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H_∞ Filtering for Singular Bilinear Systems with Application to a Single-link Flexible-joint Robot

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Abstract: In this paper, we consider the H_∞ filters design for singular bilinear systems. The approach is based on the parameterized solution of a set of constrained Sylvester equations. The exponential convergence and l_2 gain attenuation problems are solved by using the bounded real lemma, which leads to linear matrix inequalities (LMI) formulation. Finally, a detailed design procedure is given for the estimation of the states of a flexible joint robot, which demonstrates the effectiveness of the proposed method.

Keywords: Bilinear systems, filter design, H_∞, LMI, uncertainties.

1. INTRODUCTION

Bilinear systems have been firstly used to represent many physical systems when linear models are inadequate. They present the advantages to be nonlinear systems but are close to the linear ones. Then a great deal of work has been devoted to the analysis and design techniques for bilinear systems. In the other hand we know that we must have access to all the state of a system to construct control laws or monitoring schemes for this system. Consequently, several results have been devoted to the problem of observers design for bilinear systems (see [1-3]). Unfortunately, physical systems are often subjected to bounded energy perturbations, and then in order to give a good estimation of their states, we must take into account an H_∞ criterion. Several methods for H_∞ filter design have been proposed in the full order case and even in the full order case and even in the reduce order case (see [4-13]).

The singular systems, meanwhile, have been introduced to describe systems for which standard state space representations are not applicable. Recently, these types of systems have attracted many attentions of the control community. These systems have a great significance both in theory and from application point of views (see [14-16] and the references therein). They are designed as singular, descriptor, generalized, implicit or semi-state systems (see [14-16] and the references therein). Applications of these systems are encountered in chemical processes, mineral processes, electrical and economical systems [17], … Therefore, the observers design for singular systems are of considerable interest (see [14,18,19]). A great deal of works has been devoted to the observers or filters design for descriptor systems (see [14,18]). In [19], an extension to the H_∞ filter design for Lipschitz singular systems has been presented. But there are less works for singular bilinear systems ones. This is one of the main motivations of our paper. Notice that even if the approach seems the same as in [19], the presence of the bilinearities introduced by the control inputs made the filter design different.

In this paper, an H_∞ filtering method is proposed to reconstruct the state of a class of singular bilinear systems with bounded input. The proposed approach is based on a new parameterization of the solutions of a set of algebraic Sylvester equations, which are derived from the unbiasedness of the estimation error. The unbiasedness conditions ensure that the error is independent of the states of the system. Notice that the control inputs of the system are treated as structured uncertainties by doing a change of variable in order to avoid the bilinearities they imply. Thus, under some conditions the problem of H_∞ filtering for the class of systems considered can be seen as a particular case static output feedback problem. Sufficient conditions for existence and convergence of the proposed filter are given in terms of linear matrix inequalities (LMI).

This paper is organized as follows. The problem is stated in Section 2. In Section 3, we will give the conditions for the existence of the unbiased H_∞ filter. Then a model of single-link flexible joint robot is used to demonstrate the effectiveness of the proposed method in Section 4, this is an interesting industrial motivation. Finally, Section 5 concludes the paper.

2. PROBLEM STATEMENT AND BASIC ASSUMPTIONS

Consider the following bilinear system

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$$E\dot{x}(t) = A_0x(t) + Bu(t) + \sum_{i=1}^{m} u_iA_ix(t) + D_1w(t), \tag{1a}$$

$$y = Cx(t) + D_2w(t) \tag{1b}$$

with the initial semi state $x(0) = x_0$, where $x(t) \in \Re^n$ the semi state vector, $u(t) = [u_1^T \ u_2^T \ \cdots \ u_m^T]^T \in \Re^m$ is the known inputs, $w(t) \in \Re^{n_w}$ is the disturbance signal satisfying $w(t) \in \mathcal{L}_2$ and $y(t) \in \Re^p$ is the measurement output. Matrix $E \in \Re^{n_E \times n}$ is a singular one. Matrices $A_i \in \Re^{n_E \times n}$ for $i = 0,1\dots,n$, $B \in \Re^{n_E \times m}$, $C \in \Re^{p \times n}$, $D_1 \in \Re^{n_E \times n_w}$ and $D_2 \in \Re^{p \times n_w}$ are real. The control inputs $u(t)$ are continuous and bounded, i.e., $u(t) \in \Gamma$, where

$$\Gamma := \{u(t) \in \Re^m / u_{i,\min} \le u_i(t) \le u_{i,\max},\ i = 1,2,\dots,m\}.$$

In this paper, a change of variables is introduced by considering each $u_i(t)$ as a "structured uncertainty" (see [20]), then $u_i(t)$ can be rewritten as follows:

$$u_i(t) = \alpha_i + \sigma_i\delta_i(t), \tag{2}$$

where $\alpha_i \in \Re$ and $\sigma_i \in \Re$ are given by

$$\alpha_i = \frac{u_{i,\min} + u_{i,\max}}{2} \quad \text{and} \quad \sigma_i = \frac{u_{i,\min} - u_{i,\max}}{2},$$

for $i = 1,\dots,m$, $\alpha_0 = 1$ and $\sigma_0 = 0$.

