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Stability Map of Multiple Time Delayed Fractional Order Systems

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Abstract: In this paper, a novel method is presented to study the stability map of linear fractional order systems with multiple delays against uncertainties in delays. It is evident from the literature that the stability question of this class of dynamics has not been resolved yet. The backbone of the new methodology is inspired by an advanced clustering with frequency sweeping technique which enables the exhaustive determination of stability switching curves in the space of the delays. The proposed method detects all the stability regions exactly, in the parametric space of the time delays. An illustrative example is presented to confirm the proposed method results.

Keywords: Fractional delay systems, frequency sweeping, multiple time delay, stability map.

1. INTRODUCTION

The stability of every dynamic system is a basic question and a fundamental issue. In confronting the time-delay systems, we are curious to know what will happen if the amount of delay increases and how the stability feature will change. Regarding the systems with multiple time delays, the stability map of the system can be expressed as stable and unstable regions in the two- or three-dimensional space.

In this paper, one of the most important and unresolved problems of fractional delay systems is studied: the asymptotic stability of a general class of fractional order systems with multiple time delays, against delay uncertainties (this means that the time delays are constant but their true values are not exactly known). The problem is known to be notoriously complex, primarily because the systems are infinite dimensional due to delays. Multiplicity of the delays in this study complicates the analysis even further. And “fractional order” feature of the systems makes the problem much more challenging compared to integer order systems.

The original idea in this strategy is derived based on the method (advanced clustering with frequency sweeping (ACFS)) reported in [1] to achieve the stability map of integer order systems with multiple time delays. ACFS does not impose any restrictions in the number of delays, and it can directly extract the 2-D cross sections of the stability views in any two delay domain. The main objective in 2D stability analysis is to construct all the potential stability switching curves (PSSC) which partition the delay space into stable and unstable regions. Obviously, the accuracy and completeness of the analysis strongly depends on finding all the existing PSSC without any approximations [1]. There has been a large effort to deal with this problem, for the standard case (integer order systems); see [1]-[4], and others.

The researchers of [5] may be the pioneer to consider stability of the fractional order time delay system with single-delay. They have developed the Ruth-Hurwitz criteria for analyzing the stability of some special delay systems to those involve fractional power $\sqrt{s}$. For single delay case; in [6], necessary and sufficient conditions for BIBO stability of the retarded fractional order delay systems and sufficient conditions for some neutral types have been introduced. Recently, Pakzad *et al*. in [7] have presented an analytical method for finding the stability regions of fractional delay systems with single and commensurate delay in one-dimensional parametric space of delay, and with uncertain parameters in both time-delay space and coefficient space in [8], which uses the bilinear Rekasius transformation to eliminate the exponential type transcendental term in the characteristic equations. In addition, they have successfully extended an analytical algorithm based on ‘Direct Method’ (presented by Walton and Marshal [9]) for testing the stability of such systems [10].

In this paper, we extend the approach of [1] to fractional order systems with multiple delays. necessary and sufficient conditions which yield the exact lower and upper bounds of the crossing frequency set (CFS) can be computed via an automated sequential formula. These bounds are crucial as they determine the sweeping range of the only parameter, the frequency that ACFS sweeps.

The paper is organized as follows. Section 2, contains the problem statement. In Section 3, ACFS method is presented which extracts the 2-D PSSC for fractional delay systems, Section 4 brings an example to illustrate the results presented and finally Section 5 concludes the work.

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2. PRELIMINARIES AND DEFINITIONS

Standard notation has been used throughout the article:
$\mathbb{N},\mathbb{Z}$: sets of natural and integer numbers.
$\mathbb{R}(\mathbb{R}^+,\mathbb{R}^-)$: set of real (positive real, negative real) numbers.
$\mathbb{C}$: set of complex numbers.
$j=\sqrt{-1}$: the imaginary unit.
$\Re(z)$: real part of a complex number z
$\Im(z)$: imaginary part of a complex number z.
$|z|,\angle z$: magnitude and argument of a complex number z
$R_{T_\ell}(p_1,p_2)$: denotes the resultant of bivariate polynomials $p_1(T_1,T_2)$ and $p_2(T_1,T_2)$ with eliminating T_ℓ where $\ell=1,2$.

