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A Comment on "Exponential Stability of Nonlinear Delay Equation with Constant Decay Rate via Perturbed System Method"

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Abstract: In this paper, we point out that inequality (7) of [5] is not correct. A feasible modified and corrected version of the main result is presented. Furthermore, some numerical examples are given to illustrate the applicability of the modified result.

Keywords: Exponential stability, nonlinear inequality, time delay, time-varying systems.

1. INTRODUCTION

We consider the following delay system presented in [5],

$$\begin{aligned} \dot{x}(t) &= -ax(t) + Af(x(t)) + Bg(x(t-\tau)), \ t \ge t_0, \\ x(t) &= \varphi(t-t_0), \quad t_0 - \tau \le t \le t_0, \end{aligned} \tag{1}$$

where $x = [x_1, x_2, ..., x_n]^T$ denotes the state vector, $a > 0$ is a constant decay rate, $A = (a_{ij})_{n\times n}$ and $B = (b_{ij})_{n\times n}$ are real matrices. f and g are continuous vector-value functions over $\mathbb{R}^n$ with $f(0) = g(0) = 0$,

$$\varphi = \{\varphi(s) : -\tau \le s \le 0\} \in C([-\tau, 0]; \mathbb{R}^n).$$

We agree that the perturbed system method presented in [5] for exponential stability of nonlinear delay system is interesting. It tries to show that the delay system will remain exponential stability, provided the time lag is small enough. However, the proof of Theorem 1 in [5] contains a mistake, so that the result presented in this Theorem is not correct. The goal of this paper is to present a new and corrected version of this theorem. A counterexample is given to prove that the inequality used in [5] is not correct and can not be considered as a new or other version of Gronwall inequality. Furthermore, examples are provided to demonstrate the less conservatism of the obtained results based on Gronwall inequality.

First, we show that the use of the Gronwall inequality in the proof of Theorem 1 in [5] is not correct. Indeed the authors obtain:

$$\begin{aligned} |x(t) - y(t)| \le (\alpha \| A \| + \beta \| B \|) \int_{t_0}^{t} e^{a(s-t)} |x(s) - y(s)| ds \\ + \beta \| B \| \int_{t_0}^{t} e^{a(s-t)} |x(s) - x(s-\tau)| ds, \end{aligned} \tag{2}$$

and then, they showed that

$$\begin{aligned} |x(t) - y(t)| \le \beta \| B \| a^{-1} \exp((\alpha \| A \| \\ + \beta \| B \|)(1 - e^{-a(t-t_0)})) \\ \times \int_{t_0}^{t} e^{a(s-t)} |x(s) - x(s-\tau)| ds, \end{aligned} \tag{3}$$

which gives (7): If $t_0 \le t \le t_0 + 2\delta$,

$$\begin{aligned} |x(t)| \le |y(t)| + \beta \| B \| a^{-1} \exp((\alpha \| A \| \\ + \beta \| B \|)(1 - e^{-2a\delta})) \\ \times \int_{t_0}^{t} e^{a(s-t)} |x(s) - x(s-\tau)| ds. \end{aligned}$$

Remark 1: The previous inequality (3) (or (7) obtained in [5]) is not correct because the use of Gronwall inequality contains an error, there's no version of Gronwall inequality to establish the passage from (2) to (3), and the following counterexample illustrate this. In fact, the inequality used by the authors in [5] can not be considered as a new or other version of Gronwall inequality as explained in the following.