With the new "uncertain" variable $\delta_i(t) \in \bar{\Gamma} \subset \Re^m$ where the polytope $\bar{\Gamma}$ is defined as

$$\bar{\Gamma} := \{\delta(t) \in \Re^m / \underbrace{\delta_{i,\min} \le \delta_i(t) \le \delta_{i,\max}}_{\text{for } i=1,2,\dots,m}\} \text{ and where}$$

$$\delta_{i,\min} = -1 \quad \text{and} \quad \delta_{i,\max} = +1.$$

Now, as in [8,19], let $\Phi \in \Re^{r \times n_E}$ be a full row rank matrix such that

$$\Phi[E \ \ A_i] = 0, \quad \text{for} \quad i = 1,2,\dots,m$$

then, from (1), we obtain

$$\Phi A_0x(t) + \Phi D_1w(t) = -\Phi Bu(t).$$

Now, consider the following reduced-order filter for system (1)

$$\dot{\zeta}(t) = \left(N_0 + \sum_{i=1}^{m} u_iN_i\right)\zeta(t) + Hu(t) + \left(J_0 + \sum_{i=1}^{m} u_iJ_i\right)y(t), \tag{3a}$$

$$\hat{x}(t) = P\zeta(t) - Q\Phi Bu(t) + G_0y(t) \tag{3b}$$

with the initial condition $\zeta(0) = \zeta_0$. Vector $\zeta(t) \in \Re^q$ represents the state vector of the filter and $\hat{x}(t) \in \Re^n$ is the estimate of $x(t)$. N_i, J_i, H, P, Q, and G_0 are unknown matrices of appropriate dimensions, which must be determined such that $\hat{x}(t)$ asymptotically converges to $x(t)$ when $w(t) = 0$ and, for $w(t) \ne 0$, we solve

$$\min \sup_{w \in L_2 - \{0\}} \left(\frac{\|e(t)\|_2}{\|w(t)\|_2}\right).$$

Let $\varepsilon(t) = \zeta(t) - TEx(t)$ be the error between $\zeta(t)$ and $TEx(t)$, then we have the following error system:

$$\begin{aligned}\dot{\varepsilon}(t) &= \left(N_0 + \sum_{i=1}^{m} u_iN_i\right)\varepsilon(t) + \left(J_0C + N_0TE - TA_0\right)x(t) \\ &\quad + \left(H - TB\right)u(t) + \left(\sum_{i=1}^{m} u_i\left(N_iTE + J_iC - TA_i\right)\right)x(t) \\ &\quad + \left(J_0D_2 - TD_1 + \sum_{i=1}^{m} u_iJ_iD_2\right)w(t),\end{aligned} \tag{4a}$$

$$\begin{aligned}e(t) &= P\varepsilon(t) + \left([P \quad Q \quad G_0]\begin{bmatrix} TE \\ \Phi A_0 \\ C \end{bmatrix} - I_n\right)x(t) \\ &\quad + (Q\Phi D_1 + G_0D_2)w(t).\end{aligned} \tag{4b}$$

Now, if

$$\begin{aligned}&i) \quad N_0TE - TA_0 + J_0C = 0 \\ &ii) \quad N_iTE - TA_i + J_iC = 0 \quad \text{for} \quad 1 \le i \le m \\ &iii) \quad H = TB \\ &iv) \quad [P \quad Q \quad G_0]\begin{bmatrix} TE \\ \Phi A_0 \\ C \end{bmatrix} = I_n\end{aligned} \tag{5}$$

then the error (4) is independent of x explicitly, and we can rewritten the error (4) as

$$\begin{aligned}\dot{\varepsilon}(t) &= \left(N_0 + \sum_{i=1}^{m} u_iN_i\right)\varepsilon(t) + \left(J_0D_2 - TD_1\right)w(t) \\ &\quad + \sum_{i=1}^{m} u_iJ_iD_2w(t),\end{aligned} \tag{6a}$$

$$e(t) = P\varepsilon(t) + (Q\Phi D_1 + G_0D_2)w(t). \tag{6b}$$

Notice that equations $i) - iv)$ of (5) are in the form of a set of constrained Sylvester equations. In the sequel, we assume that

Assumption 1:

$$rank\begin{bmatrix} E \\ \Phi A_0 \\ C \end{bmatrix} = n.$$

Remark 1: One can see that the dimension of the filter (3) is $q \le n$. Then, the presented approach unifies the filter design for the full-order $q = n$, the reduced-order $q = n - p$ and the minimal order filter.

The design of filter (3) of dimension q is reduced to find the matrices T, N_i, J_i, H, P, Q and G_0 which satisfy the constraints $i) - iv)$ of (5).

3. MAIN RESULTS

In this section, a new method is presented to design filter (3) for system (1) guaranteeing that $e(t)$ asymptotically converge to zero when for $w(t) = 0$ and for

$w(t) \neq 0$, $\|e(t)\|_2 < \gamma \|w(t)\|_2$, where γ is a scalar corresponding to a prescribed performance. In order to ensure the unbiasedness of the filter (3), we must give a solution of the Sylvester matrix equations given by *i*) – *iv*) of (5). For this, let us first consider the following proposition.

Proposition 1: Let R_0 a full row rank matrix such that

$$rank\begin{bmatrix} R_0 \\ \Phi A_0 \\ C \end{bmatrix} = rank\begin{bmatrix} \tilde{T}E \\ \Phi A_0 \\ C \end{bmatrix}$$

then there always exists a matrix $\tilde{T}$ such that

$$\tilde{T} = R_0 \begin{bmatrix} E \\ \Phi A_0 \\ C \end{bmatrix}^+ \begin{bmatrix} I \\ 0 \end{bmatrix}$$

Proof: Let R_0 be any full row rank matrix such that

$$rank\begin{bmatrix} R_0 \\ \Phi A_0 \\ C \end{bmatrix} = rank\begin{bmatrix} \tilde{T}E \\ \Phi A_0 \\ C \end{bmatrix}$$

then there always exist matrix parameters, K_0 and $\tilde{T}$ such that

$$\tilde{T}E = R_0 - K_0 \begin{bmatrix} \Phi A_0 \\ C \end{bmatrix}$$

or equivalently

$$\begin{bmatrix} \tilde{T} & K_0 \end{bmatrix} \begin{bmatrix} E \\ \Phi A_0 \\ C \end{bmatrix} = R_0 \tag{7}$$

Then, under Assumption 1, the solution to (7) is

$$\begin{bmatrix} \tilde{T} & K_0 \end{bmatrix} = R_0 \begin{bmatrix} E \\ \Phi A_0 \\ C \end{bmatrix}^+$$

In this case, we have

$$\tilde{T} = R_0 \begin{bmatrix} E \\ \Phi A_0 \\ C \end{bmatrix}^+ \begin{bmatrix} I \\ 0 \end{bmatrix}$$

and

$$K_0 = R_0 \begin{bmatrix} E \\ \Phi A_0 \\ C \end{bmatrix}^+ \begin{bmatrix} 0 \\ I \end{bmatrix}$$

Before continuing, we also give the following lemmas, which give the conditions to get solutions to some parts of (5).

