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H_∞ Control of Singular Systems via Delta Operator Method

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Abstract: This paper studies the problem of state feedback H_∞ control for singular systems through delta operator approach. A necessary and sufficient condition is presented such that a singular delta operator system is admissible with a prescribed H_∞ performance, which can provide a unified framework of the existing H_∞ performance analysis results for both continuous case and discrete case. The existence condition and explicit expression of a desirable H_∞ controller are also obtained for singular delta operator systems. The proposed design method can be used for both singular continuous systems and singular discrete systems directly. The corresponding design procedures, which simplify the classical approaches, are discussed and presented. All obtained conditions in this paper are in the form of strict linear matrix inequalities whose feasible solutions can be found by standard linear programming method. Numerical examples are provided to illustrate the effectiveness of the theoretical results obtained in this paper.

Keywords: Admissibility, H_∞ performance, linear matrix inequality (LMI), singular delta operator systems, state feedback.

1. INTRODUCTION

Compared with the state-space systems, singular systems can describe dynamics and algebraic constraints of state variables simultaneously, and therefore they are more suitable to describe many practical systems. Singular systems have been appeared in many different research areas such as economic systems, highly interconnected large-scale systems, constrained mechanical systems and electrical networks, etc (see, e.g., [1]). Many fundamental results have been developed for singular systems during the past decades, for example, see [1-21] and the references therein. Moreover, singular system methods are also adopted to study certain types of state-space systems in order to obtain less conservative results [22,23]. It is worth pointing out that both analysis and control of singular systems are much more complicated than those of state-space systems because admissibility (i.e., regularity, stability, and impulse elimination for the continuous case or regularity, stability and causality for the discrete case) is a basic requirement for the control of singular systems, whereas for the control of state-space systems, the concern is mainly the stability.

The problems of H_∞ performance analysis and synthesis for singular systems are of both practical and theoretical importance and have received considerable attention in the past decades. Many significant results have been reported in literature. The systematic results of H_∞ performance analysis and synthesis for singular continuous systems and singular discrete systems, respectively, can be found in [2]. Recently, certain different novel LMI-based bounded real lemmas (i.e., H_∞ performance analysis results) for singular discrete systems have been developed in [12,13]. In particular, the problem of state feedback H_∞ control for singular discrete systems with or without uncertainty has been considered in [11,13]. However, current framework for the study of singular systems is either for the continuous case or the discrete case, and these two cases are considered separately and independently. There is no direct relationship established in literature between singular continuous systems and singular discrete systems.

A significant amount of discrete systems are often obtained from continuous systems through state sampling. Standard shift operator is usually used to describe discrete systems. The main drawback of this setting is that the corresponding discrete version does not converge smoothly to its continuous counterpart as the sampling period approaches to zero [24]. This creates a gap between continuous systems and their corresponding discrete models, which results in the disconnection of controller design for a continuous system and its discrete version. In order to overcome this problem, a delta operator method was proposed in [25]. It is shown that the delta operator model not only can provide a unified framework for both-state space continuous systems and their discrete models but also can establish a direct

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connection between continuous systems and their corresponding discrete systems so that the relationship between these two kinds of systems becomes more transparent in terms of sampling periods [26]. Furthermore, the delta operator requires a smaller word length when implemented in fixed-point digital control processors than the shift operator does and the delta operator method is also significantly less sensitive than the shift operator method at high sampling rates [27]. There have been many results for state-space systems through the delta operator method. Some solutions to the problems of H_2 control, guaranteed cost control, and H_∞ filtering were given in [26] for delta operator systems (DOSs). The problem of robust stabilization was considered in [28] for delay DOSs. An observer-based sliding mode controller was synthesized for uncertain DOSs in [29]. A novel delta operator Kalman filter was designed in [30] and the problem of guaranteed cost control was considered for networked control systems based on the above filter [31]. The problems of robust H_∞ control and fault-tolerant control for fuzzy systems via the delta operator method were studied, respectively, in [32] and [33].

As we just mentioned, the delta operator method has been adopted extensively to study various types of state-space systems. The main advantage by using the delta operator method for controller design is that the design procedure can be unified for a continuous system and its corresponding discrete model and we only need to focus on one of them under the delta operator framework. Recently, the delta operator model has been set up for a singular continuous system which will converge to the corresponding continuous system as the sampling period approaches to zero [17,18]. Thus this indicates that the delta operator model, similar to state-space situation, can also provide a unified framework for a singular continuous system and its discrete model. Moreover, under the delta operator framework, the admissibility can be characterized by the same condition for both systems. Existing research results about singular delta operator systems (SDOSs) can be found in [17-21]. Various admissibility conditions were discussed in [18]. The analysis results of controllability and observability were given in [19] and [20], respectively. The problem of admissible control was considered in [17,21], respectively. However, all of the current existing results about H_∞ performance analysis and synthesis of singular systems are considered for the continuous case and the discrete case separately and independently. This usually renders the control design procedure to be either more complicated or redundant. To the best of our knowledge, the problem of H_∞ control for singular systems under the delta operator framework remains unknown in current literature.