Example: We show that, with $m = 1$, $u(t) = t$ and $v(t) = 1$ for all $t \ge 0$ that satisfies

$$u(t) \le m \int_0^t e^{(s-t)} u(s) ds + \int_0^t e^{(s-t)} v(s) ds, \tag{4}$$

may not satisfies

$$u(t) \le \exp(m(1 - e^{-t})) \times \int_0^t e^{(s-t)} v(s) ds. \tag{5}$$

Indeed, we have

$$\int_0^t e^{(s-t)} v(s) ds = 1 - e^{-t}$$

and

$$\int_0^t e^{(s-t)} u(s) ds = t - 1 + e^{-t}.$$

Then, (4) is given by

$$t = u(t) \le \int_0^t e^{(s-t)} u(s) ds + \int_0^t e^{(s-t)} v(s) ds = t, \quad \forall t \ge 0.$$

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Suppose that (5) is correct, this implies that

$$u(t) \le e^{1-e^{-t}} \times \int_0^t e^{(s-t)} v(s) ds. \quad (6)$$

Hence, (6) gives that

$$t \le (1-e^{-t}) e^{1-e^{-t}} \le e, \ \forall t \ge 0,$$

which is a contradiction.

The above counterexample prove that there's no version of Gronwall inequality to establish the passage (4) to (5) for u and v nonnegative continuous functions. Thus, the expression (3) (or (7) in [5]) can not be obtained by any argument.

We give in the following section a modified and corrected version of the original result proposed by [5].

2. EXPONENTIAL STABILITY

We always assume that the functions f and g satisfy the two following assumptions:

(H_1) There exist positive constants α and β, such that

$$| f(x) - f(y) | \le \alpha | x - y |$$

and

$$| g(x) - g(y) | \le \beta | x - y |$$

hold for any $x, y \in \mathbb{R}^n$, where $|z|$ denotes the Euclidean norm of a vector z.

The corresponding crisp system associated with (1) is of the form:

$$\dot{y}(t) = -ay(t) + Af(y(t)) + Bg(y(t)), \ t \ge t_0, \quad y(t_0) = \varphi(0). \quad (7)$$

One can see that under the standing hypothesis (H_1) (1) (respectively, (7)) has a unique solution denoted by $x(t; t_0, \varphi)$ on $t \ge t_0 - \tau$ (respectively, $y(t; t_0, \varphi(0))$ on $t \ge t_0$).

For the purpose of this paper, we propose another standing hypothesis:

(H_2) Equation (7) is globally exponentially stable. That is, there exists a pair of constants K and γ such that

$$| y(t; t_0, \varphi(0)) | \le K | \varphi(0) | e^{-\gamma(t-t_0)}, \ \forall t \ge t_0.$$

We will show that under assumptions (H_1) and (H_2), equation (1) remains globally exponentially stable provided τ is small enough.

There exist many Lemmas which carry the name of Gronwall's Lemma (see [1-3]). A main class may be identified is the integral inequality. The original Lemma proved by Gronwall [4] in 1919, is the following.

Gronwall Lemma: Let $z : [a, a+h] \to \mathbb{R}$ be a continuous function that satisfies the inequality

$$0 \le z(x) \le \int_a^x (A + Mz(s)) ds$$

for all $a \le x \le a+h$, where $A, M \ge 0$ are constants. Then

$$0 \le z(x) \le Ahe^{Mh}$$

for all $a \le x \le a+h$.

The above Lemma can be formulated by the following famous inequality, which is called the Gronwall inequality:

Let $u(t)$ be a continuous function defined on the interval $[t_0, t_1]$ and

$$u(t) \le a + b \int_{t_0}^t u(s) ds,$$

where a and b are nonnegative constants. Then, for all $t \in [t_0, t_1]$, we have

$$u(t) \le ae^{b(t-t_0)}.$$

After more than 20 years, Bellman [2] in 1943 extended the last inequality, which reads in the following:

Let a be a positive constant, $u(t)$ and $b(t)$, $t \in [t_0, t_1]$ be real-valued continuous functions, $b(t) \ge 0$, satisfying

$$u(t) \le a + \int_{t_0}^t b(s)u(s) ds, \ t \in [t_0, t_1].$$

Then, for all $t \in [t_0, t_1]$, we have

$$u(t) \le a \exp\left(\int_{t_0}^t b(s) ds \right).$$

The somewhat more general extensions of the original Gronwall inequality can be found in [3].

