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Sampled-data Collective Rotating Consensus for Second-order Networks under Directed Interaction

Yintao Wang* and Qi Sun

Abstract: This paper investigates collective rotating motions of second-order multi-agent systems in 3-D under a sampled-data setting. A rotating consensus protocol was proposed and conditions on sampling period, damping gain, communication topology and rotating angle such that the vehicles will eventually move on a straight-line path, cylindrical spirals and logarithmic, respectively, were derived. In particular, when the vehicles move along circular orbits, the relative radii of the orbits (respectively, the relative phases of the vehicles on their orbits) are equal to the relative magnitudes (respectively, the relative phases) of the components of a right eigenvector associated with the critical eigenvalue of the nonsymmetric Laplacian matrix. Simulations are performed to validate the theoretical results.

Keywords: Collective motion, cooperative control, rotating consensus, sampled-data, second-order networks.

1. INTRODUCTION

Enabled by recent technological advances, the autonomous agents that can cooperatively perform complex tasks are rapidly becoming a reality. In particular, there have been considerable progresses reported in the literatures on robotics and sensor networks regarding as coverage control [1], surveillance [2], and environmental monitoring [3]. In many cooperative situations, all team members are required to reach an agreement on a common value including positions, phases, velocities and altitudes by negotiating with their neighbors. Such problems are called consensus and which play important roles in achieving coordinated behavior through local interactions.

The topics on consensus include the dynamics of agents, interaction topology of the network, finite-time convergence, time-delays, etc. In [4], the authors showed that information consensus under dynamically changing interaction topologies can be achieved asymptotically if the union of the directed interaction graphs has a spanning tree frequently enough as the system evolves. By using a comparison based tool, the authors [5] studied the finite-time consensus for single-integrator kinematics with unknown inherent nonlinear dynamics. Considering the limited memory, computation, and communication capabilities of the agents, [6] studied the consensus for agents over finite fields which assumed that agents can process only values from a finite alphabet. In [7], the authors considered consensus for second-order networks that takes into account the constraints of velocity and proposed a distributed control law only using the neighbors' positions and the each agent's own velocity. In [8], a synchronized tracking control law was proposed for multiple agents with high-order dynamic system modeled by a chain form, whereas the desired trajectory is only available for a portion of the team members. Another area of interest is consensus for systems with unknown inherent time delays. Cui *et al.* [9] considered the consensus problem for the general high order multi-agent systems with both the communication delay and input delay. Conditions were derived in [9] under which the consensus for these high order linear systems with time-varying, arbitrarily large yet bounded, and even unknown communication and inputs delays could be achieved, by the proposed state feedback and observer based output feedback controller. Furthermore, by only using the relative outputs of neighboring agents, a truncated reduced-order observer based protocol was proposed in [10] under which the consensus of high-order linear systems under directed interaction is achieved. Similarly, considering the fact that physical state of agents is not always completely accessible because of environmental noise or difficulty in measurement, Hu *et al.* [11] constructed a parallel Luenberger observer and the output of each observer is designed to estimate the local control input, moreover, LMI-based and optimization based methods were used to design the controller gains and observer parameters, respectively. Existing research in the aforementioned works mainly concerns with the translational behaviors of the agents. In fact, a class of collective circular motions widely exist in nature including flocks of birds flying along a circular orbit, foraging ants around a piece

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Yintao Wang and Qi Sun are with the School of marine science and technology, Northwestern Polytechnical University, Xi'an 710072, P. R. China (e-mails: wangyintao@nwpu.edu.cn, q.sun.cse@gmail.com).
* Corresponding author.

of rice, a swirling growing epiphyte colony, and panic escaping fish school around a predator, etc. These collective behaviors can be applied to formation flight of satellites, circular mobile sensor networks and so on. However, rare results are derived to generate such motions currently. One of the earliest contributions was given in [12], where circular motions are obtained with a virtual reference beacon. Following this line, more control algorithms were developed to gain collective stable circular motions with allowable equilibrium configurations [13,14]. In [13], a group of mobile agents were studied where each agent pursues the leading neighbor along the line of sight rotated by a common offset angle, resulting in a circular motion. In particular, motivated by the applications of autonomous underwater vehicles (AUVs) in oceanographic sampling, a novel rotating formation control problem was solved in [14] to make all agents circle around a common point with some special structures at an unit speed. The aforementioned cyclic pursuit formation controller [13,14] is based on a fixed network topology, especially, represented by a circulant matrix. The result was extended in [15] by introducing a rotation matrix to an existing second-order consensus protocol and the conditions under which rendezvous, circular patterns, and logarithmic spiral patterns can be achieved were derived, however, the center of the final trajectories can only be fixed. Along this research line, the latest work is referred to in [16] and [17] where control protocols were proposed to make all agents surround a common point with a desired formation structure, in 2D and 3D spaces under undirected graph, respectively.

Most of the aforementioned works are assumed that all information is transmitted continuously. However, due to the unreliability of information channels, the capability of transmission bandwidth of networks, the sensing ability of each agent and the total cost, it is quite difficult or expensive to ensure the continuity of information transmission. Hence it is more practical to take account of intermittent information transmission. The discrete-time coordination algorithms were primarily studied for systems with single-integrator dynamics [4,18,19], and consensus problems were addressed for second-order agents in a sampled-data setting in [20-22]. In these references, although conditions to ensure consensus were derived in discrete-time settings and the effect of sampled-data control on stability of vehicles were considered explicitly, there are few research results on rotating consensus problems of second-order agents. Based on the above considerations, we investigate rotating consensus problems of continuous-time second-order agents in a sampled-data setting. The main contributions of this work are twofold. Firstly, we proposed a sampled-data-based discrete rotating consensus algorithm for networked systems with second-order dynamic associated by directed interactions, and we derive sufficient conditions on the network topology, the sampling period, the damping factor and the Euler angle such that different collective motions can be achieved. Secondly, by introducing a velocity consensus item to an existing rotating consensus algorithm proposed in [15] for second-order dynamics, all agents finally converge to a desired plane in 3D and keep moving on with a synchronized velocity rather than moving surround a fixed center like [15].

This paper is organized as follows. In Section 2, we present some notations and some concepts in graph theory used throughout this note, we also formulate the problem to be studied. Section 3 states the main results, i.e. the convergence results are analyzed for the rotating consensus algorithm proposed. Numerical results are presented in Section 4 to illustrate the effectiveness of the theoretical results. Finally, conclusions and future research works are given in Section 5.