Since SDOSs can provide a unified framework for both singular continuous systems and singular discrete systems, one would expect that the approach by using the delta operator method should be able to simplify the procedure of H_∞ performance analysis and synthesis. Motivated by this, in this paper, we will consider the problem of state feedback H_∞ control for singular systems via the delta operator method. The main contributions of this paper are given as follows: (1) A necessary and sufficient condition is derived for a SDOS to be admissible with a prescribed H_∞ performance, which can provide a unified expression of the existing H_∞ performance analysis results for both singular continuous systems and singular discrete systems. (2) The existence condition and explicit expression of a desirable H_∞ controller are given for SDOSs and the obtained method can be used directly to design a state feedback H_∞ controller for both singular continuous systems and singular discrete systems. The detailed design procedures, which simplify the classical approaches, are discussed and presented. All conditions in this paper are in the form of strict LMIs whose feasible solutions can be found in an efficient way by standard linear programming method.

The remainder of this paper is organized as follows. Section 2 introduces some preliminaries and presents the problem formulation. Section 3 shows the main results. Section 4 provides some numerical examples to illustrate our theoretical outcomes, and Section 5 concludes this paper.

Throughout this paper, R^n and $R^{m\times n}$ denote the spaces of n-dimensional real vectors and $m\times n$ real matrices, respectively. Matrix $P>0$ (or $P<0$) means that P is symmetric and positive definite (or negative definite). The superscript T means the transpose of a vector or a matrix. The shorthand $diag(M_1, M_2, \cdots, M_s)$ denotes a block diagonal matrix with diagonal blocks being the matrices $M_1, M_2, \cdots, M_s$. $D_{\text{int}}(a,r)$ is the interior of the region in the complex plane with the center at $(a,0)$ and the radius r. The identity matrix with dimension $r\times r$ is denoted by I_r. $\lambda(A,B)=\{z\,|\det(zA-B)=0\}$ stands for the eigenvalue set of B relative to A. δ is the delta operator defined by

$$\delta x(t)=\begin{cases}\dfrac{dx(t)}{dt}, & h=0,\\ \dfrac{x(t+h)-x(t)}{h}, & h\neq 0,\end{cases}$$

where h is the sampling period.

2. PRELIMINARIES AND PROBLEM FORMULATION

Consider the following singular continuous system

$$\begin{aligned}&E\dot{x}(t)=A_s x(t)+F_s\omega(t)+B_s u(t),\\ &z(t)=Cx(t),\end{aligned}\tag{1}$$

where $x(t)\in R^n$ is the state, $\omega(t)\in R^p$ is the disturbance input, $u(t)\in R^m$ is the control input, $z(t)\in R^q$ is the regulated output, $E\in R^{n\times n}$ satisfies $rank(E)=r<n$, A_s, B_s, F_s, C are known real matrices with appropriate dimensions.

For a prescribed sampling period $h>0$, one can set up the delta operator model for the system (1) as follows

$$\begin{aligned} E\delta x(t_k) &= Ax(t_k) + F\omega(t_k) + Bu(t_k), \\ z(t_k) &= Cx(t_k), \end{aligned} \tag{2}$$

where $x(t_k) \in R^n$ is the state, $u(t_k) \in R^m$ is the control input, $\omega(t_k) \in R^p$ is the disturbance input, $z(t_k) \in R^q$ is the regulated output. t_k denotes the time $t = kh$. E, C are the same as that in the system (1), A, B and F are known real matrices obtained from the system (1) associated with the sampling period $h > 0$. It has been known that the system (2) approaches to the system (1) as the sampling period h tends to zero [17,18].

Consider the following SDOS

$$E\delta x(t_k) = Ax(t_k). \tag{3}$$

Definition 1 [18]**:** The system (3) is said to be regular if $\det(\zeta E - A)$ is not identically zero. The system (3) is said to be causal if $\deg(\det(\zeta E - A)) = rank(E)$. The system (3) is said to be stable if $\lambda(E, A) \subset D_{\text{int}}(-h^{-1}, h^{-1})$. The system (3) is said to be admissible if it is regular, causal and stable.

Lemma 1 [18]**:** The system (3) is admissible if and only if there exist matrices $W > 0$ and Q such that

$$hA^T WA + A^T P + P^T A < 0, \tag{4}$$

where $P = WE + SQ$ and S is any matrix of full column rank and satisfies $E^T S = 0$.

Let the controller to be designed in this paper is a state feedback one as

$$u(t_k) = Kx(t_k). \tag{5}$$

Then the closed-loop system of the system (2) under the controller (5) is

$$\begin{aligned} E\delta x(t_k) &= A_c x(t_k) + F\omega(t_k), \\ z(t_k) &= Cx(t_k), \end{aligned} \tag{6}$$

where $A_c = A + BK$.

The purpose of this paper is to give the design method of the gain matrix K in the controller (5), such that the closed-loop system (6) is admissible with an H_∞ performance γ, i.e., the system (6) satisfies the following requirements [11]

1) When $\omega(t_k) = 0$, the system (6) is admissible.

2) When $\omega(t_k) \neq 0$, under initial condition $x(0) = 0$, the system (6) satisfies an H_∞ performance γ, i.e.,

$$J = \sum_{k=0}^{\infty} (z^T(t_k) z(t_k) - \gamma^2 \omega^T(t_k)\omega(t_k)) < 0.$$

For the derivation of our main results, we present the following lemmas.

