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Eigenvalue Assignment in Linear Descriptor Systems using Dynamic Compensators

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Abstract: An approach for eigenvalue assignment in strongly controllable and observable linear descriptor systems using dynamic compensators is proposed. Parametric expressions for the controller coefficient matrices are given. The approach assigns the full number of distinct finite closed-loop eigenvalues, guarantees the closed-loop regularity and overcomes the defects of some previous works. In addition, using the proposed eigenvalue assignment approach, a sufficient condition for generic eigenvalue assignability using dynamic compensators is proved.

Keywords: Descriptor systems, dynamic compensators, eigenvalue assignment, regularity.

1. INTRODUCTION

Consider the following linear descriptor system

$$E\dot{x} = Ax + Bu, \quad y = Cx, \tag{1}$$

where $x \in \mathbf{R}^n$, $u \in \mathbf{R}^m$, $y \in \mathbf{R}^p$ are, respectively, the state vector, the input vector and the output vector; $E, A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times m}$, $C \in \mathbf{R}^{p\times n}$ are known matrices with $\operatorname{rank}(E) = n_0 \le n$, $\operatorname{rank}(B) = m$ and $\operatorname{rank}(C) = p$. Assume that the system (1) is strongly controllable and observable, i.e., the following conditions hold [1,2]:

$$\operatorname{rank}[\lambda E - A \quad B] = \operatorname{rank}\begin{bmatrix} \lambda E - A \\ C \end{bmatrix} = n, \ \forall \lambda \in \mathbf{C}, \tag{2}$$

$$\operatorname{rank}[E \quad AV_\infty \quad B] = \operatorname{rank}[E^T \quad A^T T_\infty \quad C^T]^T = n, \tag{3}$$

where V_∞ and T_∞ are $n \times (n - n_0)$ matrices defined by $EV_\infty = 0,\ T_\infty^T E = 0,\ \operatorname{rank}(V_\infty) = \operatorname{rank}(T_\infty) = n - n_0$. If a dynamic compensator of order q

$$\begin{cases} \dot{z} = K_{22}z + K_{21}y \\ u = K_{12}z + K_{11}y \end{cases} \tag{4}$$

is applied to system (1), then the closed-loop system is

$$\begin{bmatrix} E & 0 \\ 0 & I_q \end{bmatrix}\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A + BK_{11}C & BK_{12} \\ K_{21}C & K_{22} \end{bmatrix}\begin{bmatrix} x \\ z \end{bmatrix}, \tag{5}$$

where $z \in \mathbf{R}^q$ is the compensator state vector and K_{ij}, $i, j = 1, 2$ are four controller coefficient matrices of appropriate dimensions. This paper studies the problem of eigenvalue assignment (Problem EA) in the system (1) using the dynamic compensator (4). The problem can be stated as: Given the strongly controllable and observable system (1), find real controller coefficient matrices K_{ij}, $i, j = 1, 2$ such that the finite generalized eigenspectrum of the closed-loop system (5) equals an arbitrary self-conjugate set Λ of $n_0 + q$ distinct complex numbers.

Eigenvalue assignment in linear descriptor systems is a very important problem in descriptor systems theory and has been studied by a lot of researchers [3-30]. Among these reported results on eigenvalue assignment in linear descriptor systems, only a few are concentrated on the dynamic output feedback case [8,9,16]. Wang *et al.* [8] studied the problem of eigenvalue assignment in descriptor systems using dynamic compensators of the form of (4). They proved that for a strongly controllable and observable system (1) there exists a dynamic compensator of order $q = n_0$ such that the closed-loop eigenvalues are assigned arbitrarily to prespecified locations. However, the requirement of $q = n_0$ is too conservative. Sakr and Khalifa [16] also studied the problem of eigenvalue assignment in descriptor systems using dynamic compensators of the form of (4). The basis of this work was a demonstration of the equivalence between eigenvalue assignment in descriptor systems using dynamic compensators and in state space systems using static output feedbacks. From this result, they identified the order of the dynamic compensator required for full assignment of an almost arbitrary set of closed-loop eigenvalues as

$$q \ge \max\{0, n_0 - m - p + 1\}. \tag{6}$$

This result improves considerably the one obtained in [8]. Unfortunately, an example shows that the equivalence of eigenvalue assignability obtained in [16] may not hold (see Remark 2). Shayman [9] studied the problem of

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eigenvalue assignment in descriptor systems using alternative dynamic compensators which are not in the form of (4). He showed that for a completely controllable and observable system (1) there exists a dynamic compensator of order $q=\nu-1$ (ν is the largest homogeneous index) such that the closed-loop eigenvalues are assigned arbitrarily close to prespecified locations. However, his theory is not applicable to strongly controllable and observable systems.