2. PRELIMINARIES AND PROBLEM FORMULATION

2.1. Graph theory

It is a natural way to model the interaction among a group of n agents by a directed graph $G=(V,E)$, where the agent set and the edge set can be denoted by $V=\{1,2,...,n\}$ and $E\subseteq V^2$ respectively. An edge denoted as (i,j) means that information can be sent from agent i to agent j, but not necessarily vice versa. That is, agent i is a neighbor of agent j. We use N_j to denote the neighbor set of agent j. A directed path is a sequence of edges of the form $(i_1,i_2),(i_2,i_3),...,$ where $i_k\in V$, $k=1,2,\cdots$. Moreover, a directed graph has a directed spanning tree if there exists at least one agent that has directed paths to all other agents. Usually, we use A to represent the weighted adjacency matrix, and each entry of A denoted as a_{ij} is defined such that a_{ij} is positive weight if $(j,i)\in E$, while $a_{ij}=0$ if $(j,i)\notin E$. The Laplacian matrix $L=[\ell_{ij}]\in R^{n\times n}$ with $\ell_{ii}=\sum_{j=1,j\neq i}^{n}a_{ij}$ and $\ell_{ij}=-a_{ij}$, $i\neq j$. In particular, we let $a_{ii}=0$, $i=1,...,n$, (i.e., agent i is not a neighbor of itself). It is straightforward to verify that L has at least one eigenvalue equal to zero with a corresponding right eigenvector $\mathbf{1}_n$, where $\mathbf{1}_n$ is the $n\times 1$ column vector with its entries are all ones.

2.2. Rotating consensus algorithm for second-order dynamics in a sampled-data setting

Consider vehicles with dynamics by

$$\dot{r}_i=v_i,\quad \dot{v}_i=u_i,\quad i=1,...,n, \tag{1}$$

where $r_i\in\mathbb{R}^m$ and $v_i\in\mathbb{R}^m$ represents the position vector and velocity vector of the ith agent respectively, and $u_i\in\mathbb{R}^m$ is the control input vector. Rewriting (1) in a matrix form

$$\dot{x}(t)=Ax(t)+Bu(t), \tag{2}$$

where t denotes the evolving time of (1) and

$$A=\begin{bmatrix}0&1\\0&0\end{bmatrix},\quad B=\begin{bmatrix}0\\1\end{bmatrix},\quad x=[r,v]^T.$$

With the knowledge of linear time invariant systems [23], the discretization form of (2) can be written as

$$x(k+1) = Gx(k) + Hu(k), \tag{3}$$

where $k = 0,1,...$ denotes the discrete-times index and T denotes the sampling-data period with $x(k) = [x(t)]_{t=kT}$, $u(k) = [u(t)]_{t=kT}$, $G = e^{AT}$, $H = \int_0^T e^{A\tau} B \mathrm{d}\tau$.
With zero-order hold, it follows that

$$u_i(t) = u_i[k], \quad kT \le t < (k+1)T. \tag{4}$$

Bringing (4) and $G = e^{AT}$, $H = \int_0^T e^{A\tau} B \mathrm{d}\tau$ into (3) will get

$$\begin{aligned} r_i[k+1] &= r_i[k] + Tv_i[k] + \frac{T^2}{2} u_i[k], \\ v_i[k+1] &= v_i[k] + Tu_i[k]. \end{aligned} \tag{5}$$

To achieve the collective rotating motions, in this note, we propose a distributed sampled-date rotating consensus algorithm for (5) as

$$u_i[k] = -\sum_{j=1}^{n} a_{ij} R(r_i[k] - r_j[k]) - \alpha \sum_{j=1}^{n} a_{ij}(v_i[k] - v_j[k]), \quad i = 1,...,n, \tag{6}$$

where α is a positive damping gain to be designed and $R \in \mathbb{R}^{m \times m}$ denotes a Cartesian coordinate coupling matrix defined in [15].

In this paper, we mainly formulate and concentrate on the collective control for a team of agents moving in a 3-D mission space, it thus follows that $r_i \in \mathbb{R}^3$, $v_i \in \mathbb{R}^3$ and $u_i \in \mathbb{R}^3$. However, all results to be proposed and analyzed still hold for $r_i \in \mathbb{R}^m$, $v_i \in \mathbb{R}^m$ and $u_i \in \mathbb{R}^m$ by use of the properties of the Kronecker product.

3. CONVERGENCE ANALYSIS OF THE SAMPLED-DATA ROTATING CONSENSUS ALGORITHM

In this section, we will analyze the convergence properties of (6) under fixed directed interaction, and derive the necessary and sufficient conditions to achieve the desired collective rotating motions. Before moving on, we need the following Notation and lemmas:

Lemma 1 [24]**:** Let $A \in \mathbb{R}^{p \times p}$, $B \in \mathbb{R}^{q \times q}$, $C \in \mathbb{R}^{p \times p}$, and $D \in \mathbb{R}^{q \times q}$, then $(A \otimes B)(C \otimes D) = AC \otimes BD$, where $\otimes$ denotes the Kronecker product. If λ is an eigenvalue of A and $x \in \mathbb{C}^p$ is a corresponding eigenvector of A, and μ is an eigenvalue of B with $y \in \mathbb{C}^q$ being the corresponding eigenvector of B, then $\lambda\mu$ is an eigenvalue of $A \otimes B$ and $x \otimes y$ is the corresponding eigenvector of $A \otimes B$.

Lemma 2 [24]**:** If $A \in \mathbb{C}^{n \times n}$ and if λ, μ are any two eigenvalues of A with $\lambda \neq \mu$, then any left eigenvector of A corresponding to μ is orthogonal to any right eigenvector of A corresponding to λ.

Lemma 3 [25]**:** Let L be the nonsymmetric Laplacian matrix associated with weighted directed graph G. Then L has at least one zero eigenvalue and all other eigenvalues have positive real parts. Furthermore, L has exactly one zero eigenvalue and all its nonzero eigenvalues have positive real parts if and only if the directed graph G has a directed spanning tree. In addition, there exists $\mathbf{1}_n$ satisfying $L\mathbf{1}_n = 0$ and $\mathbf{p} \in \mathbb{R}^n$, satisfying $\mathbf{p} > 0$, $\mathbf{p}^T L = 0$ and $\mathbf{p}^T \mathbf{1}_n = 1$. That is, $\mathbf{1}_n$ and $\mathbf{p}$ are, respectively, the right and left eigenvectors of L associated with the zero eigenvalue.

Notation: Let μ_i, $i = 1,...,n$ be the ith eigenvalue of $-L$ with associated right and left eigenvectors ω_i and ν_i. Let $\arg(\mu_i) = 0$ for $\mu_i = 0$ and $\arg(\mu_i) = \pi$ for $\mu_i < 0$ and $\mu_i \in \mathbb{R}$, where $\arg(\cdot)$ denotes the phase of a number. Denote $\arg(\mu_i) \in (\pi/2, \pi) \cup (\pi, 3\pi/2)$ for all other μ_i with non-zero imaginary parts. Without loss of generality, suppose that μ_i, $i = 1,...,n$ is labeled as $\mu_i = 0$, $\mu_{2,...,k} \in \mathbb{R}$, where k denotes $-L$ has k eigenvalues being on real axis, and $\mu_{k+1,...,n}$ represent other eigenvalues with non-zero imaginary parts. It follows from Lemma 3 that $\mu_i = 0$, $\omega_1 = \mathbf{1}_n$, and $\nu_1 = \mathbf{p}$.