Lemma 1: Under Assumption 1, the constrained Sylvester equations *i*) and *iv*) of (5) have a solution if and only if

$$rank\begin{bmatrix} \tilde{T}E \\ \Phi A_0 \\ C \\ I_n \end{bmatrix} = rank\begin{bmatrix} \tilde{T}E \\ \Phi A_0 \\ C \end{bmatrix} = n.$$

In this case, the general solution is given by

$$\begin{bmatrix} N_0 & \Psi_0 & J_0 \\ P & Q & G_0 \end{bmatrix} = \begin{bmatrix} \tilde{T}A_0 \\ I_n \end{bmatrix} \Omega_0^+ - \begin{bmatrix} Z_0 \\ Y_0 \end{bmatrix} (I - \Omega_0 \Omega_0^+) \tag{8}$$

where

$$\tilde{T} = T + \Psi_0 \Phi = R_0 \begin{bmatrix} E \\ \Phi A_0 \\ C \end{bmatrix}^+ \begin{bmatrix} I \\ 0 \end{bmatrix}, \quad \Omega_0 = \begin{bmatrix} \tilde{T}E \\ \Phi A_0 \\ C \end{bmatrix}$$

in which Ψ_0 and $\begin{bmatrix} Z_0 \\ Y_0 \end{bmatrix}$ are arbitrary matrices of appropriate dimension.

Proof: Define the matrix $\tilde{T} = T + \Psi_0 \Phi$ where Ψ_0 is an arbitrary matrix [19], then equations *i*) and *iv*) of (5) can be rewritten as

$$\begin{bmatrix} N_0 & \Psi_0 & J_0 \\ P & Q & G_0 \end{bmatrix} \begin{bmatrix} \tilde{T}E \\ \Phi A_0 \\ C \end{bmatrix} = \begin{bmatrix} \tilde{T}A_0 \\ I_n \end{bmatrix} \tag{9}$$

Equation (9) has a solution if and only if

$$rank\begin{bmatrix} \tilde{T}E \\ \Phi A_0 \\ C \\ \tilde{T}A_0 \\ I_n \end{bmatrix} = rank\begin{bmatrix} \tilde{T}E \\ \Phi A_0 \\ C \end{bmatrix} = n.$$

If the Assumption 1 is verified, then the general solution to (9) is given by (8).

Lemma 2: The constrained Sylvester equation *ii*) of (5) has a solution if and only if for $1 \le i \le m$

$$rank\begin{bmatrix} \tilde{T}E \\ C \\ \tilde{T}A_i \end{bmatrix} = rank\begin{bmatrix} \tilde{T}E \\ C \end{bmatrix}$$

and the general solution is given by

$$[N_i \quad J_i] = \tilde{T}A_i\Omega^+ - Z_i(I - \Omega\Omega^+), \text{ for } 1 \le i \le m \tag{10}$$

where $\Omega = \begin{bmatrix} \tilde{T}E \\ C \end{bmatrix}$ and Z_i are arbitrary matrices of appropriate dimension for $1 \le i \le m$.

Proof: By substituting $\tilde{T}$ in *ii*) of (5), we obtain

$$\begin{bmatrix} N_i & J_i \end{bmatrix} \begin{bmatrix} \tilde{T}E \\ C \end{bmatrix} = \tilde{T}A_i, \text{ for } 1 \le i \le m \tag{11}$$

Equation (11) has a solution if and only if

$$rank\begin{bmatrix}\tilde{T}E\\ C\\ \tilde{T}A_i\end{bmatrix} = rank\begin{bmatrix}\tilde{T}E\\ C\end{bmatrix}$$

then the general solution to (11) is given by (10).

Before giving the complete design method of the H_∞ filter (3), we define the following matrices which are derived from (8) and (10), respectively

$$\Lambda_{N_0} = \tilde{T}A_0\Omega_0^+\begin{bmatrix}I\\0\\0\end{bmatrix},\quad \Delta_{N_0} = \left(I-\Omega_0\Omega_0^+\right)\begin{bmatrix}I\\0\\0\end{bmatrix},$$

$$\Lambda_{\Psi_0} = \tilde{T}A_0\Omega_0^+\begin{bmatrix}0\\I\\0\end{bmatrix},\quad \Delta_{\Psi_0} = \left(I-\Omega_0\Omega_0^+\right)\begin{bmatrix}0\\I\\0\end{bmatrix},$$

$$\Lambda_{J_0} = \tilde{T}A_0\Omega_0^+\begin{bmatrix}0\\0\\I\end{bmatrix},\quad \Delta_{J_0} = \left(I-\Omega_0\Omega_0^+\right)\begin{bmatrix}0\\0\\I\end{bmatrix},$$

$$\Lambda_P = I_n\Omega_0^+\begin{bmatrix}I\\0\\0\end{bmatrix},\quad \Delta_P = \left(I-\Omega_0\Omega_0^+\right)\begin{bmatrix}I\\0\\0\end{bmatrix},$$

$$\Lambda_Q = I_n\Omega_0^+\begin{bmatrix}0\\I\\0\end{bmatrix},\quad \Delta_Q = \left(I-\Omega_0\Omega_0^+\right)\begin{bmatrix}0\\I\\0\end{bmatrix},$$

$$\Lambda_{G_0} = I_n\Omega_0^+\begin{bmatrix}0\\0\\I\end{bmatrix},\quad \Delta_{G_0} = \left(I-\Omega_0\Omega_0^+\right)\begin{bmatrix}0\\0\\I\end{bmatrix},$$

and

$$\Lambda_{N_i} = \tilde{T}A_i\Omega^+\begin{bmatrix}I\\0\end{bmatrix},\quad \Delta_{N_i} = \left(I-\Omega\Omega^+\right)\begin{bmatrix}I\\0\end{bmatrix},$$

$$\Lambda_{J_i} = \tilde{T}A_i\Omega^+\begin{bmatrix}0\\I\end{bmatrix},\quad \Delta_{J_i} = \left(I-\Omega\Omega^+\right)\begin{bmatrix}0\\I\end{bmatrix}.$$