Lemma 2 [34]**:** For matrices $Q = Q^T$, $R = R^T$ and S,

$$\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} < 0$$

is equivalent to $R < 0$ and $Q - SR^{-1}S^T < 0$ or $Q < 0$ and $R - S^T Q^{-1} S < 0$.

Lemma 3 [29]**:** For any time function $x(t)$, $y(t)$ and a sampling period $h > 0$, there exists

$$\delta(x(t)y(t)) = \delta x(t)y(t) + x(t)\delta y(t) + h\delta x(t)\delta y(t).$$

Lemma 4 [2]**:** The system (1) with $u(t) = 0$ is admissible with an H_∞ performance γ if and only if there exist matrices $W > 0$ and Q such that

$$\begin{bmatrix} P^T A_s + A_s^T P + C^T C & P^T F_s \\ F_s^T P & -\gamma^2 I_p \end{bmatrix} < 0. \tag{7}$$

where $P = WE + SQ$ and S is any matrix of full column rank and satisfies $E^T S = 0$.

3. MAIN RESULTS

Consider the unforced system of the system (2) as

$$\begin{aligned} E\delta x(t_k) &= Ax(t_k) + F\omega(t_k), \\ z(t_k) &= Cx(t_k). \end{aligned} \tag{8}$$

First we present the result of H_∞ performance analysis for the system (8) as follows.

Theorem 1: The system (8) is admissible with an H_∞ performance γ if and only if there exist matrices $W > 0$ and Q such that

$$\begin{bmatrix} A^T P + P^T A + C^T C & P^T F & A^T W \\ F^T P & -\gamma^2 I_p & F^T W \\ WA & WF & -h^{-1} W \end{bmatrix} < 0, \tag{9}$$

where $P = WE + SQ$ and S is any matrix of full column rank and satisfies $E^T S = 0$.

Proof: (Sufficiency) Assume that the inequality (9) holds. Then from $W > 0$ and Lemma 2 we have that (9) is equivalent to the following inequality

$$\begin{bmatrix} A^T P + P^T A + C^T C + hA^T WA & P^T F + hA^T WF \\ F^T P + hF^T WA & -\gamma^2 I_p + hF^T WF \end{bmatrix} < 0. \tag{10}$$

From (10) and $C^T C \geq 0$, it is easy to obtain

$$hA^T WA + A^T P + P^T A < 0.$$

Then from Lemma 1 we have that the system (8) with $\omega(t_k) = 0$ is admissible.

Let

$$V(x(t_k)) = x^T(t_k) E^T WEx(t_k). \tag{11}$$

Then it follows from $W > 0$ that $V(x(t_k)) \geq 0$ holds for any $k \geq 0$. By Lemma 3, we have

$$\begin{aligned} \delta V(x(t_k)) &= \delta x^T(t_k) E^T WEx(t_k) + x^T(t_k) E^T WE\delta x(t_k) \\ &\quad + h\delta x^T(t_k) E^T WE\delta x(t_k) \\ &= (Ax(t_k) + F\omega(t_k))^T WEx(t_k) \\ &\quad + x^T(t_k) E^T W(Ax(t_k) + F\omega(t_k)) \\ &\quad + h(Ax(t_k) + F\omega(t_k))^T W(Ax(t_k) + F\omega(t_k)) \end{aligned}$$

$$= \begin{bmatrix} x^T(t_k) & \omega^T(t_k) \end{bmatrix} \Xi \begin{bmatrix} x(t_k) \\ \omega(t_k) \end{bmatrix},$$

where

$$\Xi = \begin{bmatrix} A^T WE + E^T WA + hA^T WA & E^T WF + hA^T WF \\ F^T WE + hF^T WA & hF^T WF \end{bmatrix}.$$

From $E^T S = 0$ we have

$$\delta x^T(t_k) E^T SQx(t_k) + x^T(t_k) Q^T S^T E \delta x(t_k)$$
$$= \begin{bmatrix} x^T(t_k) & \omega^T(t_k) \end{bmatrix} \Lambda \begin{bmatrix} x(t_k) \\ \omega(t_k) \end{bmatrix} = 0,$$

where

$$\Lambda = \begin{bmatrix} A^T SQ + Q^T S^T A & Q^T S^T F \\ F^T SQ & 0 \end{bmatrix}.$$

From $x(0)=0$ we obtain $V(x(0)) = 0$. Then

$$J = \sum_{k=0}^{\infty} (z^T(t_k) z(t_k) - \gamma^2 \omega^T(t_k) \omega(t_k))$$
$$\le \sum_{k=0}^{\infty} (z^T(t_k) z(t_k) - \gamma^2 \omega^T(t_k) \omega(t_k)) + h^{-1} V(x(t_\infty))$$
$$= \sum_{k=0}^{\infty} (z^T(t_k) z(t_k) - \gamma^2 \omega^T(t_k) \omega(t_k) + \delta V(x(t_k)))$$
$$+ \delta x^T(t_k) E^T SQx(t_k) + x^T(t_k) Q^T S^T E \delta x(t_k)$$
$$= \sum_{k=0}^{\infty} \begin{bmatrix} x^T(t_k) & \omega^T(t_k) \end{bmatrix} \Sigma \begin{bmatrix} x(t_k) \\ \omega(t_k) \end{bmatrix},$$

where

$$\Sigma = \Xi + \Lambda + diag(C^T C, -\gamma^2 I_p).$$

From (10) we have $\Sigma < 0$, then we can obtain $J < 0$ which means that the system (8) satisfies an H_∞ performance γ. Therefore we have proved that the system (8) is admissible with an H_∞ performance γ.