Lemma 4 [15]**:** Given a rotation matrix $R \in R^{3 \times 3}$, let $\mathbf{a} = [a_1, a_2, a_3]^T$ and θ denote, respectively, the Euler axis and Euler angle. The eigenvalues of R are $\sigma_1 = 1$, $\sigma_2 = e^{\iota\theta}$, and $\sigma_3 = e^{-\iota\theta}$, where ι denotes the imaginary unit, with the associated right eigenvectors given by, respectively, $\varsigma_1 = \mathbf{a}$,

$$\begin{aligned} \varsigma_2 = [&(a_2^2 + a_3^2)\sin^2(\theta/2), \\ &(-a_1 a_2)\sin^2(\theta/2) + \iota a_3 \sin(0a/2)|\sin(\theta/2)|, \\ &(-a_1 a_3)\sin^2(\theta/2) - \iota a_2 \sin(\theta/2)|\sin(\theta/2)|]^T \end{aligned}$$

and $\varsigma_3 = \bar{\varsigma}_2$, where $\bar{\cdot}$ denotes the complex conjugate of a number. The associated left eigenvectors are, respectively, $\varpi_1 = \varsigma_1$, $\varpi_2 = \varsigma_2$ and $\varpi_3 = \varsigma_3$.

Corollary 1: Let R, ς_i, and ϖ_i be defined in Lemma 1, then $\sum_{i=1}^{3} \frac{1}{\varpi_i^T \varsigma_i} \varsigma_i \varpi_i^T = I_3$.

Proof: Noting that ς_i and ϖ_i are the right and left eigenvectors associated with σ_i, $i = 1,2,3$. Define two matrices as

$$\begin{bmatrix} \frac{\varsigma_1}{\varpi_1^T \varsigma_1} & \frac{\varsigma_2}{\varpi_2^T \varsigma_2} & \frac{\varsigma_3}{\varpi_3^T \varsigma_3} \end{bmatrix} \text{ and } \begin{bmatrix} \varpi_1^T \\ \varpi_2^T \\ \varpi_3^T \end{bmatrix}.$$

According to Lemma 2, $\varpi_i^T \varsigma_k = 0$ for each $i \neq k$, then it follows that

$$\begin{bmatrix} \varpi_1^T \\ \varpi_2^T \\ \varpi_3^T \end{bmatrix} R \begin{bmatrix} \frac{\varsigma_1}{\varpi_1^T \varsigma_1} & \frac{\varsigma_2}{\varpi_2^T \varsigma_2} & \frac{\varsigma_3}{\varpi_3^T \varsigma_3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{\iota\theta} & 0 \\ 0 & 0 & e^{-\iota\theta} \end{bmatrix},$$

where it can be found that the diagonal entries of diagonal matrix are exactly the eigenvalues of R. Therefore,

$$\begin{bmatrix} \frac{\varsigma_1}{\varpi_1^T \varsigma_1} & \frac{\varsigma_2}{\varpi_2^T \varsigma_2} & \frac{\varsigma_3}{\varpi_3^T \varsigma_3} \end{bmatrix} \text{ and } \begin{bmatrix} \varpi_1^T \\ \varpi_2^T \\ \varpi_3^T \end{bmatrix}$$

are a pair of invertible linear transformation. Then

$$\begin{bmatrix} \frac{\varsigma_1}{\varpi_1^T \varsigma_1} & \frac{\varsigma_2}{\varpi_2^T \varsigma_2} & \frac{\varsigma_3}{\varpi_3^T \varsigma_3} \end{bmatrix} \begin{bmatrix} \varpi_1^T \\ \varpi_2^T \\ \varpi_3^T \end{bmatrix} = \sum_{i=1}^{3} \frac{1}{\varpi_i^T \varsigma_i} \varsigma_i \varpi_i^T = I_3.$$

Denote $r = [r_1^T, \dots, r_n^T]^T$ and $v = [v_1^T, \dots, v_n^T]^T$ as the column stack vector of r_i, v_i, $i = 1, \dots, n$, respectively. We can rewrite the closed loop system of (5) using (6) as

$$\begin{bmatrix} r[k+1] \\ v[k+1] \end{bmatrix} = \underbrace{\begin{bmatrix} I_{3n} - \frac{T^2}{2} L \otimes R & (TI_n - \frac{T^2}{2} \alpha L) \otimes I_3 \\ -TL \otimes R & (I_n - \alpha TL) \otimes I_3 \end{bmatrix}}_{\Omega} \begin{bmatrix} r[k] \\ v[k] \end{bmatrix}, \tag{7}$$

where I_{3n} and I_3 denote the identity matrix with different dimensions respectively, and L is the nonsymmetric Laplacian matrix associated with G. To analyze the convergence property of (7), we firstly derive the eigenvalue and the corresponding eigenvectors of the system matrix Ω by the following lemma.

Lemma 5: According to the Notation we have denoted, the ith eigenvalue of $-L$ is given by μ_i with associated right and left eigenvectors ω_i and v_i, respectively. Denote the three eigenvalues of R as $\sigma_1 = 1$, $\sigma_2 = e^{\iota\theta}$ and $\sigma_3 = e^{-\iota\theta}$ with the corresponding right and left eigenvectors given by Lemma 1. Then the eigenvalues of Ω are given by

$$\zeta_{6(i-1)+2\ell-1} = 1 + \frac{T^2}{4}\mu_i\sigma_\ell + \frac{T\alpha\mu_i}{2} + \frac{\sqrt{4 + 4T^2\mu_i\sigma_\ell + \left(\frac{T^2}{2}\mu_i\sigma_\ell - T\alpha\mu_i\right)^2}}{2},$$

where $i = 1, \dots, n$, $\ell = 1, 2, 3$ with the corresponding right and left eigenvectors given by

$$\begin{bmatrix} \frac{T(\zeta_{6(i-1)+2\ell-1} + 1)}{2(\zeta_{6(i-1)+2\ell-1} - 1)} \omega_i \otimes \varsigma_\ell \\ \omega_i \otimes \varsigma_\ell \end{bmatrix} \text{ and } \begin{bmatrix} v_i \otimes \varpi_\ell \\ \left(\frac{T}{\zeta_{6(i-1)+2\ell-1} - 1} + \frac{\alpha}{\sigma_\ell} \right) v_i \otimes \varpi_\ell \end{bmatrix},$$

respectively. And the eigenvalues

$$\zeta_{6(i-1)+2\ell} = 1 + \frac{T^2}{4}\mu_i\sigma_\ell + \frac{T\alpha\mu_i}{2} - \frac{\sqrt{4 + 4T^2\mu_i\sigma_\ell + \left(\frac{T^2}{2}\mu_i\sigma_\ell - T\alpha\mu_i\right)^2}}{2},$$

where $i = 1, \dots, n$, $\ell = 1, 2, 3$ with the corresponding right and left eigenvectors given by

$$\begin{bmatrix} \frac{T(\zeta_{6(i-1)+2\ell} + 1)}{2(\zeta_{6(i-1)+2\ell} - 1)} \omega_i \otimes \varsigma_\ell \\ \omega_i \otimes \varsigma_\ell \end{bmatrix} \text{ and } \begin{bmatrix} v_i \otimes \varpi_\ell \\ \left(\frac{T}{\zeta_{6(i-1)+2\ell} - 1} + \frac{\alpha}{\sigma_\ell} \right) v_i \otimes \varpi_\ell \end{bmatrix},$$

respectively.