In this paper, an approach for eigenvalue assignment in the strongly controllable and observable system (1) using the dynamic compensator (4) is proposed. Parametric expressions for the controller coefficient matrices $K_{ij}, i, j=1,2$ are given. The approach assigns n_0+q distinct finite closed-loop eigenvalues, guarantees the closed-loop regularity and overcomes the defects of some previous works [8,9,16]. In addition, it is shown using the proposed eigenvalue assignment approach that (6) is a sufficient condition for generic eigenvalue assignability using dynamic compensators.

2. SOLUTION TO PROBLEM EA

It is known from [30] that if $\max\{m,p\}\geq n_0$ then eigenvalue assignment for the system (1) can be achieved by static output feedback. Due to this reason, we assume in this paper that $\max\{m,p\}<n_0$.

Denote

$$\xi=\begin{bmatrix}x\\z\end{bmatrix},\ E_c=\begin{bmatrix}E&0\\0&I_q\end{bmatrix},\ A_c=\begin{bmatrix}A&0\\0&0\end{bmatrix},$$
$$B_c=\begin{bmatrix}B&0\\0&I_q\end{bmatrix},\ C_c=\begin{bmatrix}C&0\\0&I_q\end{bmatrix},\ K_c=\begin{bmatrix}K_{11}&K_{12}\\K_{21}&K_{22}\end{bmatrix}.$$

Then (5) can be written as

$$E_c\dot{\xi}=(A_c+B_cK_cC_c)\xi. \tag{7}$$

Let Λ be a self-conjugate set of n_0+q distinct complex numbers $\lambda_1,\lambda_2,\cdots,\lambda_{n_0+q}$ such that the decomposition $\Lambda=\Lambda_1\cup\Lambda_2$ exists, where $\Lambda_1=\{\lambda_1,\lambda_2,\cdots,\lambda_{p+q}\}$ and $\Lambda_2=\{\lambda_{p+q+1},\lambda_{p+q+2},\cdots,\lambda_{n_0+q}\}$ are self-conjugate subsets. Denote the right eigenvector of the closed-loop system (7) associated with eigenvalue $\lambda_i\in\Lambda_1$ by v_i and the left eigenvector associated with eigenvalue $\lambda_j\in\Lambda_2$ by t_j. Then we have by definition,

$$(A_c+B_cK_cC_c-\lambda_iE_c)v_i=0,\quad i=1,2,\cdots,p+q, \tag{8}$$

$$(A_c+B_cK_cC_c-\lambda_jE_c)^Tt_j=0,\quad j=p+q+1,p+q+2,\cdots,n_0+q. \tag{9}$$

Let

$$w_i=K_cC_cv_i,\quad i=1,2,\cdots,p+q,$$
$$z_j=K_c^TB_c^Tt_j,\ j=p+q+1,p+q+2,\cdots,n_0+q.$$

Then (8) and (9) become

$$(A_c-\lambda_iE_c)v_i+B_cw_i=0,\quad i=1,2,\cdots,p+q, \tag{10}$$

$$(A_c-\lambda_jE_c)^Tt_j+C_c^Tz_j=0,\quad j=p+q+1,p+q+2,\cdots,n_0+q. \tag{11}$$

Let $\Xi(\lambda)$ and $\Pi(\lambda)$ be matrices whose columns span the nullspaces $\ker([A-\lambda E\ B])$ and $\ker([A^T-\lambda E^T\ C^T])$ respectively. Partition $\Xi(\lambda)$ and $\Pi(\lambda)$ into

$$\Xi(\lambda)=\begin{bmatrix}N(\lambda)\\D(\lambda)\end{bmatrix},\ \Pi(\lambda)=\begin{bmatrix}M(\lambda)\\L(\lambda)\end{bmatrix},\quad D(\lambda)\in\mathbf{C}^{m\times m},\ L(\lambda)\in\mathbf{C}^{p\times p} \tag{12}$$

and let

$$N_c(\lambda)=\begin{bmatrix}N(\lambda)&0\\0&I_q\end{bmatrix},\ D_c(\lambda)=\begin{bmatrix}D(\lambda)&0\\0&\lambda I_q\end{bmatrix}, \tag{13}$$

$$M_c(\lambda)=\begin{bmatrix}M(\lambda)&0\\0&I_q\end{bmatrix},\ L_c(\lambda)=\begin{bmatrix}L(\lambda)&0\\0&\lambda I_q\end{bmatrix}. \tag{14}$$