(Necessity): Assume that the system (8) is admissible with an H_∞ performance γ. Then invertible matrices L and R can always be found such that [1]

$$LER = diag(I_r, 0), \quad LAR = diag(A_1, I_{n-r}), \tag{13}$$

and $\lambda(I_r, A_1) \subset D_{\text{int}}(-h^{-1}, h^{-1})$.

Let

$$LF = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}, \; CR = [C_1 \quad C_2], \; R^{-1} x(t_k) = \begin{bmatrix} x_1(t_k) \\ x_2(t_k) \end{bmatrix}. \tag{14}$$

Then the system (8) is equivalent to the following system

$$\begin{aligned} \delta x_1(t_k) &= A_1 x_1(t_k) + F_1 \omega(t_k), \\ 0 &= x_2(t_k) + F_2 \omega(t_k), \\ z(t_k) &= C_1 x_1(t_k) + C_2 x_2(t_k), \end{aligned} \tag{15}$$

which is also the same as

$$\begin{aligned} \delta x_1(t_k) &= A_1 x_1(t_k) + F_1 \omega(t_k), \\ z(t_k) &= C_1 x_1(t_k) - C_2 F_2 \omega(t_k). \end{aligned} \tag{16}$$

From the definition of $\delta x(t_k) = h^{-1}(x(t_{k+1}) - x(t_k))$ we have $x(t_{k+1}) = x(t_k) + h\delta x(t_k)$. Then, the system (16) can also be written as

$$\begin{aligned} x_1(t_{k+1}) &= (I_r + hA_1) x_1(t_k) + hF_1 \omega(t_k), \\ z(t_k) &= C_1 x_1(t_k) - C_2 F_2 \omega(t_k) \end{aligned} \tag{17}$$

and $\lambda(I_r, I_r + hA_1) \subset D_{\text{int}}(0,1)$.

Then the system (8) is admissible with an H_∞ performance γ is equivalent to that the discrete system (17) is stable with an H_∞ performance γ, which holds if and only if there exists a matrix $W_1 > 0$ such that [2]

$$\Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{12}^T & \Phi_{22} \end{bmatrix} < 0, \tag{18}$$

where we use $hJ < 0$ instead of $J < 0$ and

$$\Phi_{11} = (I_r + hA_1)^T W_1 (I_r + hA_1) - W_1 + hC_1^T C_1,$$
$$\Phi_{12} = h(I_r + hA_1)^T W_1 F_1 - hC_1^T C_2 F_2,$$
$$\Phi_{22} = hF_2^T C_2^T C_2 F_2 + h^2 F_1^T W_1 F_1 - \gamma^2 h I_p.$$

By multiplying (18) with h^{-1} we can obtain the following inequality

$$\Gamma = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{12}^T & \Gamma_{22} \end{bmatrix} < 0, \tag{19}$$

where

$$\Gamma_{11} = A_1^T W_1 + W_1 A_1 + hA_1^T W_1 A_1 + C_1^T C_1,$$
$$\Gamma_{12} = W_1 F_1 + hA_1^T W_1 F_1 - C_1^T C_2 F_2,$$
$$\Gamma_{22} = F_2^T C_2^T C_2 F_2 + hF_1^T W_1 F_1 - \gamma^2 I_p.$$

Then, a sufficiently small scalar $\beta > 0$ can always be found such that

$$\Gamma + diag(0, \beta F_2^T F_2) < 0. \tag{20}$$

Let $X = C_2^T C_2 + \beta I_{n-r}$ and

$$W = L^T diag(W_1, h^{-1} X) L, \tag{21}$$

$$Q^T = R^{-T} \begin{bmatrix} -C_1^T C_2 \\ -X \end{bmatrix} \Upsilon^{-T}, \quad S = L^T \begin{bmatrix} 0 \\ \Upsilon \end{bmatrix}, \tag{22}$$

where $\Upsilon \in R^{(n-r)\times(n-r)}$ is any invertible matrix.

Now it is straightforward to derive $X > 0$ from $\beta > 0$ and $C_2^T C_2 \ge 0$. Thus we have $W > 0$ from $h > 0$, $W_1 > 0$, $X > 0$ and the invertibility of the matrix L. We can also obtain that the matrix S is of full column rank and satisfies $E^T S = 0$.