Proof: Suppose that ζ is an eigenvalue of Ω with an associated right eigenvector $\begin{bmatrix} f \\ g \end{bmatrix}$, where $f, g \in C^{3n}$. It thus follows that

$$\begin{bmatrix} I_{3n} - \frac{T^2}{2} L \otimes R & (TI_n - \frac{T^2}{2} \alpha L) \otimes I_3 \\ -TL \otimes R & (I_n - \alpha TL) \otimes I_3 \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix} = \zeta \begin{bmatrix} f \\ g \end{bmatrix},$$

which follows that

$$f + Tg - \frac{T^2}{2}(L \otimes Rf + \alpha L \otimes I_3 g) = \zeta g, \tag{8}$$

$$-\frac{T}{2}(L \otimes Rf + \alpha L \otimes I_3 g) = \frac{T}{2}(\zeta + 1) g. \tag{9}$$

Then we can easily get

$$f = \frac{T(\zeta + 1)}{2(\zeta - 1)} g. \tag{10}$$

Substituting (10) to (9) yields

$$L \otimes \left(\frac{T^3(\zeta + 1)}{4(\zeta - 1)} R + \frac{T^2}{2} \alpha I_3 \right) g = \frac{T}{2}(1 - \zeta) g,$$

which implies that g is actually the right eigenvector of matrix $L \otimes \left(\frac{T^3(\zeta+1)}{4(\zeta-1)} R + \frac{T^2}{2} \alpha I_3 \right)$ associated with the eigenvalue $\frac{T}{2}(1 - \zeta)$. For simplicity, we denote $M = L \otimes \left(\frac{T^3(\zeta+1)}{4(\zeta-1)} R + \frac{T^2}{2} \alpha I_3 \right)$. It follows from Lemma 1 that the eigenvalue of M corresponding to the ith eigenvalue of $-L$ and the ℓth eigenvalue of R is $-\frac{T^3(\zeta+1)}{4(\zeta-1)} \mu_i \sigma_\ell - \frac{T^2}{2} \alpha \mu_i$. Let the following equation $-\frac{T^3(\zeta+1)}{4(\zeta-1)} \mu_i \sigma_\ell - \frac{T^2}{2} \alpha \mu_i = \frac{T}{2}(1 - \zeta)$, then we will have

$$\zeta^2 - \left(\frac{T^2}{2}\mu_i\sigma_\ell + T\alpha\mu_i + 2 \right)\zeta + 1 - \frac{T^2}{2}\mu_i\sigma_\ell + T\alpha\mu_i = 0, \quad \ell = 1, 2, 3. \tag{11}$$

Noting that each ζ satisfying with (11), so it is actually the eigenvalue of Ω. Moreover, each eigenvalue of $-L$ corresponds to six eigenvalues of Ω by substituting σ_ℓ, $\ell = 1, 2, 3$, to (11). Also noting that q is actually the eigenvector of M, then following from Lemma 1 we can conclude that $g = \omega_i \otimes \varsigma_\ell$, and by substituting it to (10) we obtain $f = \frac{T(\zeta+1)}{2(\zeta-1)} \omega_i \otimes \varsigma_\ell$, which indicates that the right eigenvectors of Ω is

$$\begin{bmatrix} \frac{T(\zeta + 1)}{2(\zeta - 1)} \omega_i \otimes \varsigma_\ell \\ \omega_i \otimes \varsigma_\ell \end{bmatrix}.$$

The analysis of finding left eigenvectors of Ω is similar to the right one.

Lemma 6 [26]**:** The polynomial

$$s^2 + as + b = 0, \tag{12}$$

where $a, b \in C$. All roots of (1) are included in the unit circle if and only if all roots of

$$(1+a+b)\rho^2 + 2(1-b)\rho + 1 - a + b = 0 \tag{13}$$

are in the open left half plane, where (2) is derived through a variable transformation by substituting $s = (\rho+1)/(\rho-1)$ to (12).

Lemma 7: The roots of polynomial (11) are within the unit circle if the following conditions are satisfied:

1) $(\alpha, T) \in \{\frac{1}{2} < \frac{\alpha}{T}, D_i < 0\}$, where

$$D_i = \left(1 - \frac{2\alpha}{T}\right)^2 \left(\frac{2\alpha}{T} + \frac{4\Re(\mu_i)}{T^2|\mu_i|^2}\right) + \frac{16\Im^2(\mu_i)}{|\mu_i|^2 T^4} \tag{14}$$

and $\Im(\cdot)$ denotes the imaginary part of a number.

2) Under the assumption of 1, if $|\theta| < \theta_d^c$, where $\theta_d^c = \min_i \theta_i^c$ and θ_i^c is the solution of

$$\begin{aligned} &\frac{8}{|\mu_i|^2}\sin^2(\theta + \arg(\mu_i)) + 2\alpha^3 T \sin(\theta)\sin(2\theta) \\ &+ \frac{12\alpha T}{|\mu_i|}\sin(\theta)\sin(\theta + \arg(\mu_i)) - \frac{8\alpha T}{|\mu_i|}\cos(\arg(\mu_i)) \\ &+ \frac{8\alpha^2}{|\mu_i|}\cos\theta\cos(\arg(\mu_i)) + \frac{2T^2}{|\mu_i|}\cos(\theta + \arg(\mu_i)) \\ &+ \alpha T^3\cos\theta - 4\alpha^2 T^2\cos^2\theta + 4\alpha^3 T\cos^3\theta = 0, \\ &\qquad i = 1, \dots, n. \end{aligned} \tag{15}$$

Proof: Substituting $\zeta = (\eta+1)/(\eta-1)$ to (11), we will have

$$\eta^2 - \left(1 - \frac{2\alpha}{T\sigma_\ell}\right)\eta - \frac{4}{T^2\sigma_\ell\mu_i} - \frac{2\alpha}{T\sigma_\ell} = 0, \ \ \ell = 1,2,3. \tag{16}$$

Note from Lemma 6, all roots of (11) are within unit circle if and only if the roots of (16) are all on open left half plane. We will show nextly that the roots of (16) are all on open left half plane under the conditions 1 and 2.

Without loss of generality, we can suppose that μ_i is labeled as $\Re(\mu_1) = 0$ for $\mu_1 = 0$, $\Re(\mu_i) < 0$, $\Im(\mu_i) = 0$ for $i = 2, \dots, k$ and $\Re(\mu_i) < 0$, $\Im(\mu_i) \neq 0$ for $i = k+1, \dots, n$, where $\Re(\cdot)$ denotes the real part of a number. Noting that for $\mu_1 = 0$, (11) can be written as

$$\zeta^2 - 2\zeta + 1 = 0,$$

where it can be easily verified that the eigenvalues of Ω corresponding to μ_1 is exactly on the unit circle.