Then the general parametric solutions for v_i, w_i satisfying (10) and t_j, z_j satisfying (11) are given by

$$v_i=N_c(\lambda_i)f_i,\quad w_i=D_c(\lambda_i)f_i,\quad i=1,2,\cdots,p+q, \tag{15}$$

$$t_j=M_c(\lambda_j)g_j,\quad z_j=L_c(\lambda_j)g_j,\quad j=p+q+1,p+q+2,\cdots,n_0+q, \tag{16}$$

where $f_i\in\mathbf{C}^{m+q}$, $i=1,2,\cdots,p+q$, $g_j\in\mathbf{C}^{p+q}$, $j=p+q+1,p+q+2,\cdots,n_0+q$ are two groups of free parameter vectors. Let

$$V=\begin{bmatrix}v_1&v_2&\cdots&v_{p+q}\end{bmatrix},\quad W=\begin{bmatrix}w_1&w_2&\cdots&w_{p+q}\end{bmatrix}. \tag{17}$$

Lemma 1: Let E_c, A_c, B_c, C_c, K_c, V_∞, T_∞ be matrices as described previously and $\breve{B}=[B\ 0_{n\times q}]$, $\breve{C}=[C^T\ 0_{n\times q}]^T$. Then $\deg\det(\lambda E_c-A_c-B_cK_cC_c)=\operatorname{rank}(E_c)$ if and only if

$$\det(T_\infty^T(A+\breve{B}K_c\breve{C})V_\infty)=\det(T_\infty^T(A+BK_{11}C)V_\infty)\neq 0.$$

Proof: Apply Theorem 3.2 of Fletcher [11] to the matrix pair $(E_c, A_c+B_cK_cC_c)$ directly.

Theorem 1: Given the strongly controllable and observable system (1) with $\max\{m,p\}<n_0$. Then Problem EA has solutions if there exist vectors $f_1,f_2,\cdots,f_{p+q}$ in $\mathbf{C}^{m+q}$ and vectors $g_{p+q+1},g_{p+q+2},\cdots,g_{n_0+q}$ in $\mathbf{C}^{p+q}$ satisfying the following conditions:

(a) $f_i=\overline{f_j}$ for $\lambda_i=\overline{\lambda_j}\in\Lambda_1$ and $g_k=\overline{g_l}$ for $\lambda_k=\overline{\lambda_l}\in\Lambda_2$;

(b) $C_cN_c(\lambda_i)f_i$, $i=1,2,\cdots,p+q$ are linearly independent in $\mathbf{C}^{p+q}$;

(c) $M_c(\lambda_j)g_j$, $j=p+q+1,p+q+2,\cdots,n_0+q$ are linearly independent in $\mathbf{C}^{n+q}$;

(d) $g_j^TM_c^T(\lambda_j)E_cN_c(\lambda_i)f_i=0$, $i=1,2,\cdots,p+q$, $j=p+q+1,p+q+2,\cdots,n_0+q$;

(e) $\det(T_\infty^T(A+\breve{B}W(C_cV)^{-1}\breve{C})V_\infty)\neq 0$.

When the conditions (a)-(e) are met, all solutions to

Problem EA are given by

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} = K_c = W(C_cV)^{-1}, \tag{18}$$

where V, W are given by (12)-(17).

Proof: Following a similar line as in the proof of Theorem 1 in [30] and using Lemma 1, we can prove the theorem.

Remark 1: If the decomposition $\Lambda = \Lambda_1' \cup \Lambda_2'$ exists, where $\Lambda_1' = \{\lambda_1, \lambda_2, \cdots, \lambda_{m+q}\}$ and $\Lambda_2' = \{\lambda_{m+q+1}, \lambda_{m+q+2} \cdots, \lambda_{n_0+q}\}$ are self-conjugate subsets, then the solution to Problem EA can be obtained by replacing (E_c, A_c, B_c, C_c) in Theorem 1 by its dual $(E_c^T, A_c^T, C_c^T, B_c^T)$.