Lemma 1: Let u and v be continuous and nonnegative functions defined on $J = [t_0, +\infty)$, and let η be a continuous, positive and nondecreasing function defined on J; then

$$u(t) \le \eta(t) + \int_{t_0}^t v(s)u(s) ds, \quad t \in J,$$

implies that

$$u(t) \le \eta(t) \exp\left(\int_{t_0}^t v(s) ds \right), \quad t \in J.$$

A complete description of the modified result may be as follows:

Theorem 1: Suppose that both assumptions (H_1) and (H_2) hold. Then, (1) is globally exponentially stable provided

$$\tau < \min(0.5\delta, \tau^*),$$

where

$$\delta = \gamma^{-1}(ln(K) - ln(p)) > 0$$

and $\tau^* > 0$ is the unique positive root to the equation $C_1(\tau^*) - 1 = 0$, in which $p \in (0,1)$ is a free parameter, and

$$C_1(\tau) = [Ke^{-\gamma(\delta-\tau)} + \mu_2\delta + \tau^2\beta \| B \| \mu_1 + 2\mu_1 a^{-1} e^{-a(\delta-2\tau)}(1 - e^{-a\tau})] e^{\mu_2\delta} = 1,$$

where

$$\mu_1 = \beta \| B \| \exp(2(\alpha \| A \| + \beta \| B \|)\delta),$$

and

$$\mu_2 = \tau\mu_1(a + \alpha \| A \| + \beta \| B \|).$$

Before starting the proof, we note the following remark concerning the existence and positive uniqueness of the root τ^*.

Remark 2: Let us define a function

$$F(\tau) = C_1(\tau) - 1.$$

Because

$$F(0) = Ke^{-\gamma\delta} - 1 = p - 1 < 0$$

and

$$F(+\infty) = +\infty,$$

there exists at least one root to equation $F(\tau) = 0$. On the other hand, it is easy to show that $F(\tau)$ is strictly monotonously increasing over $[0, +\infty[$ with respect to τ. Therefore, there exists a unique positive root τ^* to equation $C_1(\tau^*) = 1$ and for any $\tau \in [0, \tau^*[$, one sees that $C_1(\tau) < 1$.

Proof: We divide the proof of Theorem 1 into two steps:

Step 1: Fix the initial data t_0 and $\varphi \in C([-\tau, 0]; \mathbb{R}^n)$.

Write $x(t, t_0, \varphi) = x(t)$ and $y(t, t_0, \varphi(0)) = y(t)$. From (1), it follows that

$$e^{at} x(t) = e^{at_0} x(t_0) + \int_{t_0}^{t} e^{as} Af(x(s))ds + \int_{t_0}^{t} e^{as} Bg(x(s-\tau))ds.$$

From (7), we deduce

$$e^{at} y(t) = e^{at_0} y(t_0) + \int_{t_0}^{t} e^{as} Af(y(s))ds + \int_{t_0}^{t} e^{as} Bg(y(s))ds.$$

Therefore,

$$| e^{at}(x(t) - y(t)) | \le (\alpha \| A \| + \beta \| B \|) \int_{t_0}^{t} e^{as} | x(s) - y(s) | ds + \beta \| B \| \int_{t_0}^{t} e^{as} | x(s) - x(s-\tau) | ds.$$

By means of Gronwall inequality, we have

$$| e^{at}(x(t) - y(t)) | \le \beta \| B \| \exp((\alpha \| A \| + \beta \| B \|)(t - t_0)) \times \int_{t_0}^{t} e^{as} | x(s) - x(s-\tau) | ds.$$

Then, we obtain

$$| x(t) - y(t) | \le \beta \| B \| \exp((\alpha \| A \| + \beta \| B \|)(t - t_0)) \times \int_{t_0}^{t} e^{a(s-t)} | x(s) - x(s-\tau) | ds.$$

Hence, if $t_0 \le t \le t_0 + 2\delta$, it follows that

$$| x(t) | \le | y(t) | + \beta \| B \| \exp(2(\alpha \| A \| + \beta \| B \|)\delta) \times \int_{t_0}^{t} e^{a(s-t)} | x(s) - x(s-\tau) | ds. \tag{8}$$