Then, from $i) - ii)$ of (5), we have the filter matrices given by

$$\begin{aligned}
&N_0 = \Lambda_{N_0} - Z_0\Delta_{N_0},\quad \Psi_0 = \Lambda_{\Psi_0} - Z_0\Delta_{\Psi_0},\\
&J_0 = \Lambda_{J_0} - Z_0\Delta_{J_0},\quad P = \Lambda_P - Y_0\Delta_P,\\
&G_0 = \Lambda_{G_0} - Y_0\Delta_{G_0},\quad Q = \Lambda_Q - Y_0\Delta_Q\\
&N_i = \Lambda_{N_i} - Z_i\Delta_{N_i},\quad J_i = \Lambda_{J_i} - Z_i\Delta_{J_i},\quad \text{for } 1\le i\le m.
\end{aligned}$$

So, the design of the filter given by (3) is reduced to find the parameters Z_0, Z_i, and Y_0.

3.1. H_∞ filter design

In this section, the goal is to propose an approach to design the matrix Z_0, Z_i, and Y_0 to ensure the error convergence and to fulfill H_∞ specification. By introducing the change of variable given by (2) into (6), we obtain

$$\begin{aligned}
\dot{\varepsilon}(t) &= \left(N_0 + \sum_{i=1}^{m}\left(\alpha_i + \sigma_i\delta_i(t)\right)N_i\right)\varepsilon(t)\\
&\quad + \left(J_0D_2 - TD_1\right)w(t) \qquad (12a)\\
&\quad + \sum_{i=1}^{m}\left(\alpha_i + \sigma_i\delta_i(t)\right)J_iD_2w(t),
\end{aligned}$$

$$e(t) = P\varepsilon(t) + (Q\Phi D_1 + G_0D_2)w(t), \qquad (12b)$$

which can be rewritten as

$$\begin{aligned}
\dot{\varepsilon}(t) &= \left(N_0 + \bar{N}\bar{\alpha}_\varepsilon + \bar{N}\bar{\sigma}_\varepsilon\Delta_\varepsilon(\delta)\bar{H}_\varepsilon\right)\varepsilon(t)\\
&\quad + \left(J_0D_2 + \bar{J}\bar{\alpha}_wD_2 - \tilde{T}D_1 + \Psi_0\Phi D_1\right)w(t)\\
&\quad + \bar{J}\bar{D}_2\Delta_w(\delta)\bar{H}_ww(t),\\
e(t) &= P\varepsilon(t) + (Q\Phi D_1 + G_0D_2)w(t).
\end{aligned}$$

Using the previous developments and the following notations

$$\begin{aligned}
&\bar{N} = [N_1 \quad N_2 \quad \cdots \quad N_m],\\
&\bar{\alpha}_w = [\alpha_1I_p \quad \cdots \quad \alpha_mI_p]^T,\\
&\bar{\sigma}_\varepsilon = \text{bdiag}[\sigma_1I_q \quad \cdots \quad \sigma_mI_q],\\
&\bar{D}_2 = \text{bdiag}[\sigma_1D_2 \quad \cdots \quad \sigma_mD_2],\\
&\Delta_w(\delta) = \text{bdiag}([\delta_1I_{n_w} \quad \cdots \quad \delta_mI_p]),\\
&\bar{H}_w = [I_{n_w} \quad \cdots \quad I_{n_w}]^T,
\end{aligned}$$

we obtain

$$\begin{aligned}
\dot{\varepsilon}(t) &= \left((\Lambda_{N_0} + \Lambda_{\bar{N}}\bar{\alpha}_\varepsilon) - Z\begin{bmatrix}\Delta_{N_0}\\ \Delta_{\bar{N}}\bar{\alpha}_\varepsilon\end{bmatrix}\right)\varepsilon(t)\\
&\quad + \left(\Lambda_{\bar{N}}\bar{\sigma}_\varepsilon - \bar{Z}\Delta_{\bar{N}}\bar{\sigma}_\varepsilon\right)\Delta_\varepsilon(\delta)\bar{H}_\varepsilon\varepsilon(t)\\
&\quad + \left(\Lambda_{J_0}D_2 - \tilde{T}D_1 + \Lambda_{\Psi_0}\Phi D_1 + \Lambda_{\bar{J}}\bar{\alpha}_wD_2\right)w(t)\\
&\quad - Z\begin{bmatrix}\Delta_{J_0}D_2 + \Delta_{\Psi_0}\Phi D_1\\ \Delta_{\bar{J}}\bar{\alpha}_wD_2\end{bmatrix}w(t)\\
&\quad + (\Lambda_{\bar{J}}\bar{D}_2 - \bar{Z}\Delta_{\bar{J}}\bar{D}_2)\Delta_w(\delta)\bar{H}_ww(t),\\
e(t) &= (\Lambda_P - Y_0\Delta_P)\varepsilon(t) + (\Lambda_Q\Phi D_1 + \Lambda_{G_0}D_2)w(t)\\
&\quad - Y_0(\Delta_Q\Phi D_1 + \Delta_{G_0}D_2)w(t),
\end{aligned}$$

where

$$\begin{aligned}
&Z = [Z_0 \quad \bar{Z}],\quad \bar{Z} = [Z_1 \cdots Z_m]^T,\\
&\Lambda_{\bar{N}} = [\Lambda_{N_1} \cdots \Lambda_{N_m}],\quad \Delta_{\bar{N}} = \text{bdiag}\left([\Delta_{N_1} \cdots \Delta_{N_m}]\right),\\
&\Lambda_{\bar{J}} = [\Lambda_{J_1} \cdots \Lambda_{J_m}],\quad \Delta_{\bar{J}} = \text{bdiag}\left([\Delta_{J_1} \cdots \Delta_{J_m}]\right).
\end{aligned}$$

