$, constants $\lambda_s, \lambda_u > 0$ such that (8) for all $x \in \mathbb{R}^n$, and furthermore, the following inequalities holds:

$$\begin{aligned}&|x| \geq \varphi_1(\|u\|) + \varphi_2(\|h_i(x)\|) \\ &\Rightarrow \begin{cases} \dfrac{\partial^T V_i}{\partial x} f_i(x,u) \leq \lambda_u V_i(x), & i \in \mathcal{I}_u, \\ \dfrac{\partial^T V_i}{\partial x} f_i(x,u) \leq -\lambda_s V_i(x), & i \in \mathcal{I}_s. \end{cases}\end{aligned} \qquad (39)$$

Then, under the switching law **S1** the switched system (3) and (4) is IOSS for any switching signal with ADT

$$\tau_a > \tau_a^* = \frac{a}{\lambda^*} \qquad (40)$$

where a is given as (11), $\lambda^* \in (0, \lambda_s)$ is an arbitrarily chosen number, λ_s and λ_u are given as (32).

Proof: The proof of Theorem 3 follows the lines of the proof of Theorem 1. Define $\nu(t)$ as well as $\check{t}_1$ as in the proof of Theorem 1. Furthermore, define $\xi(t) := \alpha_1^{-1}(\mu^{N_0}\alpha_1(\nu(t)))$.

Under the switching law **S1**, according to the proof of Theorem 2 we arrive at for $t_0 \leq \check{t}_1$,

$$\begin{aligned}|x(t)| &\leq \alpha_1^{-1}(\mu^{N_0} e^{-\lambda^*(t-t_0)}\alpha_1(|x(t_0)|)) \\ &:= \beta(|x_0|, t-t_0). \end{aligned} \qquad (41)$$

Similar to Theorem 1, we obtain that for all $t \geq \check{t}_1$ it holds that $|x(t)| \leq \xi(t)$. That is, for each $\tau > \check{t}_1$ such that $|x(\tau)| > \nu(\tau)$, define the previous exit time $\hat{t}$ of the trajectory $x(.)$ from the ball B_ν as in (25). Then, using Theorem 2 on the interval $[\hat{t}, \tau]$, we arrive at the following inequality similar to (26):

$$\begin{aligned}|x(\tau)| &\leq \alpha_1^{-1}(\mu^{N_0} e^{-\lambda^*(\tau-\hat{t}\,)}\alpha_1(\nu(\hat{t}\,))) \\ &\leq \alpha_1^{-1}(\mu^{N_0}\alpha_1(\nu(\hat{t}\,)) \\ &= \xi(\hat{t}\,) \leq \xi(\tau). \end{aligned} \qquad (42)$$

Hence we conclude that for all $t \geq t_0$

$$\begin{aligned}|x(t)| &\leq \beta(|x_0|,\ t-t_0) + \xi(t) \qquad (43)\\ &\leq \beta(|x_0|,\ t-t_0) + \gamma_1(\|u\|_{[t_0,t)}) + \gamma_2(\|y\|_{[t_0,t)})\end{aligned}$$

where

$$\gamma_1(r) := \alpha_1^{-1}(\mu^{N_0}\alpha_1(2\varphi_1(r))),$$
$$\gamma_2(r) := \alpha_1^{-1}(\mu^{N_0}\alpha_1(2\varphi_2(r))),$$

which means the switched system (3) and (4) is IOSS according to (7) and since t_0 was arbitrary. □

4. AN ILLUSTRATIVE EXAMPLE

In this section, we give two illustrative examples to show how to use the results obtained in this paper to analyze the IOSS property of the switched nonlinear systems.

Example 1: Consider the following system [24]

$$\dot{x} = f_i(x,u) \qquad (44)$$

$h_i(x) = x_1 - x_2$ with $i \in \mathcal{I} = \{1,2\}$, and

$$f_1(x,u) = \begin{pmatrix} -x_1 + \sin(x_1 - x_2) \\ -x_2 + 0.8\sin(x_1 - x_2) \end{pmatrix},$$
$$f_2(x,u) = \begin{pmatrix} -x_1 + \sin(x_1 - x_2) + 0.5u \\ -x_2 + \sin(x_1 - x_2) + 0.5u \end{pmatrix}.$$

For the switched system (44), the authors have shown the ADT $\tau_a' = 1$ in [24].