Remark 2: In [16], the system (1) needs to be transformed to the following form

$$\begin{cases} \begin{bmatrix} I_{n_0} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \\ y = [C_1 \quad C_2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \end{cases} \tag{19}$$

where $x_1 \in \mathbf{R}^{n_0}$ and A_4 is nonsingular. If the static output feedback $u = Ky$ (For simplicity, we only consider the case $q = 0$.) is applied to the system (19), then the closed-loop system is obtained as:

$$\begin{bmatrix} I_{n_0} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 + B_1KC_1 & A_2 + B_1KC_2 \\ A_3 + B_2KC_1 & A_4 + B_2KC_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{20}$$

Sakr and Khalifa [16] proved that the finite eigenvalues of the descriptor system (20) are the same as those of the following state space system

$$\dot{x}_1 = (A_0 + B_0\hat{K}C_0)x_1, \tag{21}$$

where $A_0 = A_1 - A_2A_4^{-1}A_3$, $B_0 = B_1 - A_2A_4^{-1}B_2$, $C_0 = C_1 - C_2A_4^{-1}A_3$, $K = (I_m - \hat{K}C_2A_4^{-1}B_2)^{-1}\hat{K}$. Unfortunately, this result may not hold. To see this, consider the system (19) with

$$A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \; A_2 = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \; A_3 = [0 \quad 0 \quad 1], \; A_4 = 1,$$

$$B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \; B_2 = [0 \quad 1], \; C_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; C_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and consider the assignment of the set $\{-1, \pm i\}$. For this system, we have

$$A_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 1 & 0 \end{bmatrix}, \; B_0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 0 \end{bmatrix}, \; C_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

It is easy to verify that the set $\{-1, \pm i\}$ can be assigned to the state space system $\dot{x}_1 = A_0x_1 + B_0u$, $y_1 = C_0x_1$ by the output feedback $u = \hat{K}y_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} y_1$. However, it is easy to verify that the set $\{-1, \pm i\}$ cannot be assigned to the example descriptor system by any output feedback $u = Ky$.

Definition 1 [31]**:** A subset of $\mathbf{R}^{s \times t}$ is a Zariski open set of $\mathbf{R}^{s \times t}$ if it is nonempty and its complement is the set of solutions in $\mathbf{R}^{s \times t}$ to a finite set of polynomial equations.

Lemma 2: Let $X \in \mathbf{R}^{s \times t}$ and $F_i(X)$, $i = 1, 2, \cdots, d$ be respectively $h_i \times l_i$, $i = 1, 2, \cdots, d$ matrix functions whose elements are rational functions of the elements of X. Define

$$\aleph = \{X \mid X \in \mathbf{R}^{s \times t}, F_i(\mathrm{X}) \text{ is well defined} \\ \text{and} \operatorname{rank}(F_i(X)) = \min\{h_i, l_i\}, i = 1, 2, \cdots, d\}.$$

If $\aleph$ is nonempty, then it is a Zariski open set of $\mathbf{R}^{s \times t}$.

Proof: Let

$$\aleph_i = \{X \mid X \in \mathbf{R}^{s \times t}, F_i(\mathrm{X}) \text{ is well defined} \\ \text{and} \operatorname{rank}(F_i(X)) = \min\{h_i, l_i\}\}.$$

Then $\aleph = \bigcap_{i=1}^{d} \aleph_i$. It is clear that if $\aleph$ is nonempty, then $\aleph_i, i = 1, 2, \cdots, d$ are all nonempty. From Lemma 6 in [29], $\aleph_i, i = 1, 2, \cdots, d$ are all Zariski open sets of $\mathbf{R}^{s \times t}$. Since the intersection of a finite number of Zariski open sets is also a Zariski open set, $\aleph$ is a Zariski open set of $\mathbf{R}^{s \times t}$.

In the classical algebro-geometric literature, a property depending on a point of $\mathbf{R}^{s \times t}$ is often said to be "generic" if the set of points where it is true contains a nonempty Zariski open subset of $\mathbf{R}^{s \times t}$.