Finally, we obtain the following error dynamics

$$\begin{aligned}
\dot{\varepsilon}(t) &= (\mathbb{A} - Z\mathbb{C})\varepsilon(t) + \left((\tilde{\mathbb{A}} - Z\tilde{\mathbb{C}})\Delta_\varepsilon(\delta)\overline{H_\varepsilon}\right)\varepsilon(t)\\
&\quad + (\mathbb{B} - Z\mathbb{G})w(t) + \left((\tilde{\mathbb{B}} - Z\tilde{\mathbb{G}})\Delta_w(\delta)\overline{H_w}\right)w(t),
\end{aligned} \qquad (13a)$$

$$e(t) = (\Lambda_P - Y_0\Delta_P)\varepsilon(t) + (\mathbb{F} - Y\mathbb{H})w(t), \qquad (13b)$$

where

$$\mathbb{A} = \Lambda_{N_0} + \Lambda_{\bar{N}}\bar{\alpha}_\varepsilon, \quad \mathbb{C} = \begin{bmatrix} \Delta_{N_0} \\ \Delta_{\bar{N}}\bar{\alpha}_\varepsilon \end{bmatrix},$$

$$\tilde{\mathbb{C}} = \begin{bmatrix} 0 \\ \Delta_{\bar{N}}\bar{\sigma}_\varepsilon \end{bmatrix}, \quad \tilde{\mathbb{A}} = \Lambda_{\bar{N}}\bar{\sigma}_\varepsilon,$$

$$\mathbb{B} = \Lambda_{J_0}D_2 - \tilde{T}D_1 + \Lambda_{\Psi_0}\Phi D_1 + \Lambda_{\bar{J}}\bar{\alpha}_w D_2,$$

$$\mathbb{G} = \begin{bmatrix} \Delta_{J_0}D_2 + \Delta_{\Psi_0}\Phi D_1 \\ \Delta_{\bar{J}}\bar{\alpha}_w D_2 \end{bmatrix}, \quad \tilde{\mathbb{B}} = \Lambda_{\bar{J}}\bar{D}_2, \quad \tilde{\mathbb{G}} = \begin{bmatrix} 0 \\ \Delta_{\bar{J}}\bar{D}_2 \end{bmatrix},$$

$$\mathbb{F} = \Lambda_{G_0}D_2 + \Lambda_Q\Phi D_1, \quad \mathbb{H} = \Delta_Q\Phi D_1 + \Delta_{G_0}D_2.$$

Then, the design of the H_∞ filter is resolved from the following theorem in terms of LMI.

Theorem 1: Under Assumption 1 and given $\gamma > 0$, there exists a filter of the form (3) such that the error $e(t)$ given by (6) is asymptotically stable for $w(t) = 0$ and $\|e(t)\|_2 < \gamma\|w(t)\|_2$ if there exist symmetric positive definite matrices X, S_ε, S_w and matrices $\mathbb{Z}$ and Y_0 such that the LMI (14) is satisfied, and positive scalars $\mu_{i,\varepsilon}$ and $\mu_{i,w}, i = 1, 2, ..., m$. Here, '*' is the transpose of the off-diagonal part. Then, the gain Z is then obtained by $Z^T = \mathbb{Z}X^{-1}$.

$$\begin{bmatrix} \mathbb{A}^T X - \mathbb{C}^T\mathbb{Z} + X\mathbb{A} - \mathbb{Z}^T\mathbb{C} & * & * & * & * & * & * \\ \mathbb{B}^T X - \mathbb{G}^T\mathbb{Z} & -\gamma^2 I & * & * & * & * & * \\ \tilde{\mathbb{A}}^T X - \tilde{\mathbb{C}}^T\mathbb{Z} & 0 & -S_\varepsilon & * & * & * & * \\ \tilde{\mathbb{B}}^T X - \tilde{\mathbb{G}}^T\mathbb{Z} & 0 & 0 & -S_\omega & * & * & * \\ \Lambda_P - Y_0\Delta_P & \mathbb{F} - Y_0\mathbb{H} & 0 & 0 & -I & * & * \\ S_\varepsilon\bar{H}_\varepsilon & 0 & 0 & 0 & 0 & -S_\varepsilon & * \\ 0 & S_\omega\bar{H}_\omega & 0 & 0 & 0 & 0 & -S_\omega \end{bmatrix} < 0, \quad (14)$$

$$S_\varepsilon = \text{bdiag}(\mu_{1,\varepsilon}I, \cdots, \mu_{m,\varepsilon}I) > 0,$$
$$S_w = \text{bdiag}(\mu_{1,w}I, \cdots, \mu_{m,w}I) > 0.$$