On the other hand, if $t \ge t_0 + \tau$ one gets

$$\begin{aligned} &\int_{t_0+\tau}^{t} e^{a(s-t)} | x(s) - x(s-\tau) | ds \\ &\le \int_{t_0+\tau}^{t} e^{a(s-t)} ds \int_{s-\tau}^{s} [(a + \alpha \| A \|) | x(r) | + \beta \| B \| \| x(r-\tau) |]dr \\ &\le (a + \alpha \| A \|) \int_{t_0+\tau}^{t} e^{a(s-t)} \int_{s-\tau}^{s} | x(r) | dr + \beta \| B \| \int_{t_0+\tau}^{t} e^{a(s-t)} \int_{s-\tau}^{s} | x(r-\tau) | dr. \end{aligned} \tag{9}$$

By changing the order of integration one obtains:

Case 1: When $t_0 + 2\tau \ge t \ge t_0 + \tau$,

$$\begin{aligned} &\int_{t_0+\tau}^{t} e^{a(s-t)} ds \int_{s-\tau}^{s} | x(r) | dr \\ &\le \tau \left[\int_{t_0}^{t-\tau} | x(r) | dr + \int_{t-\tau}^{t_0+\tau} | x(r) | dr + \int_{t_0+\tau}^{t} | x(r) | dr \right] \\ &\le \tau \int_{t_0}^{t} | x(r) | dr. \end{aligned}$$

Case 2: When $t \ge t_0 + 2\tau$,

$$\int_{t_0+\tau}^{t} e^{a(s-t)} ds \int_{s-\tau}^{s} | x(r) | dr \le \int_{t_0+\tau}^{t} ds \int_{s-\tau}^{s} | x(r) | dr \le \tau \int_{t_0}^{t} | x(r) | dr.$$

Therefore, for any $t \ge t_0 + \tau$, we have

$$\int_{t_0+\tau}^{t} e^{a(s-t)} ds \int_{s-\tau}^{s} | x(r) | dr \le \tau \int_{t_0}^{t} | x(r) | dr, \tag{10}$$

and

$$\int_{t_0+\tau}^{t} e^{a(s-t)} ds \int_{s-\tau}^{s} | x(r-\tau) | dr \le \tau \int_{t_0}^{t} | x(r) | dr + \tau^2 (\sup_{t_0-\tau \le s \le t_0} | x(s) |). \tag{11}$$

Consequently, substituting (10) and (11) into (9) one obtains that, if $t \ge t_0 + \tau$,

$$\int_{t_0+\tau}^{t} e^{a(s-t)} | x(s) - x(s-\tau) | ds \le \tau(a + \alpha \| A \| + \beta \| B \|) \times \int_{t_0}^{t} | x(r) | dr + \beta\tau^2 \| B \| \times (\sup_{t_0-\tau \le s \le t_0} | x(s) |). \tag{12}$$

We now restrict $t_0 - \tau + \delta \le t \le t_0 - \tau + 2\delta$.

Substituting (12) into (8) and using hypothesis (H_2), it follows that

$$| x(t) | \le Ke^{-\gamma(\delta-\tau)} | \varphi(0) | + \beta \| B \| \exp(2(\alpha \| A \| + \beta \| B \|)\delta)$$

$$\times\tau(a+\alpha\|A\|+\beta\|B\|)\int_{t_0}^{t}|x(s)|\,ds$$
$$+\tau^2\beta^2\|B\|^2\exp(2(\alpha\|A\|+\beta\|B\|)\delta)$$
$$\times\sup_{t_0-\tau\le s\le t_0}|x(s)|$$
$$+\beta\|B\|\exp(2(\alpha\|A\|+\beta\|B\|)\delta)$$
$$\times\int_{t_0}^{t_0+\tau}e^{a(s-t)}|x(s)-x(s-\tau)|\,ds. \quad (13)$$