Theorem 2: Let the strongly controllable and observable system (1) with $\max\{m, p\} < n_0$ be given. If (6) holds, then for generic matrix triples (A, B, C), there exists, for almost each self-conjugate set Λ of $n_0 + q$ distinct complex numbers (It is assumed that the decomposition $\Lambda = \Lambda_1 \cup \Lambda_2$ or $\Lambda = \Lambda_1' \cup \Lambda_2'$ exists (see Theorem 1 and Remark 1)), a dynamic compensator of order q in the form of (4) such that the generalized eigenspectrum of the closed-loop system (5) (or, equivalently, (7)) is Λ.

Proof: Assume that the decomposition $\Lambda = \Lambda_1 \cup \Lambda_2$ exists. It is desired to prove that for generic matrix triples (A, B, C) and almost each self-conjugate set Λ the set

$$\kappa_c = \{K_c \mid K_c \text{ given by (18)} \\ \text{with (a)} - \text{(e) in Thoerem 1 satisfied}\}$$

is nonempty.

We rewrite the condition (d) in Theorem 1 as the following linear system of equations with the vectors $f_1, f_2, \cdots, f_{p+q}$ as unknown variables:

$$\Phi_i f_i = 0, \quad i = 1, 2, \cdots, p + q \tag{22}$$

with

$$\Phi_i = \begin{bmatrix} g_{p+q+1}^T M_c^T(\lambda_{p+q+1}) E_c N_c(\lambda_i) \\ g_{p+q+2}^T M_c^T(\lambda_{p+q+2}) E_c N_c(\lambda_i) \\ \vdots \\ g_{n_0+q}^T M_c^T(\lambda_{n_0+q}) E_c N_c(\lambda_i) \end{bmatrix}.$$

Since all Φ_i are $(n_0-p)\times(m+q)$ matrices with $n_0-p<m+q$, the system of algebraic equations (22) always has nontrivial solutions. Denote the real number λ_i by σ_i and a pair of conjugate complex numbers λ_i and λ_j by $\lambda_i=\bar{\lambda}_j=\sigma_i+\sigma_j i$. Let $\Sigma=[\sigma_1\ \sigma_2\cdots\sigma_{n_0+q}]^T \in \mathbf{R}^{n_0+q}$. We identify the space of the matrix quadruples (A,B,C,Σ) with the linear space $\mathbf{R}^{n(n+m+p)+n_0+q}$. It is easy to see that all the vectors $f_1, f_2, \cdots, f_{p+q}, g_{p+q+1}, g_{p+q+2}, \cdots, g_{n_0+q}$ satisfying the conditions (a) and (d) in Theorem 1 depend rationally on (A,B,C,Σ). If we denote (A,B,C,Σ) by Θ, then we can write $f_i=f_i(\Theta)$, $i=1,2,\cdots,p+q$, $g_j=g_j(\Theta)$, $j=p+q+1,\ p+q+2,\cdots, n_0+q$. Let

$$V(\Theta)=\left[N_c(\lambda_1)f_1(\Theta)\quad\cdots\quad N_c(\lambda_{p+q})f_{p+q}(\Theta)\right],$$

$$W(\Theta)=\left[D_c(\lambda_1)f_1(\Theta)\quad\cdots\quad D_c(\lambda_{p+q})f_{p+q}(\Theta)\right],$$

$$T(\Theta)=\left[M_c(\lambda_{p+q+1})g_{p+q+1}(\Theta)\ \cdots\ M_c(\lambda_{n_0+q})g_{n_0+q}(\Theta)\right],$$

$$H(\Theta)=T_\infty^T\left(A+\breve{B}W(\Theta)(C_cV(\Theta))^{-1}\breve{C}\right)V_\infty.$$

Then, the conditions (b), (c), and (e) in Theorem 1 become respectively the following conditions:

Condition C1: $\operatorname{rank}(C_cV(\Theta))=p+q$;

Condition C2: $\operatorname{rank}(T(\Theta))=n_0-p$;

Condition C3: $\operatorname{rank}(H(\Theta))=n-n_0$.

When Conditions C1-C3 are met, we have

$$K_c=W(\Theta)(C_cV(\Theta))^{-1}. \tag{23}$$

Then, the set κ_c can be equivalently written as

$$\kappa_c=\{K_c\,|\,K_c \text{ given by (23) with Conditions C1}-\text{C3 satisfied}\}.$$

It is desired to prove that for generic Θ the set κ_c is nonempty. For this purpose, it suffices to prove that the set

$$S=\left\{\Theta\,|\,\Theta\in\mathbf{R}^{n(n+m+p)+n_0+q}, \text{Conditions C1}-\text{C3 holds}\right\}$$

is a Zariski open set of $\mathbf{R}^{n(n+m+p)+n_0+q}$. Notice that $C_cV(\Theta), T(\Theta)$ and $H(\Theta)$ are matrix functions whose elements are rational functions of the elements of $\Theta\in\mathbf{R}^{n(n+m+p)+n_0+q}$. Then, from Lemma 2, it suffices to prove that the set S is nonempty. In the following, we will produce an example to show that the set S is nonempty.