Proof: Let

$$\begin{bmatrix} p_\varepsilon \\ p_w \end{bmatrix} = \begin{bmatrix} \Delta_\varepsilon(\delta) & 0 \\ 0 & \Delta_w(\delta) \end{bmatrix}\begin{bmatrix} p_\varepsilon \\ q_w \end{bmatrix}, \quad (15)$$

then, we can rewritten the error (14) as

$$\dot{\varepsilon}(t) = (\mathbb{A} - Z\mathbb{C})\varepsilon(t) + \left([\mathbb{B} \quad \tilde{\mathbb{A}} \quad \tilde{\mathbb{B}}] - Z[\mathbb{G} \quad \tilde{\mathbb{C}} \quad \tilde{\mathbb{G}}]\right)\begin{pmatrix} w \\ p_\varepsilon \\ p_w \end{pmatrix}, \quad (16)$$

$$\begin{bmatrix} e(t) \\ q_\varepsilon \\ q_w \end{bmatrix} = \begin{bmatrix} \Lambda_P - Y_0\Delta_P \\ \bar{H}_\varepsilon \\ 0 \end{bmatrix}\varepsilon(t) + \begin{bmatrix} \mathbb{F} - Y\mathbb{H} & 0 & 0 \\ 0 & 0 & 0 \\ \bar{H}_w & 0 & 0 \end{bmatrix}\begin{bmatrix} w \\ p_\varepsilon \\ p_w \end{bmatrix}. \quad (17)$$

By considering system giving by equations (15)-(14)-(17) as a diagonal norm-bounded linear differential inclusion (see [20]), it can be rewritten as

$$\dot{\varepsilon}(t) = A_e\varepsilon(t) + [B_e \quad H_1]\begin{pmatrix} w \\ p \end{pmatrix},$$

$$\hat{e}(t) = \begin{bmatrix} C_e \\ E_1 \end{bmatrix}\varepsilon(t) + \begin{bmatrix} D_e & H_2 \\ E_2 & E_3 \end{bmatrix}\begin{bmatrix} w \\ p \end{bmatrix},$$

where

$$A_e = \mathbb{A} - Z\mathbb{C}, \quad B_e = \mathbb{B} - Z\mathbb{G},$$

$$H_1 = \begin{bmatrix} \tilde{\mathbb{A}} - Z\tilde{\mathbb{C}} & \tilde{\mathbb{B}} - Z\tilde{\mathbb{G}} \end{bmatrix}, \quad C_e = \Lambda_P - Y_0\Delta_P,$$

$$E_1 = \begin{bmatrix} \bar{H}_\varepsilon \\ 0 \end{bmatrix}, \quad D_e = \mathbb{F} - Y\mathbb{H}, \quad H_2 = [0 \ \ 0],$$

$$E_2 = \begin{bmatrix} 0 \\ \bar{H}_w \end{bmatrix}, \quad E_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$p = \begin{bmatrix} p_\varepsilon \\ p_w \end{bmatrix}, \quad \hat{e}(t) = \begin{bmatrix} e(t) \\ p_\varepsilon \\ q_w \end{bmatrix},$$

and we can then introduce the following auxiliary system similarly to [21]

$$\dot{\varepsilon}(t) = A_e\varepsilon(t) + [\gamma^{-1}B_e \quad H_1 S^{\frac{-1}{2}}]\begin{pmatrix} w \\ p \end{pmatrix}, \quad (18a)$$

$$\hat{e}(t) = \begin{bmatrix} C_e \\ S^{\frac{1}{2}}E_1 \end{bmatrix}\varepsilon(t) + \begin{bmatrix} \gamma^{-1}D_e & H_2 S^{\frac{-1}{2}} \\ \gamma^{-1}S^{\frac{1}{2}}E_2 & S^{\frac{1}{2}}E_3 S^{\frac{-1}{2}} \end{bmatrix}\begin{bmatrix} w \\ p \end{bmatrix}, \quad (18b)$$

where $S = \begin{bmatrix} S_\varepsilon & 0 \\ 0 & S_w \end{bmatrix}$.

By using the bounded real lemma (see [20-22]), system (18) is exponentially convergent for

$w(t) \neq 0$ and $\|e(t)\|_2 < \gamma\|w(t)\|_2$,

if there exist symmetric positive definite matrices X, S, and a scalar γ such that the inequality (19) is satisfied.

$$\left[\begin{matrix} A_e^T X + XA_e + C_e^T C_e + E_1^T S E_1 & \\ \gamma^{-1}B_e^T X + \gamma^{-1}D_e^T C_e + \gamma^{-1}E_2^T S E_1 & \gamma^{-2}D_e^T D_e + \gamma^{-2}E_2^T S E_2 \\ S^{\frac{-1}{2}}H_1^T + S^{\frac{-1}{2}}H_2^T C_e + S^{\frac{-1}{2}}E_3^T S E_1 & S^{\frac{-1}{2}}\gamma^{-1}H_2^T D_e + \gamma^{-1}S^{\frac{-1}{2}}E_3^T E_2 E_3^T S E_1 \end{matrix}\right.$$

$$\gamma^{-1}XB_e + \gamma^{-1}C_e^T D_e + \gamma^{-1}E_1^T S E_2$$

$$\begin{bmatrix} XH_1S^{\frac{-1}{2}}+C_e^TH_2S^{\frac{-1}{2}}+E_1^TSE_3S^{\frac{-1}{2}} \\ \gamma^{-1}D_e^TH_2S^{\frac{-1}{2}}+\gamma^{-1}S^{\frac{-1}{2}}E_2SE_3S^{\frac{-1}{2}} \\ \gamma^{-1}H_2^TH_2S^{\frac{-1}{2}}+S^{\frac{-1}{2}}+S^{\frac{-1}{2}}E_3^TSE_3S^{\frac{-1}{2}} \end{bmatrix}<0. \quad (19)$$

By pre-multiplying and post-multiplying inequality (19) by $\mathrm{bdiag}(I,\gamma I,S^{1/2})$, and by using the Schur complement, we obtain:

$$\begin{bmatrix} A_e^TX+XA_e & XB_e & XH_1 & C_e^T & E_1^TS \\ B_e^TX & -\gamma^2I & 0 & D_e^T & E_2^TS \\ H_1^TX & 0 & -S & H_2^T & E_3S \\ C_e & D_e & H_2 & -I & 0 \\ SE_1 & SE_2 & SE_3 & 0 & -S \end{bmatrix}<0 \quad (20)$$

by substituting A_e, B_e, H_1, C_e, E_1, D_e, H_2, E_2, E_3 by their values, and using $\mathbb{Z}=Z^TX$, we obtain the LMI (14). Then, Theorem 1 is proved.