From (13), (21) and (22) we have

$$P = WE + SQ$$
$$= L^T diag(W_1, h^{-1} X) L L^{-1} diag(I_r, 0) R^{-1}$$

$$+L^T\begin{bmatrix}0\\ \Upsilon\end{bmatrix}\Upsilon^{-1}\begin{bmatrix}-C_2^TC_1 & -X\end{bmatrix}R^{-1}$$
$$=L^T\begin{bmatrix}W_1 & 0\\ -C_2^TC_1 & -X\end{bmatrix}R^{-1}.$$

From (13), (14), (21) and the expression of P we can obtain

$$\begin{aligned}&hA^TWA+A^TP+P^TA+C^TC\\&=R^{-T}diag(hA_1^TW_1A_1,X)R^{-1}\\&\quad+R^{-T}\begin{bmatrix}A_1^T & 0\\ 0 & I_{n-r}\end{bmatrix}\begin{bmatrix}W_1 & 0\\ -C_2^TC_1 & -X\end{bmatrix}R^{-1}\\&\quad+R^{-T}\begin{bmatrix}W_1 & -C_1^TC_2\\ 0 & -X\end{bmatrix}\begin{bmatrix}A_1 & 0\\ 0 & I_{n-r}\end{bmatrix}R^{-1}\\&\quad+R^{-T}\begin{bmatrix}C_1^T\\ C_2^T\end{bmatrix}\begin{bmatrix}C_1 & C_2\end{bmatrix}R^{-1}\\&=R^{-T}diag(\Gamma_{11},-\beta I_{n-r})R^{-1},\end{aligned}$$

and

$$\begin{aligned}P^TF+hA^TWF&=R^{-T}\begin{bmatrix}W_1 & -C_1^TC_2\\ 0 & -X\end{bmatrix}LL^{-1}\begin{bmatrix}F_1\\ F_2\end{bmatrix}\\&\quad+hR^{-T}\begin{bmatrix}A_1^TW_1 & 0\\ 0 & h^{-1}X\end{bmatrix}LL^{-1}\begin{bmatrix}F_1\\ F_2\end{bmatrix}\\&=R^{-T}\begin{bmatrix}\Gamma_{12}\\ 0\end{bmatrix}.\end{aligned}$$

From (14) and (21) we have

$$\begin{aligned}&hF^TWF-\gamma^2I_p\\&=h\begin{bmatrix}F_1^T & F_2^T\end{bmatrix}diag(W_1,h^{-1}X)\begin{bmatrix}F_1\\ F_2\end{bmatrix}-\gamma^2I_p\\&=\Gamma_{22}+\beta F_2^TF_2.\end{aligned}$$

By pre-multiplying and post-multiplying the inequality (20) with $diag(R^{-T},I_p)$ and $diag(R^{-1},I_p)$, respectively, we can derive

$$\begin{bmatrix}R^{-T}\Gamma_{11}R^{-1} & R^{-T}\Gamma_{12}\\ \Gamma_{12}^TR^{-1} & \Gamma_{22}+\beta F_2^TF_2\end{bmatrix}<0. \tag{23}$$

Then from (23) and the above equations we have that the inequality (10) holds, which is also equivalent to (9) from $W>0$ and Lemma 2. This is the end of proof.

From the definition of $\delta x(t_k)=h^{-1}(x(t_{k+1})-x(t_k))$, the system (2) can also be written as the following singular discrete system

$$\begin{aligned}&Ex(t_{k+1})=A_zx(t_k)+F_z\omega(t_k)+B_zu(t_k),\\&z(t_k)=Cx(t_k),\end{aligned} \tag{24}$$

where $A_z=E+hA$, $F_z=hF$ and $B_z=hB$.

Then from Theorem 1 we can derive the following corollary.

Corollary 1: The system (24) with $u(t_k)=0$ is admissible with an H_∞ performance γ if and only if there exist matrices $W>0$ and Q such that

$$\begin{bmatrix}\Omega & Q^TS^TF_z & A_z^TW\\ F_z^TSQ & -\gamma^2I_p & F_z^TW\\ WA_z & WF_z & -W\end{bmatrix}<0, \tag{25}$$

where $\Omega=A_z^TSQ+Q^TS^TA_z-E^TWE+C^TC$ and S is any matrix of full column rank and satisfies $E^TS=0$.

Proof: Since the system (24) and the system (2) are the same system, by Theorem 1 the system (2) with $u(t_k)=0$ is admissible with an H_∞ performance γ if and only if there exist matrices $W>0$ and Q such that

$$\begin{bmatrix}A^TP+P^TA+h^{-1}C^TC & P^TF & A^TW\\ F^TP & -h^{-1}\gamma^2I_p & F^TW\\ WA & WF & -h^{-1}W\end{bmatrix}<0, \tag{26}$$

where $P=WE+SQ$ and we use $h^{-1}J<0$ instead of $J<0$.

From $A_z=E+hA$ and $F_z=hF$ we can obtain

$$A=h^{-1}(A_z-E)=h^{-1}A_E,\qquad F=h^{-1}F_z. \tag{27}$$

By substituting (27) into the inequality (26) and multiplying h to both sides of (26) we can derive

$$\begin{bmatrix}A_E^TP+P^TA_E+C^TC & P^TF_z & A_E^TW\\ F_z^TP & -\gamma^2I_p & F_z^TW\\ WA_E & WF_z & -W\end{bmatrix}<0. \tag{28}$$

From $W>0$ and Lemma 2 we have that (28) is equivalent to the following inequality

$$\begin{bmatrix}A_E^TP+P^TA_E+C^TC & P^TF_z\\ F_z^TP & -\gamma^2I_p\end{bmatrix}+\begin{bmatrix}A_E^T\\ F_z^T\end{bmatrix}W[A_E\ \ F_z]<0. \tag{29}$$

By some simple computation, one can get

$$\begin{aligned}&A_E^TP+P^TA_E+C^TC+A_E^TWA_E\\&=(A_z-E)^T(WE+SQ)+(WE+SQ)^T(A_z-E)\\&\quad+C^TC+(A_z-E)^TW(A_z-E)\\&=A_z^TWA_z+A_z^TSQ+Q^TS^TA_z-E^TWE+C^TC,\end{aligned}$$

and

$$\begin{aligned}F_z^TP+F_z^TWA_E&=F_z^T(WE+SQ)+F_z^TW(A_z-E)\\&=F_z^TWA_z+F_z^TSQ.\end{aligned}$$

Then from $W>0$, Lemma 2 and the above equations we can derive that (29) is equivalent to the inequality (25). This is the end of proof.