Since $\operatorname{rank}(E)=n_0$, there exist two nonsingular matrices P, Q such that

$$PEQ=\begin{bmatrix} I_{n_0} & 0 \\ 0 & 0\end{bmatrix}.$$

Define matrices A, B and C as follows:

$$A=P^{-1}\begin{bmatrix} A_1 & 0 \\ 0 & I_{n-n_0}\end{bmatrix}Q^{-1},\quad B=P^{-1}\begin{bmatrix} B_1 \\ 0_{(n-n_0)\times m}\end{bmatrix},$$
$$C=\begin{bmatrix} C_1 & 0_{p\times(n-n_0)}\end{bmatrix}Q^{-1}, \tag{24}$$

where $\operatorname{rank}(B_1)=m$, $\operatorname{rank}(C_1)=p$ and the matrix triples (A_1,B_1,C_1) is both controllable and observable. Using Theorem 4 of [32], we can show that, for the descriptor system (1) with matrices A, B and C given by (24), if (6) holds, then an almost arbitrary set of n_0+q distinct closed-loop eigenvalues is assignable using dynamic compensators of order q. This means that we can always find a matrix triple $(\tilde{A},\tilde{B},\tilde{C})$ in the form of (24) and a vector $\tilde{\Sigma}$ in $\mathbf{R}^{n_0+q}$ such that $\tilde{\Theta}=(\tilde{A},\tilde{B},\tilde{C},\tilde{\Sigma})\in S$. Thus the set S is nonempty and the theorem for the case that $\Lambda=\Lambda_1\cup\Lambda_2$ exists follows.

The theorem for the case that $\Lambda=\Lambda_1'\cup\Lambda_2'$ exists can be proved by replacing (E,A,B,C) by its dual (E^T, A^T, C^T, B^T).

Remark 3: Theorem 2 provides a theoretical proof for the sufficient condition (6). The condition (6) is the best sufficient condition for eigenvalue assignability so far. On the other hand, a dimension argument shows that the necessary condition for eigenvalue assignability using dynamic compensators is

$$q(m+p)+mp\geq n_0+q. \tag{25}$$

This is a generalization to descriptor systems of the result of Willems and Hesselink [33] concerning eigenvalue assignment using dynamic compensators for state space systems, i.e., $q(m+p)+mp\geq n+q$. Thus, it is the author's belief that the lower bound on the order of dynamic compensators given by (6) is still very conservative, and there remains a wide room for further improvement.

3. AN ILLUSTRATIVE EXAMPLE

Consider a system in the form of (1) with the following coefficient matrices

$$E=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix},\quad A=\begin{bmatrix}0&0&1&0\\1&0&0&0\\0&1&0&0\\0&0&1&1\end{bmatrix},$$
$$B=[0\ \ 0\ \ 1\ \ 0]^T,\quad C=[0\ \ 0\ \ 0\ \ 1].$$

For this system we have $n=4$, $n_0=2$, $m=p=1$. It is easy to verify that the system is strongly controllable and observable (The theory developed in [9] cannot be applied since the system is not completely controllable and observable). According to Theorem 2, a first-order dynamic compensator is required in order to obtain almost arbitrary eigenvalue assignment. Now we consider the assignment of a self-conjugate set $\Lambda = \{\lambda_1, \lambda_2, \lambda_3\}$ of distinct numbers $\lambda_1, \lambda_2, \lambda_3$, where $\lambda_1 = \overline{\lambda}_2 = \alpha + \beta i$, $\lambda_3 = \gamma, \alpha, \beta, \gamma \in \mathbf{R}$. Obviously, we have the decomposition $\Lambda = \Lambda_1 \cup \Lambda_2$, where $\Lambda_1 = \{\lambda_1, \lambda_2\}$ and $\Lambda_2 = \{\lambda_3\}$.