For the first statement, considering μ_i, $i = 2, \dots, n$ and $\sigma_1 = 1$ for $\ell = 1$ firstly. Then polynomial (16) can be rewritten as

$$\eta^2 - \left(1 - \frac{2\alpha}{T}\right)\eta - \frac{4}{T^2\mu_i} - \frac{2\alpha}{T} = 0. \tag{17}$$

Let s_1 and s_2 be the roots of (17), it thus follows that

$$s_{1,2} = \frac{1 - \frac{2\alpha}{T} \pm \sqrt{\left(1 + \frac{2\alpha}{T}\right)^2 + \frac{16}{T^2\mu_i}}}{2}, \quad i = 1, \dots, n \tag{18}$$

and

$$s_1 + s_2 = 1 - \frac{2\alpha}{T}.$$

It can be verified that s_1 and s_2 either have opposite imaginary parts or virtually no imaginary parts at all. It can be seen that the necessary condition of the fact that both s_1 and s_2 having negative real parts is $1 - \frac{2\alpha}{T} < 0$. Noting that $\sqrt{(1+\frac{2\alpha}{T})^2 + \frac{16}{T^2\mu_i}}$ always has non-negative real part then s_2 is definitely on the open left half plane under the necessary condition. So it is left to show condition under which s_1 is on the open left half plane. Supposing that $a + b\imath = \sqrt{(1+\frac{2\alpha}{T})^2 + \frac{16}{T^2\mu_i}}$, it follows that

$$a^2 - b^2 = \left(1 + \frac{2\alpha}{T}\right)^2 + \frac{16\Re(\mu_i)}{T^2|\mu_i|^2}, \tag{19}$$

$$ab = \frac{16\Im(\mu_i)}{T^2|\mu_i|^2}. \tag{20}$$

Substitute (20) to (19), we can obtain

$$a^4 - \left(\left(1 + \frac{2\alpha}{T}\right)^2 + \frac{16\Re(\mu_i)}{T^2|\mu_i|^2}\right)a^2 - \frac{64\Im^2(\mu_i)}{T^4|\mu_i|^4} = 0.$$

After some computation, it yields that

$$a^2 = \frac{\left(1 + \frac{2\alpha}{T}\right)^2 + \frac{16\Re(\mu_i)}{T^2|\mu_i|^2}}{2} + \frac{\sqrt{\left[\left(1 + \frac{2\alpha}{T}\right)^2 + \frac{16\Re(\mu_i)}{T^2|\mu_i|^2}\right]^2 + \frac{256\Im^2(\mu_i)}{T^4|\mu_i|^4}}}{2}.$$

It is not difficult to find out that s_1 has negative real part if and only if $(1 - \frac{2\alpha}{T})^2 > a^2$ and $1 - \frac{2\alpha}{T} < 0$. After some manipulation, we can finally get (14). Noting that the result of this statement is much similar to Lemma 4.3 proposed in [26], while we introduce a different way to get it.

For the second statement, we need to analyze the polynomial (16) when $\ell = 2,3$. Here we only consider the case for $\sigma_\ell = e^{\imath\theta}$ (the analysis of the case $\sigma_\ell = e^{-\imath\theta}$ is exactly same). Then the polynomial (16) can be rewritten as

$$\eta^2 - \left(1 - \frac{2\alpha}{Te^{\imath\theta}}\right)\eta - \frac{4}{T^2\mu_i e^{\imath\theta}} - \frac{2\alpha}{Te^{\imath\theta}} = 0. \tag{21}$$

Let s_1 and s_2 be the roots of (21), then we will have

$$s_{1,2} = \frac{1-\frac{2\alpha}{Te^{\iota\theta}} \pm \sqrt{\left(1+\frac{2\alpha}{Te^{\iota\theta}}\right)^2 + \frac{16}{T^2 e^{\iota\theta}\mu_i}}}{2}, \quad i=1,\ldots,n. \tag{22}$$

Note that α and T have been fixed by condition 2, it thus follows from (22) that there existing a critical value of θ which makes s_1 and s_2 exactly lie on the imaginary axis. Following a similar procedure of the first statement, what we need to do next is to derive the appropriate θ_i^c for (22) corresponding to each μ_i. After some manipulation, we can finally get (15), and it can also be verified that s_1 and s_2 are in the open left half plane if $\theta < \theta_i^c$, and have positive real parts when $\theta > \theta_i^c$.

Now the main result of this paper is summarized by the following theorem.

Theorem 1: Consider a network of second-order agents with fixed topology. Suppose that associated weighted directed graph G has a directed spanning tree. Let the control algorithm for (5) be given by (6), where $r[k]=[x[k],y[k],z[k]]^T$ and $v[k]=[v_x[k],v_y[k],v_z[k]]^T$. Let μ_i, ω_i, ν_i, and $\arg(\mu_i)$ be defined in the Notation, $\mathbf{p}$ be defined in Lemma 3, and R, $\mathbf{a}=[a_1,a_2,a_3]^T$, σ_ℓ, ς_ℓ, and ϖ_ℓ be defined in Lemma 4. Then:

1) All the agents will eventually converge to and move on a straight-line path described by (23) with the consensus velocity given by (24) if and only if α, T, and θ under the assumption of Lemma 7.

$$\Big[\mathbf{p}^T\left(x[0]+v_x[0]kT\right),\ \ \mathbf{p}^T\left(y[0]+v_y[0]kT\right), \\ \mathbf{p}^T\left(z[0]+v_z[0]kT\right)\Big] \tag{23}$$

$$\left[\mathbf{p}^T v_x[0], \mathbf{p}^T v_y[0], \mathbf{p}^T v_z[0]\right], \tag{24}$$

where $x[0]$, $y[0]$, $z[0]$ and $v_x[0]$, $v_y[0]$, $v_z[0]$ is the initial positions and velocities of all agents, respectively.

2) If α and T satisfy the condition 1 of Lemma 7 and $|\theta| = \theta_d^c$ in condition 2, there exists a unique μ_m that exactly makes two eigenvalues of Ω being on the unit circle. Then all the agents will eventually move on cylindrical spirals with center line given by (23) and period $2\pi/\arg(\zeta_c)$, where ζ_c is the eigenvalue of Ω on the unit circle associated with μ_m. The radius of the cylindrical spiral of agent i is given by

$$2\left|\omega_{mi} q_c^T \begin{bmatrix} r[0] \\ v[0] \end{bmatrix}\right| \sqrt{a_2^2+a_3^2}\,\sin^2\left(\frac{\theta}{2}\right),$$

where ω_{mi} is the ith component of ω_m and

$$q_c = \left(\frac{T(\zeta_c+3)}{2(\zeta_c-1)}+\frac{\alpha}{e^{\iota\theta}}\right)\begin{bmatrix} \nu_m \otimes \varpi_2 \\ \left(\frac{T}{\zeta_c-1}+\frac{\alpha}{e^{\iota\theta}}\right)\nu_m \otimes \varpi_2 \end{bmatrix},$$

where

$$\zeta_c = \frac{2+\frac{T^2}{2}\lambda_c + T\alpha\mu_m + \sqrt{4+4T^2\lambda_c + \left(\frac{T^2}{2}\lambda_c - T\alpha\mu_m\right)^2}}{2}$$

with $\lambda_c = \mu_m e^{\iota|\theta|}$. The relative radii of the cylindrical spiral are equal to the relative magnitudes of ω_{mi}, and the relative phases of the agents on their spirals are equal to the relative phases of ω_{mi}.