Remark 1: Corollary 1 provides a necessary and sufficient condition for a singular discrete system to be admissible with an H_∞ performance γ which is obtained directly based on Theorem 1. It should be pointed out that Corollary 1 is in fact equivalent to Theorem 1 in [11] which also ensures the same performance of a singular discrete system.

Remark 2: From Lemma 4, Theorem 1 and the fact that as the sampling period h tends to zero, the system (2) approaches to the system (1), we thus know that when h tends to zero, Theorem 1 becomes Lemma 4. Moreover, the system (2) is itself a singular discrete system described by the delta operator. Thus, Theorem 1 provides a unified result of H_∞ performance analysis for both singular continuous systems and singular discrete systems.

Next, we consider the problem of H_∞ control for the system (2). From Theorem 1 we know that the closed-loop system (6) is admissible with a prescribed H_∞ performance γ if and only if there exist matrices $W>0$ and Q satisfying the inequality (9), where the matrix A is replaced by the matrix $A_c = A + BK$. In this case, the unknown matrix K is contained in A_c and A_c is companied by different matrices as W and Q^TS^T, thus it is difficult to obtain the design method of the matrix K. To solve the problem of H_∞ control, we rewrite the system (6) as the following system

$$\begin{aligned}&\bar{E}\delta y(t_k) = \bar{A}y(t_k) + \bar{F}\omega(t_k),\\ &z(t_k) = \bar{C}y(t_k),\end{aligned} \tag{30}$$

where

$$\bar{E} = \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} A & B \\ K & -I_m \end{bmatrix}, \tag{31}$$

$$\bar{F} = \begin{bmatrix} F \\ 0 \end{bmatrix}, \quad \bar{C} = [C \quad 0], \quad y(t_k) = \begin{bmatrix} x(t_k) \\ u(t_k) \end{bmatrix}. \tag{32}$$

Then we have the following result.

Theorem 2: There exists a controller (5) for the system (2) such that the closed-loop system (6) is admissible with an H_∞ performance γ, if there exist matrices $W>0$, $V>0$, Q, U and Y such that

$$\begin{bmatrix} \Psi_{11} & \Psi_{21}^T & P^TF & A^TW & Y^T \\ \Psi_{21} & \Psi_{22} & U^TS^TF & B^TW & -V \\ F^TP & F^TSU & -\gamma^2 I_p & F^TW & 0 \\ WA & WB & WF & -h^{-1}W & 0 \\ Y & -V & 0 & 0 & -h^{-1}V \end{bmatrix} < 0, \tag{33}$$

where $P = WE + SQ$ and S is any matrix of full column rank and satisfies $E^TS = 0$, J_1, J_2 are known matrices with appropriate dimensions, and

$$\begin{aligned}\Psi_{11} &= A^TP + P^TA + J_1^TY + Y^TJ_1 + C^TC,\\ \Psi_{21} &= B^TP - VJ_1 + J_2^TY + U^TS^TA,\\ \Psi_{22} &= U^TS^TB + B^TSU - VJ_2 - J_2^TV.\end{aligned}$$

In this case, the gain matrix K of the controller (5) can be designed as $K = V^{-1}Y$.

Proof: Assume that the inequality (33) holds. Let

$$\bar{W} = diag(W,V),\ \bar{S} = diag(S,V),\ \bar{Q} = \begin{bmatrix} Q & U \\ J_1 & J_2 \end{bmatrix}, \tag{34}$$

where the partitions are compatible with the structure of $\bar{E}$.

Thus, one can obtain $\bar{W} > 0$ from $W>0$ and $V>0$. Moreover, we also have that the matrix $\bar{S}$ is of full column rank from $V>0$ and that S is of full column rank. Moreover, we have

$$\bar{E}^T\bar{S} = diag(E^T, 0)diag(S,V) = diag(E^TS, 0) = 0.$$

By denoting $Y = VK$, $\bar{P} = \bar{W}\bar{E} + \bar{S}\bar{Q}$, (31), (32) and (34), we arrive at the inequality (33) which is the same as

$$\begin{bmatrix} \bar{A}^T\bar{P} + \bar{P}^T\bar{A} + \bar{C}^T\bar{C} & \bar{P}^T\bar{F} & \bar{A}^T\bar{W} \\ \bar{F}^T\bar{P} & -\gamma^2 I_p & \bar{F}^T\bar{W} \\ \bar{W}\bar{A} & \bar{W}\bar{F} & -h^{-1}\bar{W} \end{bmatrix} < 0.$$

Then from Theorem 1 we have that the system (30) (i.e., the system (6)) is admissible with an H_∞ performance γ. From $V>0$ we can get $K = V^{-1}Y$ immediately. This is the end of proof.