For the example system, we have

$$V_\infty = T_\infty = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \; N(\lambda) = \begin{bmatrix} -\lambda \\ -1 \\ -\lambda^2 \\ \lambda^2 \end{bmatrix}, \; M(\lambda) = \begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \\ -1 \end{bmatrix},$$

$$D(\lambda) = L(\lambda) = 1.$$

Then, from (15) and (16), we have

$$v_i = N_c(\lambda_i) f_i, \; w_i = D_c(\lambda_i) f_i, \; i = 1,2, \; t_3 = M_c(\lambda_3) g_3.$$

We may choose

$$f_1 = \overline{f}_2 = \begin{bmatrix} 1 \\ x_1 + x_2 i \end{bmatrix}, \; g_3 = \begin{bmatrix} 1 \\ x_3 \end{bmatrix}, \; x_i \in \mathbf{R}, \; i = 1,2,3$$

due to the lack of uniqueness of eigenvectors. It is easy to verify that the conditions (a) and (c) in Theorem 1 hold and the conditions (b) and (d) are respectively

$$2\alpha\beta x_1 - (\alpha^2 - \beta^2) x_2 \neq 0, \tag{26}$$

$$\begin{cases} -\alpha - \gamma + x_1 x_3 = 0 \\ -\beta + x_2 x_3 = 0. \end{cases} \tag{27}$$

From (27) and the condition $\beta \neq 0$ (since $\lambda_1, \lambda_2, \lambda_3$ are distinct), we have $x_3 \neq 0$ and

$$x_1 = \frac{\alpha + \gamma}{x_3}, \quad x_2 = \frac{\beta}{x_3}. \tag{28}$$

From (26) and (28), we have $\alpha^2 + 2\alpha\gamma + \beta^2 \neq 0$. From (17) and replacing v_1 and v_2 by $\mathrm{Re}(v_1)$ and $\mathrm{Im}(v_1)$, and w_1 and w_2 by $\mathrm{Re}(w_1)$ and $\mathrm{Im}(w_1)$, we have

$$V = \begin{bmatrix} -\alpha & -\beta \\ -1 & 0 \\ -\alpha^2 + \beta^2 & -2\alpha\beta \\ \alpha^2 - \beta^2 & 2\alpha\beta \\ \dfrac{\alpha + \gamma}{x_3} & \dfrac{\beta}{x_3} \end{bmatrix},$$

$$W = \begin{bmatrix} 1 & 0 \\ \dfrac{\alpha^2 + \alpha\gamma - \beta^2}{x_3} & \dfrac{2\alpha\beta + \beta\gamma}{x_3} \end{bmatrix}.$$

It is easy to verify that the condition (e) in Theorem 1 holds. From the above, we see that if $x_3 \neq 0$ and $\alpha^2 + 2\alpha\gamma + \beta^2 \neq 0$, then the conditions (a)-(e) in Theorem 1 are satisfied, and by (18) the controller coefficient matrices $K_{ij}, i, j = 1,2$ are obtained as

$$K_{11} = \frac{-x_3}{\alpha^2 + 2\alpha\gamma + \beta^2}, \qquad K_{12} = \frac{2\alpha x_3^2}{\alpha^2 + 2\alpha\gamma + \beta^2},$$

$$K_{21} = \frac{\alpha^2 + \beta^2 + 2\alpha\gamma + \gamma^2}{\alpha^2 + 2\alpha\gamma + \beta^2}, \quad K_{22} = \frac{(\alpha^2 + \beta^2)\gamma}{\alpha^2 + 2\alpha\gamma + \beta^2}.$$

Specially choosing $\alpha = -1, \; \beta = x_3 = 1, \; \gamma = -2$, we have

$$K_{11} = -\frac{1}{6}, \quad K_{12} = -\frac{1}{3}, \quad K_{21} = \frac{5}{3}, \quad K_{22} = -\frac{2}{3}.$$

4. CONCLUSION

An approach for eigenvalue assignment in the strongly controllable and observable system (1) using the dynamic compensator (4) is proposed. Parametric expressions for the controller coefficient matrices $K_{ij}, i, j = 1,2$ are given. The approach assigns $n_0 + q$ distinct finite closed-loop eigenvalues, guarantees the closed-loop regularity and overcomes the defects of some previous results. In addition, it is shown using the proposed eigenvalue assignment approach that (6) is a sufficient condition for generic eigenvalue assignability using dynamic compensators.

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