Note also that,

$$\int_{t_0}^{t}|x(s)|\,ds=\int_{t_0}^{t_0-\tau+\delta}|x(s)|\,ds+\int_{t_0-\tau+\delta}^{t}|x(s)|\,ds$$
$$\le\delta(\sup_{t_0\le s\le t_0-\tau+\delta}|x(s)|)+\int_{t_0-\tau+\delta}^{t}|x(s)|\,ds,$$

and

$$\int_{t_0}^{t_0+\tau}e^{a(s-t)}|x(s)-x(s-\tau)|\,ds\le 2a^{-1}e^{-a(\delta-2\tau)}$$
$$\times(1-e^{-a\tau})\times(\sup_{t_0-\tau\le s\le t_0+\tau}|x(s)|).$$

Substituting the last two inequality into (13), yields for $t_0-\tau+\delta\le t\le t_0-\tau+2\delta$,

$$|x(t)|\le[Ke^{-\gamma(\delta-\tau)}+\mu_2\delta+\tau^2\beta\|B\|\mu_1$$
$$+2\mu_1a^{-1}e^{-a(\delta-2\tau)}(1-e^{-a\tau})]$$
$$\times e^{\mu_2\delta}\left(\sup_{t_0-\tau\le s\le t_0-\tau+\delta}|x(s)|\right) \quad (14)$$
$$\le C_1(\tau)\left(\sup_{t_0-\tau\le s\le t_0-\tau+\delta}|x(s)|\right).$$

Note that, since $\tau<\tau^*$ we have $C_1<1$. Write

$$C_1=e^{-\varepsilon\delta}$$

with

$$\varepsilon=-\frac{1}{\delta}\ln C_1.$$

It then follows from (14) that,

$$\sup_{t_0-\tau+\delta\le t\le t_0-\tau+2\delta}|x(t)|\le\exp(-\varepsilon\delta)\sup_{t_0-\tau\le s\le t_0-\tau+\delta}|x(s)|, \quad (15)$$

which holds for any $t_0\ge 0$ and $\varphi\in C([-\tau,0];\mathbb{R}^n)$.

Step 2: Fix $t_0\ge 0$ and $\varphi\in C([-\tau,0];\mathbb{R}^n)$ arbitrarily, and let $k=1,2,\ldots$
Denote

$$\hat{x}(t_0+(k-1)\delta;t_0;\varphi)$$
$$=\{x(t_0+(k-1)\delta+s;t_0;\varphi):-\tau\le s\le 0\}.$$

Thus, by (15)

$$\sup_{t_0-\tau+k\delta\le t\le t_0-\tau+(k+1)\delta}|x(t)|$$
$$=\sup_{t_0+(k-1)\delta-\tau+\delta\le t\le t_0+(k-1)\delta-\tau+2\delta}|x(t)|$$
$$\le\exp(-\varepsilon\delta)\sup_{t_0-\tau+(k-1)\delta\le t\le t_0-\tau+k\delta}|x(t)|.$$

By induction

$$\sup_{t_0-\tau+k\delta\le t\le t_0-\tau+(k+1)\delta}|x(t)|$$
$$\le\exp(-\varepsilon k\delta)\times\sup_{t_0-\tau\le t\le t_0-\tau+\delta}|x(t)|. \quad (16)$$

It is easy to show that there exists a positive constant $C_2>0$, such that

$$\sup_{t_0-\tau\le t\le t_0-\tau+\delta}|x(t)|\le C_2\sup_{-\tau\le s\le 0}|\varphi(s)|.$$

Substituting this into (16), yields

$$\sup_{t_0-\tau+k\delta\le t\le t_0-\tau+(k+1)\delta}|x(t)|\le C_2\exp(-\varepsilon k\delta)\sup_{-\tau\le s\le 0}|\varphi(s)|. \quad (17)$$

Now, for any $t\ge t_0-\tau+\delta$, one can find a constant k, such that

$$t_0-\tau+k\delta\le t\le t_0-\tau+(k+1)\delta.$$

Thus,

$$|x(t)|\le C_2\exp(\varepsilon\delta-\varepsilon(t-t_0))\sup_{-\tau\le s\le 0}|\varphi(s)|.$$