Based on Lemma 4, Remark 2 and Theorem 2, we can further obtain the following corollary.

Corollary 2: There exists a controller $u(t) = Kx(t)$ for the system (1) such that the closed-loop system is admissible with an H_∞ performance γ, if there exist matrices $W>0$, $V>0$, Q, U and Y such that

$$\begin{bmatrix} \Pi_{11} & \Pi_{21}^T & P^TF_s \\ \Pi_{21} & \Pi_{22} & U^TS^TF_s \\ F_s^TP & F_s^TSU & -\gamma^2 I_p \end{bmatrix} < 0, \tag{35}$$

where $P = WE + SQ$ and S is any matrix of full column rank and satisfies $E^TS = 0$, J_1, J_2 are known matrices with appropriate dimensions, and

$$\begin{aligned}\Pi_{11} &= A_s^TP + P^TA_s + J_1^TY + Y^TJ_1 + C^TC,\\ \Pi_{21} &= B_s^TP - VJ_1 + J_2^TY + U^TS^TA_s,\\ \Pi_{22} &= U^TS^TB_s + B_s^TSU - VJ_2 - J_2^TV.\end{aligned}$$

In this case, the gain matrix K of the controller can be designed as $K = V^{-1}Y$.

Similarly, based on Corollary 1 and the method to obtain Theorem 2, we can obtain another result as follows.

Corollary 3: There exists a controller (5) for the system (24) such that the closed-loop system is admissible with an H_∞ performance γ, if there exist matrices $W>0$, $V>0$, Q, U and Y such that

$$\begin{bmatrix} \Omega_{11} & \Omega_{21}^T & Q^TS^TF_z & A_z^TW & Y^T \\ \Omega_{21} & \Omega_{22} & U^TS^TF_z & B_z^TW & -V \\ F_z^TSQ & F_z^TSU & -\gamma^2 I_p & F_z^TW & 0 \\ WA_z & WB_z & WF_z & -W & 0 \\ Y & -V & 0 & 0 & -V \end{bmatrix} < 0, \tag{36}$$

where $P = WE + SQ$ and S is any matrix of full column rank and satisfies $E^T S = 0, J_1, J_2$ are known matrices with appropriate dimensions, and

$$\Omega_{11} = A_z^T SQ + Q^T S^T A_z + J_1^T Y + Y^T J_1 - E^T WE + C^T C,$$
$$\Omega_{21} = B_z^T SQ - VJ_1 + J_2^T Y + U^T S^T A_z,$$
$$\Omega_{22} = U^T S^T B_z + B_z^T SU - VJ_2 - J_2^T V.$$

In this case, the gain matrix K of the controller (5) can be designed as $K = V^{-1}Y$.

Remark 3: From Theorem 2 and Corollary 2 we know that the design method of a state feedback H_∞ controller for SDOSs can be used directly for singular continuous systems. From Theorem 2 and Corollary 3 we know that the same design method can also be adopted to obtain the H_∞ control result for singular discrete systems. Thus, the design method of a suitable H_∞ controller for SDOSs in this paper can be used for both singular continuous systems and singular discrete systems directly.

4. EXAMPLES

In this section, we give some numerical examples to demonstrate the theoretical results that we have obtained in the above sections.

Example 1: Consider the system (1) with the following matrices

$$E = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}, \quad A_s = \begin{bmatrix} -5 & 3 \\ 2 & -3 \end{bmatrix}, \quad B_s = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad F_s = \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$
$$C = [1 \quad 1].$$

Let $\gamma = 3$, $S = [1 \quad -1]^T$ and solve the inequality (7), then we can find a feasible solution. Decrease the value of γ and still keep (7) having a feasible solution. Thus by Lemma 4 we can obtain that the minimal value of γ is $\gamma = 2.34$ such that the system (1) with $u(t) = 0$ is admissible with an H_∞ performance γ.

Let the sampling period be $h = 0.2$ and we can set up the delta operator model (2) of the system (1), where E, C are the same as above and

$$A = \begin{bmatrix} -4.9856 & 3.0432 \\ 2.0144 & -2.9568 \end{bmatrix},$$
$$B = \begin{bmatrix} 1.8849 \\ 2.8849 \end{bmatrix}, \quad F = \begin{bmatrix} 0.9281 \\ 1.9281 \end{bmatrix}.$$

Let γ, S as above and solve the inequality (9), then we can find a feasible solution. Decrease the value of γ and still keep (9) having a feasible solution. Thus by Theorem 1 we can obtain that the minimal value of γ is also $\gamma = 2.34$ such that the system (2) with $u(t_k) = 0$ is admissible with an H_∞ performance γ.

Let $J_1 = [1 \quad 1]^T$, $J_2 = 1$, γ and S as above and solve the inequality (33), then we can derive a feasible solution. Decrease the value of γ and still keep (33) having a feasible solution. Then by Theorem 2 we can find that the minimal value of γ is $\gamma = 0.49$ such that the closed-loop system (6) of the system (2) under the controller (5) is admissible with an H_∞ performance γ. Thus we know that the introduction of the controller (5) for the system (2) has improved the H_∞ performance of the closed-loop system significantly.