In the next, we apply our method to find the ADT for the system (44). According to the results in [24], $V_1(x) = \frac{1}{2}(x_1^2 + 1.25x_2^2)$ is an IOSS Lyapunov function for the 1-th subsystem, $V_2(x) = \frac{1}{2}(x_1^2 + x_2^2)$ is an IOSS Lyapunov function for the 2-th subsystem, and $\lambda_s = \frac{7}{4}$. By our method, since

$$|x|^2 \leq V_1(x) \leq 1.25|x|^2,\ \ |x|^2 \leq V_2(x) \leq |x|^2,$$

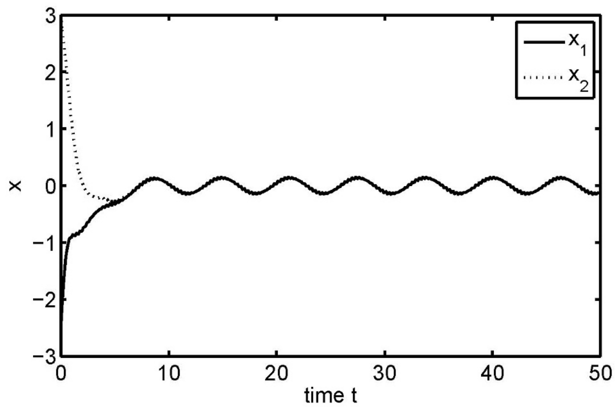

Fig. 1. The state's response.

where $|x|^2 = x_1^2 + x_2^2$, and we obtain that $\mu_1 = 1.25, \mu_2 = 1$. Thus, $\mu = \max\{\mu_1, \mu_2\} = 1.25$.

Therefore, we get

$$\tau_a^* = \frac{\ln\mu}{\lambda_s} = 0.1275 < \tau_a'.$$

According to Theorem 1, the conclusions can be obtained.

To illustrate the correctness of the above conclusion, we carry out some simulation results with the following choices. Initial Condition: $[x_1(0), x_2(0)] = [-2.5, 3]$, Input: $u = \sin(t)$, and Switching Path:

$$\sigma(t) = \begin{cases} 1, \ t \in [t_{2m}, t_{2m+1}), \ t_{2m+1} - t_{2m} = 0.1275, \\ 2, \ t \in [t_{2m+1}, t_{2m+2}), \ t_{2m+2} - t_{2m+1} = 0.2, \end{cases}$$

where $m = 0, 1, 2, \cdots$. The simulation result is given in Fig. 1, which is the response of the state under the above path $\sigma(t)$.

It can be observed from Figure 1 that the trajectory $x(t)$ is bounded. The simulation shows that Theorem 1 is very effective in analyzing the IOSS property of the switched nonlinear systems.

Example 2: Consider the following system

$$\dot{x} = f_i(x, u) \tag{45}$$

$h_i(x) = x_1 - x_2$ with $i \in \mathcal{I} = \{1, 2\}$, and

$$f_1(x,u) = \begin{pmatrix} -x_1 - x_1 x_2^2 \\ x_1^2 x_2 - 3x_2 \end{pmatrix}, \quad f_2(x,u) = \begin{pmatrix} 2x_1 + 2x_2 \\ x_1 + 3x_2 \end{pmatrix}.$$

It is easy to know that $V(x) = x^T x$ is a CLF for the switched system (45), and

$$\dot{V}|_{(1)} = 2x_1\dot{x}_1 + 2x_2\dot{x}_2|_{(1)} \le -2V(x),$$
$$\dot{V}|_{(2)} = 2x_1\dot{x}_1 + 2x_2\dot{x}_2|_{(2)} \le 8V(x).$$

According to the above results, we obtain that $\lambda_u = 8, \lambda_s = 2$ and $a = 0$. Therefore, the lower bound of ADT $\tau_a^* = 0$, i.e., the ADT can be arbitrary. Next, we choose $\lambda^* = 0.8$. Then, the switching law (**S1**) will require

$$\frac{T^s(t_0,t)}{T^u(t_0,t)} \ge \frac{\lambda_u + \lambda^*}{\lambda_s - \lambda^*} = \frac{8.8}{1.2} \approx 7.33.$$

According to Theorem 3, the switched system (45) is GAS under the above switching law (**S1**).