3) If α and T satisfy the condition 1 of Lemma 7 and $\theta_d^c < |\theta| < \max_{i,i\neq m}\theta_i^*$, there will exists a unique μ_m that exactly makes two eigenvalues of Ω being outside of the unit circle. Then all the agents will eventually move along logarithmic columnar curves with center line given by (23), and with growing rate $\Re(\zeta_s)$ and period $2\pi/\Im(\zeta_s)$, where ζ_s is the eigenvalue of Ω being outside of the unit circle associated with μ_m. The radius of the logarithmic columnar curve of agent i is given by

$$2\left|\omega_{mi} q_c^T \begin{bmatrix} r[0] \\ v[0] \end{bmatrix}\right| e^{\Re(\zeta_s)t}\sqrt{a_2^2+a_3^2}\,\sin^2\left(\frac{\theta}{2}\right),$$

where

$$q_c = \left(\frac{T(\zeta_s+3)}{2(\zeta_s-1)}+\frac{\alpha}{e^{\iota\theta}}\right)\begin{bmatrix} \nu_m \otimes \varpi_2 \\ \left(\frac{T}{\zeta_s-1}+\frac{\alpha}{e^{\iota\theta}}\right)\nu_m \otimes \varpi_2 \end{bmatrix}, \text{ and}$$

$$\zeta_s = 1+\tfrac{T^2}{4}\lambda_s + \frac{T\alpha\mu_m}{2} + \frac{\sqrt{4+4T^2\lambda_s+\left(\frac{T^2}{2}\lambda_s - T\alpha\mu_m\right)^2}}{2}$$

with $\lambda_s = \mu_m e^{\iota|\theta|}$.

The relative radii of the logarithmic columnar curves are equal to the relative magnitudes of ω_{mi} and the relative phases of the agents on their spirals are equal to the relative phases of ω_{mi}.

Proof: 1) For the first statement, suppose L being the nonsymmetric Laplacian matrix associated with directed graph G which has a directed spanning tree. Then the eigenvalues of $-L$ has exactly one zero eigenvalue and all other eigenvalues are in the open left half plane [4]. According to Lemma 7, eigenvalues of Ω are all within unit circle when the corresponding μ_i have non-positive real part. So it is left to derive the eigenvalues of Ω when the corresponding μ_i equals to zero. By substituting $\mu_i = 0$ to (11), then Ω have exactly six eigenvalues equaling to one, furthermore, it is not difficult to verify that the geometric multiplicity of these eigenvalues is three. It thus follows from Lemma 3 that we can choose $[1_n^T \otimes \varsigma_\ell^T, 0_{3n}^T]^T$ as the right eigenvector associated with eigenvalue one, and $[0_{3n}^T, T\mathbf{p}^T \otimes \varpi_\ell^T]^T$ as the according left eigenvector. Then $[0_{3n}^T, (1/T)1_n^T \otimes \varsigma_\ell^T]^T$ and $[\mathbf{p}^T \otimes \varpi_\ell^T, 0_{3n}^T]^T$ are generalized right and left eigenvectors accordingly. Noting that Ω can be written in Jordan canonical form as $\Omega = SJS^{-1}$, where J is the Jordan block diagonal matrix with the eigenvalues of Ω be the diagonal entries and the columns of S, denoted by p_k, $k = 1,\dots,6n$, can be chosen to be the right eigenvectors or generalized right eigenvectors of Ω, the rows of S^{-1}, denoted by q_k, $k = 1,\dots,6n$, can be chosen to be left eigenvectors or generalized left eigenvectors of Ω. Let

$$p_{2\ell-1} = \begin{bmatrix} 1_n \otimes \varsigma_\ell \\ 0_{3n} \end{bmatrix}, \quad p_{2\ell} = \begin{bmatrix} 0_{3n} \\ (1/T)1_n \otimes \varsigma_\ell \end{bmatrix},$$
$$q_{2\ell-1} = \begin{bmatrix} \mathbf{p} \otimes \varpi_\ell \\ 0_{3n} \end{bmatrix} \text{ and } q_{2\ell} = \begin{bmatrix} 0_{3n} \\ T \otimes \varpi_\ell \end{bmatrix}.$$

Also note that the eigenvalues of Ω are within unit circle except those six ones being on it, so it can be easily verified that $p_k^T q_k = 1$ and $p_k^T q_\ell = 0$, $k \neq \ell$. Then we will have

$$\Omega^k \to SJ^k S^{-1} \to \sum_{\ell=1}^{3} \left([p_{2\ell-1} \quad p_{2\ell}] \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \begin{bmatrix} q_{2\ell-1}^T \\ q_{2\ell}^T \end{bmatrix} \right)$$
$$\to \begin{bmatrix} 1_n p^T & kT1_n p^T \\ 0_{n\times n} & 1_n p^T \end{bmatrix} \otimes \sum_{\ell=1}^{3} (1/\varpi_\ell^T \varsigma_\ell) \varsigma_\ell \varpi_\ell^T.$$

Based on the fact that $\sum_{\ell=1}^{3} (1/\varpi_\ell^T \varsigma_\ell)\varsigma_\ell \varpi_\ell^T = I_3$ according to Corollary 1, it will be followed that

$$x_i[k] \to \mathbf{p}^T \left(x[0] + kTv_x[0] \right),$$
$$y_i[k] \to \mathbf{p}^T \left(y[0] + kTv_y[0] \right),$$
$$z_i[k] \to \mathbf{p}^T \left(z[0] + kTv_z[0] \right),$$

and

$$v_{xi}[k] \to \mathbf{p}^T v_x[0],$$
$$v_{yi}[k] \to \mathbf{p}^T v_y[0],$$
$$v_{zi}[k] \to \mathbf{p}^T v_z[0],$$

when $k \to \infty$, which shows that all agents will finally move on the straight-line path given by (23) with the consensus velocities given by (24).