In the following section, we apply the results developed in this paper to an illustrative example to show the effectiveness of our approach.

4. PRACTICAL EXAMPLE

Let us consider the example of a single-link flexible joint robot given in [23]. A slight modification of the model gives the following system

$$\begin{aligned} &\dot{x}_1(t)=x_2(t),\\ &\dot{x}_2(t)=-\frac{k}{J_m}x_1(t)+\frac{k}{J_m}x_2(t)+\frac{K_k}{J_m}x_3(t)+u(t),\\ &\dot{x}_3(t)=x_4(t),\\ &\dot{x}_4(t)=\frac{k}{J_l}x_1(t)+\frac{k}{J_l}x_3(t)+\gamma x_5(t)\\ &\qquad+\frac{mghk}{J_l}(x_3(t)+x_5(t))u(t),\\ &0=x_1+x_3+x_5+d_0(x_3(t)+x_5(t))u(t),\\ &y_1(t)x_1,\quad y_2(t)=x_2(t), \end{aligned}$$

where $x_1(t)$, $x_2(t)$, $x_3(t)$, $x_4(t)$ and $x_5(t)$ are the angular rotation of the motor, the angular velocity of the motor, the angular position of the link, the angular velocity of the link, the fast subsystem perturbation and the scalar unknown input, respectively. The obtained model is then a bilinear singular system. For illustration purpose, a set of model parameters k, J_m, J_l, B_m, K_k, g, and h is chosen to give the following singular bilinear model

$$E\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\\ \dot{x}_3(t)\\ \dot{x}_4(t)\\ \dot{x}_5(t)\end{bmatrix}=A\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\\ x_4(t)\\ x_5(t)\end{bmatrix}+\begin{bmatrix}0\\0\\0\\-0.1\\0.359\end{bmatrix}x_3(t)u(t)$$

$$+\begin{bmatrix}1\\0\\1\\0\\0.359\end{bmatrix}x_5(t)u(t)+\begin{bmatrix}0\\21.5\\0\\0\\0\end{bmatrix}u(t)+\begin{bmatrix}2\\1\\1\\0\\1\end{bmatrix}w(t),$$

$$y(t)=\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\\ x_4(t)\\ x_5(t)\end{bmatrix}+\begin{bmatrix}1\\1\end{bmatrix}w(t),$$

where

$$A=\begin{bmatrix}0&1&0&0&0\\-48.5&-1.25&48.5&0&0\\0&0&0&1&0\\19.5&0&-19.5&0&-0.1\\1&0&1&0&1\end{bmatrix},$$

$$E=\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&0\end{bmatrix} \text{ and } u(t)=0.1\sin(t).$$

For this system, the matrix $\Phi=[0\ \ 0\ \ 0\ \ 0\ \ 1]$. In this case, it is easy to see that Assumption 1 is verified. We shall design a reduced-order filter of dimension $q=3$, let

$$R_0=\begin{bmatrix}1&0&0&1&0\\0&1&0&0&1\\0&0&1&0&0\end{bmatrix} \text{ then } rank\begin{bmatrix}R_0\\ \Phi A_0\\ C\end{bmatrix}=5$$

for $\gamma=2.93$, the parameters of H_∞ filter (3) are given by

$$N_0=\begin{bmatrix}0&19.4&0.53\\-1&-24.25&-0.64\\0&0&-0.9\end{bmatrix},\quad N_1=\begin{bmatrix}0&0&-0.01\\0&0&0.023\\0&0&0.91\end{bmatrix},$$

$$J_0=\begin{bmatrix}28.74&-9.21\\-35.18&11.03\\0.017&0.018\end{bmatrix},\quad J_1=\begin{bmatrix}0&0\\0&0\\0.017&0\end{bmatrix},$$

$$H=\begin{bmatrix}0\\-10.75\\0\end{bmatrix},\quad Q\Phi B=\begin{bmatrix}0\\0\\0\\0\\0\end{bmatrix},$$

$$P=\begin{bmatrix}0&0&0\\0&0&0\\0&-1&-0.028\\1&0&-0.014\\0&1&0.028\end{bmatrix},\quad G_0=\begin{bmatrix}1&0\\0&1\\-0.47&0.5\\-0.49&0\\-0.53&-0.5\end{bmatrix}.$$

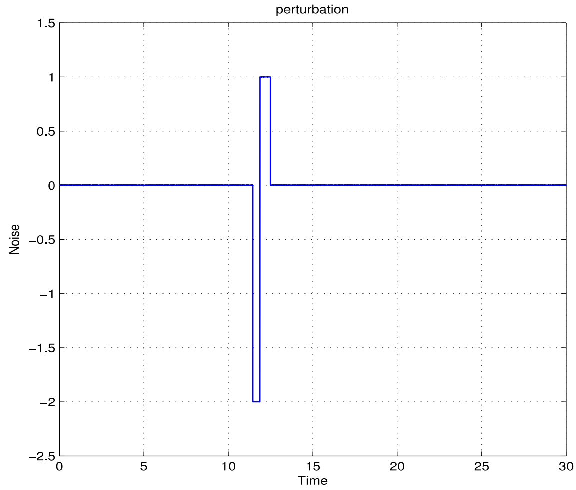

Fig. 1. The disturbance $w(t)$.