To illustrate the correctness of the above conclusion, we carry out some simulation results with the following choices. Initial Condition: $[x_1(0), x_2(0)] = [-2.5, 3]$, and Switching Path:

$$\sigma(t) = \begin{cases} 2, \ t \in [t_{2m}, t_{2m+1}), \ t_{2m+1} - t_{2m} = 0.2, \\ 1, \ t \in [t_{2m+1}, t_{2m+2}), \ t_{2m+2} - t_{2m+1} = 1.6, \end{cases}$$

where $m = 0, 1, 2, \cdots$. The simulation result is given in Fig. 2, which is the response of the state under the above path $\sigma(t)$.

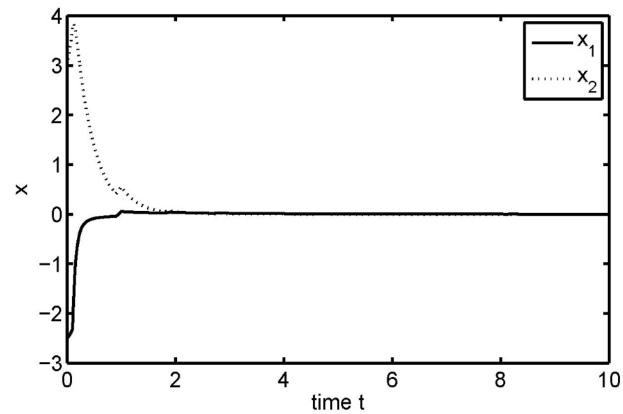

Fig. 2. The state's response.

It can be observed from Fig. 2 that the trajectory $x(t)$ converges to origin quickly. The simulation shows that Theorem 3 is very effective in analyzing the IOSS property for the switched nonlinear systems with both IOSS and non-IOSS subsystems. □

5. CONCLUSIONS

In this paper, we have investigated the IOSS stability for a class of switched nonlinear systems under ADT switching signals in two cases. First, a new IOSS result for the switched systems whose subsystems are IOSS has been obtained. Second, a new stability analysis result for the unforced nominal switched system in which parts of subsystems are unstable has been established, and a new IOSS result for the switched system in which parts of subsystems are IOSS has also been obtained. Two illustrative examples with simulation have been verified the correctness and validity of our results. Finally, we hope this improved ADT method may be used to study the stability of some other nonlinear systems in the future, such as, switched positive system, switched singular systems, etc.

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Lixia Liu was born in Weifan, China in 1990. She received her B.S. degree in Mathematics from Qilu University of Technology, Jinan, China, in 2014. She is currently a graduate student for master degree. Her research interests include nonlinear systems, switched systems.

Rongwei Guo was born in Jinan, China in 1979. He received his B.S. degree in Mathematics from University of Jinan, Jinan, China, in 2001, and his M.S. and Ph.D. degrees in Mathematics of Shanghai University and Shandong University, China, in 2004 and 2011, respectively. He joined School of Science at Qilu University of Technology in 2004, where he is currently a professor. His research interests include nonlinear systems, switched systems.

Shuping Ma was born in Rizhao, China in 1970. She received her B.S. degree in Mathematics from Shandong University, Jinan, China, in 1992, and her M.S. and Ph.D. degrees in Mathematics and System Science of Shandong University, China, in 1997 and 2000, respectively. She joined School of Mathematics and System Science at Shandong University in 2000, where she is currently a professor. Her research interests include singular systems, time-delay systems, Markov jump systems, robust control.