2) For the second statement, if $\theta = \theta_d^c$ ($\theta = -\theta_d^c$ is similar except that all agents will move in the opposite direction). Considering the fact that (11) cannot have two conjugated roots when the corresponding μ is on the open left half plane, then there must exists a unique μ_m that makes one of the roots of the corresponding polynomial (11) exactly lie on the unit circle, and based on the truth that Ω is a real matrix which also implies that Ω has exactly two conjugated eigenvalues being on the unit circle. Noting that it is possible for ζ_{6m-3} to be on the unit circle, while ζ_{6m-2} is definitely within the unit circle. For simplicity, we denote this eigenvalue as $\zeta_c = e^{\imath \arg(\zeta_c)}$, and its conjugate is given by $\bar{\zeta}_c = e^{-\imath \arg(\zeta_c)}$. Noting from Lemma 5 that the right and left eigenvectors associated with ζ_c are given by

$$\begin{bmatrix} \dfrac{T(\zeta_c+1)}{2(\zeta_c-1)} \omega_m \otimes \varsigma_2 \\ \omega_m \otimes \varsigma_2 \end{bmatrix} \text{ and } \begin{bmatrix} v_m \otimes \varpi_2 \\ \left(\dfrac{T}{\zeta_c - 1} + \dfrac{\alpha}{e^{\imath\theta}} \right) v_m \otimes \varpi_2 \end{bmatrix},$$

respectively. Similar to the first statement, we can choose

$$p_c = \begin{bmatrix} \dfrac{T(\zeta_c+1)}{2(\zeta_c-1)} \omega_m \otimes \varsigma_2 \\ \omega_m \otimes \varsigma_2 \end{bmatrix} \text{ and}$$
$$q_c = \left(\frac{T(\zeta_c+3)}{2(\zeta_c-1)} + \frac{\alpha}{e^{\imath\theta}} \right) \begin{bmatrix} v_m \otimes \varpi_2 \\ \left(\dfrac{T}{\zeta_c - 1} + \dfrac{\alpha}{e^{\imath\theta}} \right) v_m \otimes \varpi_2 \end{bmatrix}.$$

It can be verified that $q_c^T p_c = 1$. Similarly, it follows that the right and left eigenvectors p_c^* and q_c^* according to eigenvalue $\bar{\zeta}_s$ are, respectively, conjugates of p_c and q_c. Then

$$\begin{bmatrix} r[k] \\ v[k] \end{bmatrix} \to \Omega^k \begin{bmatrix} r[0] \\ v[0] \end{bmatrix} \to$$
$$\begin{bmatrix} 1_n \mathbf{p}^T & kT1_n \mathbf{p}^T \\ 0_{n\times n} & 1_n \mathbf{p}^T \end{bmatrix} \otimes I_3 \begin{bmatrix} r[0] \\ v[0] \end{bmatrix} + c[k],$$

when $k \to \infty$, where

$$c[k] = \left(e^{\imath k \arg(\zeta_c)} p_c q_c^T + e^{-\imath k \arg(\zeta_c)} p_c q_c^T \right) \begin{bmatrix} r[0] \\ v[0] \end{bmatrix}.$$

Let $c_i[k]$ denote the ith component of $c[k]$, then

$$c_{3(i-1)+\ell} = 2 \left| \frac{T(\zeta_c+3)}{2(\zeta_c-1)} \varsigma_2(\ell) \omega_{mi} q_c^T \begin{bmatrix} r[0] \\ v[0] \end{bmatrix} \right| \cdot$$
$$\cos \left\{ \begin{aligned} &\arg(\zeta_c)k \\ &+ \arg\left(\omega_{mi} q_c^T \frac{T(\zeta_c+3)}{2(\zeta_c-1)} \varsigma_2(\ell) \omega_{mi} q_c^T \begin{bmatrix} r[0] \\ v[0] \end{bmatrix} \right) \\ &+ \arg(\varsigma_2(\ell)) \end{aligned} \right\},$$

where $i = 1,\dots,n$ and $\ell = 1,2,3$. Also note that ω_{mi} and $\varsigma_2(\ell)$ denote, respectively, the ith component of ω_m and the ℓth component of ς_2. Therefore, it follows that

$$x_i[k] \to \mathbf{p}^T \left(x[0] + kTv_x[0] \right) + c_{3i-2}[k],$$
$$y_i[k] \to \mathbf{p}^T \left(y[0] + kTv_y[0] \right) + c_{3i-1}[k],$$
$$z_i[k] \to \mathbf{p}^T \left(z[0] + kTv_z[0] \right) + c_{3i}[k].$$

After some manipulations, it can be verified that

$$\| [c_{3i-2}(k), c_{3i-1}(k), c_{3i}(k)]^T \|$$
$$= 2 \left| \omega_{mi} q_c^T \begin{bmatrix} r[0] \\ v[0] \end{bmatrix} \right| \sqrt{a_2^2 + a_3^2} \sin^2\left(\frac{\theta}{2} \right),$$

which concludes that all agents will eventually move on cylindrical spirals with center line given by (23) with period of $2\pi / \arg(\zeta_c)$. The relative radii of the cylindrical spiral are equal to the relative magnitudes of ω_{mi}, and the relative phases of the agents on their spirals are equal to the relative phases of ω_{mi}.

3) For the third statement, here we only consider $\theta_d^c < \theta < \max_{i,i\neq m} \theta_i^*$ (respectively, $-\max_{i,i\neq m} \theta_i^* < \theta < -\theta_d^c$). Similar to the analysis of the second statement, there exists a unique μ_m which makes Ω have a pair of conjugate eigenvalues outside of the unit circle. Similarly, it is possible for ζ_{6m-3} to be outside of the unit circle, while ζ_{6m-2} is definitely within the unit circle. For simplicity, we denote this eigenvalue as ζ_s, and its complex conjugate is given by $\bar{\zeta}_s$. By following a similar procedure to the proof of the second statement, we can easily obtain that all the agents will eventually move along logarithmic columnar curves with center line given by (23), with growing rate $\Re(\zeta_s)$ and period $2\pi/\Im(\zeta_s)$. The radius of the logarithmic columnar curve of agent i is given by

$$2\left|\omega_{mi}q_c^T\begin{bmatrix} r[0] \\ v[0] \end{bmatrix}\right| e^{\Re(\zeta_s)t}\sqrt{a_2^2+a_3^2}\sin^2\left(\frac{\theta}{2}\right).$$

The relative radii of the logarithmic columnar curves are equal to the relative magnitudes of ω_{mi}, and the relative phases of the agents on their spirals are equal to the relative phases of ω_{mi}.

4. SIMULATION

Consider a group of four vehicles associated by a directed graph G shown in Fig. 1. Note that G has a spanning tree. Let L associated with G be defined as follows:

$$\begin{bmatrix} 1.4 & 0 & -1.0 & -0.4 \\ -1.5 & 1.5 & 0 & 0 \\ -1.0 & -0.4 & 1.4 & 0 \\ -0.5 & 0 & 0 & 0.5 \end{bmatrix}.$$

For the controller parameters, we set $\alpha = 2.0$ and $T = 0.4$, which can be verified that both satisfy with the conditions derived in Lemma 7. By simple computations, it is solved that $\theta_d^c = 1.1989$. Let R be the rotation matrix corresponding to Euler axis $1/\sqrt{14}[1,2,3]^T$ and Euler angle θ. Then, it can be calculated that the right eigenvector of $-L$ associated with $\mu = 0$ is $\omega = [-0.2119,\ -0.3684,\ -0.4709,\ 0.7731]^T$ an $\mathbf{p} = [0.3697,\ 0.0704,\ 0.2641,\ 0.2958]^T$.