But this holds for any $t_0\le t\le t_0-\tau+\delta$ as well. It follows that, equation (1) is globally exponentially stable. □

Remark 3: For computational consideration, in order to find the supper bound of delay such that (1) is globally exponentially stable provided $\tau<\hat{\tau}$, we suggest the following optimization problem:

$$(P)\begin{cases}\max\hat{\tau}=\sup\limits_{0<p<1}\left(\min\left\{\dfrac{\delta}{2},\tau^*\right\}\right),\\ s.t.,\ 1>p>0,\ C_1(\tau^*)=1,\\ \delta=\gamma^{-1}(\ln K-\ln p)>0.\end{cases}$$

Using Matlab, the problem (P) can provide the optimal value of the root of equation $C_1(\tau^*)=1$.

The result established in Theorem 1 is given for constant-delay case which still holds when the time delay is time-varying. More precisely, let $\tau:\mathbb{R}_+\to[0,\overline{\tau}]$ be a Borel measurable function, where $\overline{\tau}>0$. In this case, equation (1) is rewritten as the form

$$\dot{x}(t)=-ax(t)+Af(x(t))+Bg(x(t-\tau(t))),\ t\ge t_0,$$
$$x(t)=\varphi(t-t_0),\ t_0-\overline{\tau}\le t\le t_0, \quad (18)$$

where

$$\varphi=\{\varphi(s):-\tau\le s\le 0\}\in C([-\tau,0];\mathbb{R}^n),$$

the matrices A and B, functions f and g are the same as defined in (1).

Next, we can establish the following result when the time delay is time-varying.

Theorem 2: Suppose that both assumptions (H_1) and (H_2) hold. Then, (18) is globally exponentially stable provided

$$\sup_{t\ge t_0}\tau < \min(0.5\delta, \tau^*),$$

where $\tau^* > 0$ and δ are the same as defined in Theorem 1.

Proof: The proof is similar to that of Theorem 1, and therefore, omitted here. □

Next, we will show the validity of the modified and corrected result given in Theorem 1 via the examples utilized in [5].

3. EXAMPLES

Three examples are considered in this paragraph.

Example 1: Consider a one-dimensional differential delay equation

$$\begin{aligned}&\dot{x}(t) = -x(t) - 2\sin(x(t-\tau)),\ t \ge t_0\\&x(t) = \varphi(t-t_0), \quad t_0 - \tau \le t \le t_0,\end{aligned} \tag{19}$$

where φ is the same as defined in (1).
The corresponding differential equation has the form

$$\begin{aligned}&\dot{y}(t) = -y(t) - 2\sin(y(t)),\ t \ge t_0,\\&y(t_0) = \varphi(0).\end{aligned} \tag{20}$$

The solution of (20), denoted by $y(t,t_0,\varphi(0))$, satisfies:

$$|y(t;t_0;\varphi)| \le e^{-(t-t_0)}|\varphi(0)|, \quad t \ge t_0.$$

Hence, one sees that the standing assumptions (H_1) and (H_2) are satisfied with $\alpha = 0$, $\beta = 2$, $K = 1$ and $\gamma = 1$.
By solving problem (P) yields that

$$p = 0.835$$

(therefore, $\delta \simeq 0.1803$) and $\tau^* \simeq 0.0133$ the maximum

$$\hat{\tau} \simeq 0.0133.$$

It follows from Theorem 1 that the delay equation (19) remain exponentially stable provided $\tau < 0.0133$. However, if we apply the result in [7], the modified version of [6], the threshold value of the delay ensuring exponential stability will be 0.0093, which is much smaller that our value. Moreover, it is easy to verify that the results in [8-10] are not available for this example.