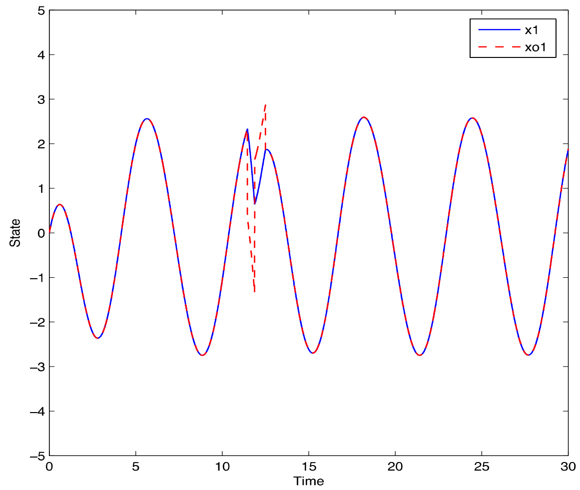

Fig. 2. $x_1(t)$ (solid lines) and $\hat{x}_1(t)$ (dashed lines).

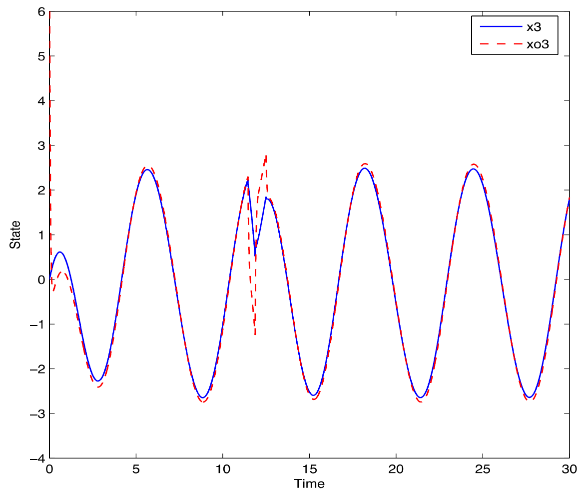

Fig. 3. $x_2(t)$ (solid lines) and $\hat{x}_2(t)$ (dashed lines).

Fig. 1 shows the applied disturbance, which appears after ten seconds. Then, Figs. 2 to Fig. 6 show the different states (solid lines) of the considered system and their estimates (dashed lines) using the H_∞ filter

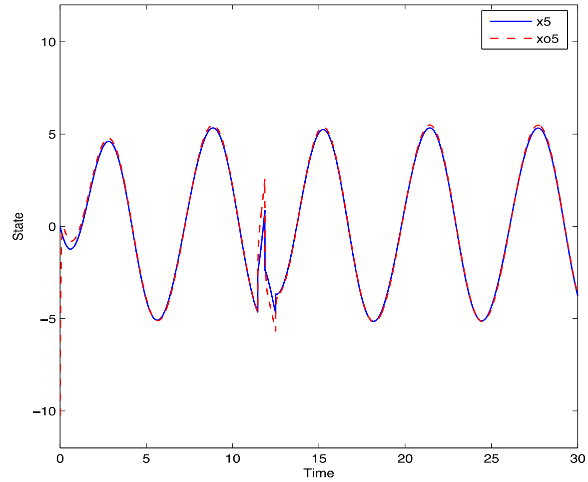

Fig. 4. $x_3(t)$ (solid lines) and $\hat{x}_3(t)$ (dashed lines).

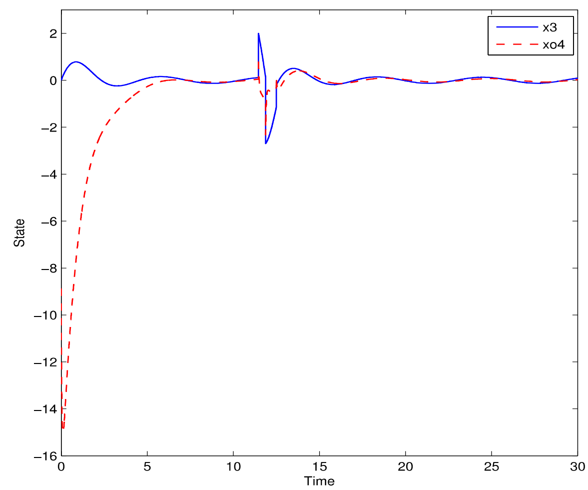

Fig. 5. $x_4(t)$ (solid lines) and $\hat{x}_4(t)$ (dashed lines).

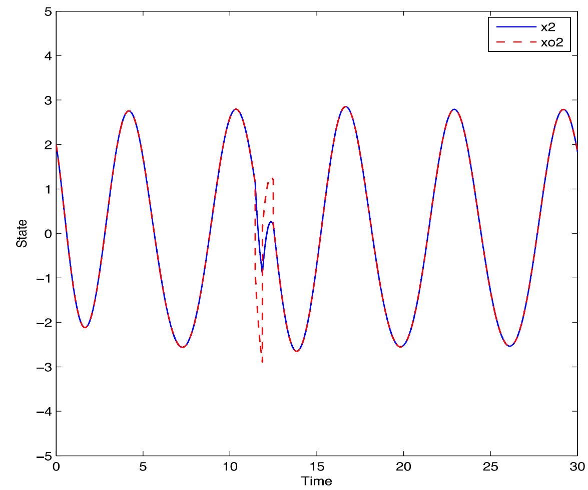

Fig. 6. $x_5(t)$ (solid lines) and $\hat{x}_5(t)$ (dashed lines).

presented in this paper. The efficiency of our filter is then proved; the states and their estimates converge to the same values before the appearance of the disturbance. When the disturbance appears, we have a non zero errors which become newly zero when the disturbance vanished.

5. CONCLUSION

In this paper, an H_∞ filter design procedure for singular bilinear systems is presented. It is based on the parameterization of the general solution of the constrained Sylvester equations. The solution is obtained from LMI formulation. A practical example is given to illustrate our approach.

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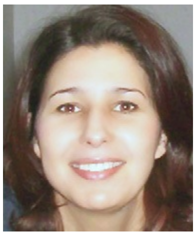

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