Figs. 2, 3, 4 show, respectively, the trajectories of the four agents using (6) with $\theta = \theta_d^c - 0.2$, $\theta = \theta_d^c$, $\theta = \theta_d^c + 0.1$. It can be seen that all agents finally move on the straight-line path given by (23) with $\theta = \theta_d^c - 0.2$, move on cylindrical spirals with $\theta = \theta_d^c$, and move along logarithmic columnar curves with $\theta = \theta_d^c + 0.1$. In particular, it can be observed from Fig. 5 that all agents will converge to and keep moving on the straight-line path given by (23) with a consensus velocity rather than rendezvous at a position. Similarly, the final velocity patterns can also be observed for other two cases.

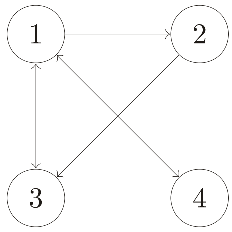

Fig. 1. Network topology for four agents. An arrow from j to i denotes that agent i can receive information from agents j. A double arrow is the simplified form of two directed arrows.

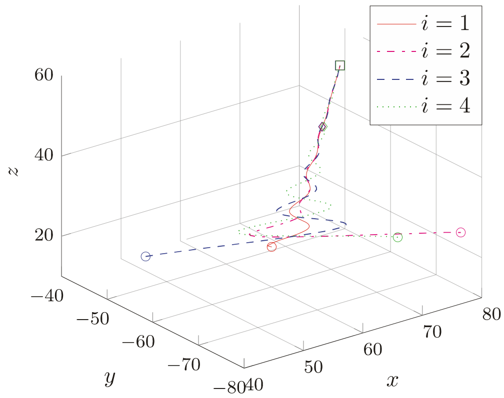

Fig. 2. Trajectories of the four agents using (6) with $\theta = \theta_d^c - 0.2$ in a sampled-data setting.

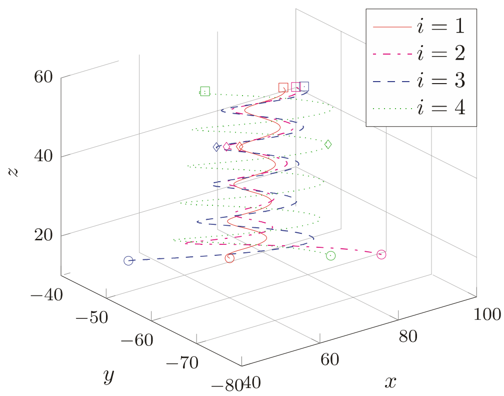

Fig. 3. Trajectories of the four agents using (6) with $\theta = \theta_d^c$ in a sampled-data setting.

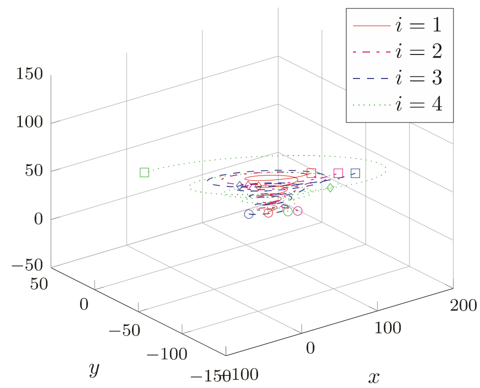

Fig. 4. Trajectories of the four agents using (6) with $\theta = \theta_d^c + 0.1$ in a sampled-data setting.

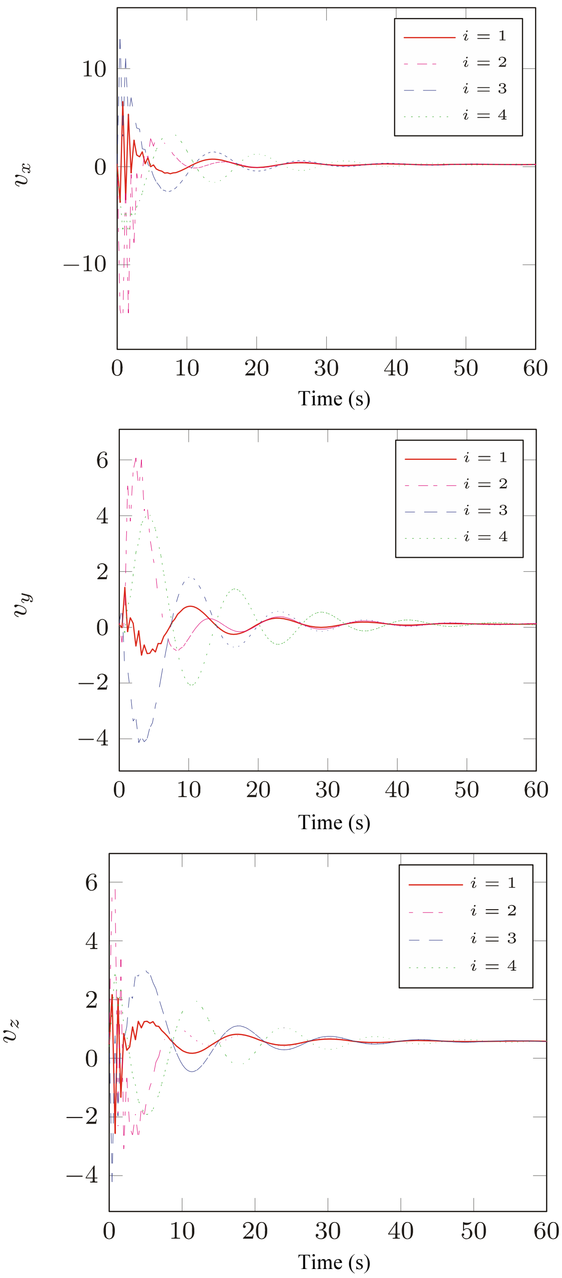

Fig. 5. Velocity evolving of the four agents using (6) with $\theta = \theta_d^c - 0.2$ in a sampled-data setting.

5. CONCLUSION

This technical note has studied rotating consensus problems of discrete second-order agents in a sampled-data setting. Using a rotation matrix and a relative damping term, we introduced a protocol and derived the conditions under which collective motions of straight-line path, cylindrical spiral and logarithmic columnar curve were achieved. With the help of matrix theory and graph skills, the convergence properties of the protocol were analyzed theoretically. Finally, simulation results are performed to demonstrate the effectiveness of the proposed algorithm. Although, rotating consensus without a leader is useful in applications such as the typical collective motion patterns given in this paper, there are many other applications that require a dynamic leader, examples include formation flying, body guard, and coordinated target tracking applications. So we will address the theoretical challenges when there exists a static or moving target, which will pose many challenging problems that warrant further research.

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Yintao Wang received his B.S., M.S., and Ph.D. degrees in Control Science and Technology from Northwestern Polytechnical University, China, in 2002, 2005, and 2011, respectively. He is currently an associate professor at the School of Marine Science and Technology of Northwestern Polytechnical University. His research interests include cooperative estimation and control, multi-agent systems.

Qi Sun received his B.S. degree in Control Science and Technology from Northwestern Polytechnical University in 2009. He is currently pursuing his M.S. degree at Northwestern Polytechnical University. His research interests include cooperative estimation and control, optimal control theory, and adaptive dynamic programming.