Let $\gamma = 0.6$ and solve (33), then we can get a feasible solution. Thus from Theorem 2 we have that there is a state feedback H_∞ controller (5) for the system (2) and the gain matrix K of the controller (5) can be designed as $K = [-0.2386 \quad -1.8631]$. Select $x_1(0) = 4$ and the state trajectory of the closed-loop system (6) with $\omega(t_k) = 0$ is shown in Fig. 1. Let $x(0) = 0$, $\omega(t_k) = 0.5$ and $\gamma_1 = |z(t_k)/\omega(t_k)|$. The time response of γ_1 of the closed-loop system (6) is shown in Fig. 2. Fig. 1 shows that the system (6) with $\omega(t_k) = 0$ is indeed admissible and Fig. 2 shows that the system (6) with $\omega(t_k) \neq 0$ indeed has an H_∞ performance 0.6.

Adopt the same $\gamma = 0.6$ and solve the inequality (35), then we can get a feasible solution. Thus by Corollary 2 there exists an H_∞ controller $u(t) = Kx(t)$ for the system (1) and the gain matrix K can be designed as $K = [-0.0363 \quad -1.7328]$.

Example 2: Consider the system (24) with the following matrices [11,13]

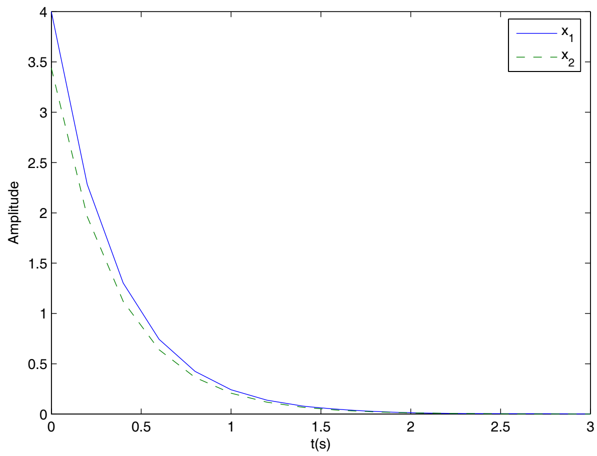

Fig. 1. The state trajectory of the system (6).

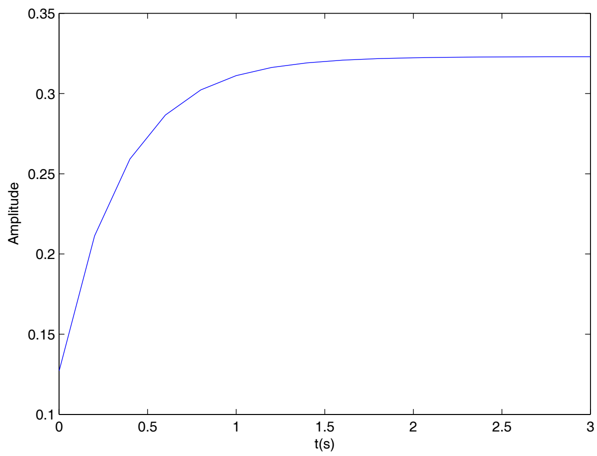

Fig. 2. The time response of γ_1 of the system (6).

$$E=\begin{bmatrix}1 & 2\\ 0 & 0\end{bmatrix},\ A_z=\begin{bmatrix}2.5 & 1\\ 1.7 & 0.8\end{bmatrix},\ B_z=\begin{bmatrix}1\\ 0\end{bmatrix},\ F_z=\begin{bmatrix}1\\ 1\end{bmatrix},$$
$$C=[1\quad 1].$$

Let $\gamma=2$, $S=[0\quad 1]^T$, $J_1=[1\quad 1]^T$, $J_2=1$ and solve the inequality (36), then we can get a feasible solution. Decrease the value of γ and still keep (36) having a feasible solution. Then by Corollary 3 we can find that the minimal value of γ is $\gamma=0.43$ such that the closed-loop system of the above system (24) under the controller (5) is admissible with an H_∞ performance γ. By using the methods in [13] and [11] to design the same H_∞ controller (5) for the system (24), respectively, we can obtain the minimal value of γ as $\gamma=0.41$ and $\gamma=1.19$, respectively.

It should be noticed that the results in [11,13] and our paper (i.e., Corollary 3) all provide only sufficient conditions to ensure the existence of an H_∞ controller (5) for the system (24). From this example, we can conclude that the design method of a state feedback H_∞ controller for singular discrete systems in our paper is similar to that in [13] and much less conservative than that in [11].

5. CONCLUDING REMARKS

In this paper, the problem of state feedback H_∞ control for singular systems via the delta operator method has been considered. A necessary and sufficient condition has been derived such that a SDOS is admissible with a prescribed H_∞ performance. The obtained condition can provide a unified expression of the existing H_∞ performance analysis results for both singular continuous systems and singular discrete systems. Moreover, a design method of a desirable H_∞ controller has also been given for SDOSs which can be used directly for both singular continuous systems and singular discrete systems. All obtained results in this paper are in the form of strict LMIs whose feasible solutions can be derived easily.

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