Example 2: Consider a two-dimensional differential delay equation

$$\begin{cases}\dot{x}_1(t) = -2x_1(t) + \sin(x_2(t-\tau))\\ \dot{x}_2(t) = -2x_2(t) - 2\sin(x_1(t-\tau)), \quad t \ge t_0.\end{cases} \tag{21}$$

The initial value is assumed to be

$$x(t) = [x_1(t), x_2(t)]^T = \varphi(t-t_0)$$

on $t_0 - \tau < t \le t_0$, where $\varphi \in C([-\tau, 0]; \mathbb{R}^n)$.
The corresponding differential equation has the form:

$$\begin{cases}\dot{y}_1(t) = -2y_1(t) + \sin(y_2(t))\\ \dot{y}_2(t) = -2y_2(t) - 2\sin(y_1(t))\end{cases} \tag{22}$$

on $t \ge t_0$ with initial value

$$y(t_0) = [y_1(t_0), y_2(t_0)]^T = \varphi(0).$$

The solution of (22), denoted by $y(t,t_0,\varphi(0))$, satisfies

$$|y(t,t_0,\varphi(0))| \le K|\varphi(0)|\exp(-\gamma(t-t_0)),\ t \ge t_0,$$

where

$$K = 1 \text{ and } \gamma = 1.$$

Note that $a = 2$, $\alpha = 0$, $\beta = 1$,

$$A = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$$

and

$$B = \begin{pmatrix}0 & 1\\ -2 & 0\end{pmatrix}.$$

By solving Problem (P) yields that

$$p = 0.84$$

(therefore, $\delta \simeq 0.1744$) and $\tau^* \simeq 0.0129$ the maximum

$$\hat{\tau} \simeq 0.0129.$$

However, if we apply the modified result [7] of the reference [6] the threshold value of the delay ensuring exponential stability will be 0.0045574, which is also much smaller that our value.

Example 3: Consider a two-neuron cellular neural network system with delay

$$\begin{cases}\dot{x}_1(t) = -2x_1(t) - 0.5f(x_1(t)) + 0.1f(x_2(t))\\ \qquad -0.1f(x_1(t-\tau)) + 0.2f(x_2(t-\tau)),\\ \dot{x}_2(t) = -2x_2(t) + 0.2f(x_1(t)) - 0.1f(x_2(t))\\ \qquad +0.2f(x_1(t-\tau)) + 0.1f(x_2(t-\tau)),\ t \ge t_0,\end{cases} \tag{23}$$

where $f(x) = 0.5(|x+1| - |x-1|)$.
The initial value is assumed to be

$$x(t) = [x_1(t), x_2(t)]^T = \varphi(t-t_0)$$

on $t_0 - \tau < t \le t_0$, where $\varphi \in C([-\tau, 0]; \mathbb{R}^n)$.
In [10,11], the authors studied the asymptotic stability of the analog of (23) respectively. The upper bounds of delay estimated in [10] and [11] are $\tau^* < 0.17$ and $\tau^* <$ 0.0279, respectively. The corresponding differential equation has the form:

$$\begin{cases}\dot{y}_1(t) = -2y_1(t) - 0.6f(y_1(t)) + 0.3f(y_2(t))\\ \dot{y}_2(t) = -2y_2(t) + 0.4f(y_1(t))\end{cases} \tag{24}$$

on $t \ge t_0$ with initial value

$$y(t_0) = [y_1(t_0), y_2(t_0)]^T = \varphi(0).$$

The solution of (24), denoted by $y(t,t_0,\varphi(0))$, satisfies

$$|y(t,t_0,\varphi(0))| \le K|\varphi(0)|\exp(-\gamma(t-t_0)),\ t \ge t_0,$$

where

$K = 1$ and $\gamma = 1.41095.$

Note that $a = 2,\ \alpha = \beta = 1,$

$$A = \begin{pmatrix} -0.5 & 0.1 \\ 0.2 & -0.1 \end{pmatrix}$$

and

$$B = \begin{pmatrix} -0.1 & 0.2 \\ 0.2 & 0.1 \end{pmatrix}.$$

By solving Problem (P) yields that

$p = 0.469$

(therefore, $\delta \simeq 0.536$) and $\tau^* \simeq 0.184$ the maximum

$\hat{\tau} \simeq 0.184,$

which is larger than those in [10,11].

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