Consider a fractional order system with the following characteristic equation:

$$CE(s,\tau_1,\tau_2)=\sum_{\ell=0}^{n}p_\ell\left(\sqrt[\alpha]{s}\right)e^{-\ell\tau_1 s}+\sum_{i=1}^{m}q_i\left(\sqrt[\alpha]{s}\right)e^{-i\tau_2 s}\,, \quad (1)$$

where parameters τ_1 and τ_2 are non-negative, such that $(\tau_1,\tau_2)\in\mathbb{R}^2$ and p_ℓ and q_i are real polynomials in complex variable $\sqrt[\alpha]{s}$ with arbitrary order (where $\alpha\in\mathbb{N}$). Note that the zeros of characteristic equation (1) are in fact the poles of the system under investigation. We find out from [6] that the transfer function of a system with a characteristic equation in the form of (1) will be H_∞ stable if, and only if, it doesn't have any pole at $\Re(s)\geq 0$ (in particular, no poles of fractional order at $s=0$).

For fractional order systems, if an auxiliary variable of $v=\sqrt[\alpha]{s}$ is used, a practical test for the evaluation of stability can be obtained. By applying this auxiliary variable in characteristic equation (1), the following relation is obtained:

$$CE(v,\tau_1,\tau_2)=\sum_{\ell=0}^{n}p_\ell(v)e^{-\ell\tau_1 v^\alpha}+\sum_{i=1}^{m}q_i(v)e^{-i\tau_2 v^\alpha}\,. \quad (2)$$

This will transform the domain of the system from a multisheeted Riemann surface into the complex plane, where the poles can be easier calculated. In this new variable, the instability region of the original system is not given by the right half-plane, but in fact by the region described as:

$$|\angle v|\leq\frac{\pi}{2\alpha} \quad (3)$$

with $v\in\mathbb{C}$, which the stable region has been displayed by hatched lines in Fig. 1. Note that under this transformation, the imaginary axis in the s-domain is mapped into the lines

$$|\angle v|=\pm\frac{\pi}{2\alpha}\,. \quad (4)$$

Let us assume that; $s=\pm j\omega$ or in other words, $s=\omega e^{\pm j\pi/2}$ are the roots of characteristic equation (1) for a $(\tau_1,\tau_2)\in\mathbb{R}^2$. Then for the auxiliary variable, the roots are defined as follows:

$$|\angle v|=\pm\frac{\pi}{2\alpha}\,. \quad (5)$$

Therefore, with the auxiliary variable $v=\sqrt[\alpha]{s}$, there is a direct relation between the roots on the imaginary axis for the s-domain with the ones having argument $\pm\pi/2\alpha$ in the v-domain.

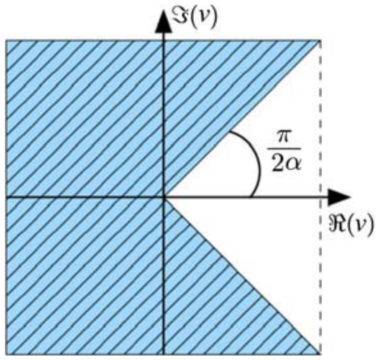

Fig. 1. The v-stability region for fractional delay systems.

3. METHODOLOGY

In this section, we propose a method that can extract the 2-D PSSC for fractional order systems with multiple time delay. In other words, we perform the stability analysis of (1) in 2-D delay parameter space.

3.1. ACFS methodology

If there exists an imaginary root of equation (1) at $s=\pm j\omega_c$ ('c' for crossing) for a given set of time delays $\tau=(\tau_1,\tau_2)$ the same imaginary root will also exist at all the countably infinite grid points of

$$\{\tau\}=(\tau_{1l},\tau_{2k})=\left(\tau_{10}+\frac{2\pi}{\omega_c}l,\tau_{20}++\frac{2\pi}{\omega_c}k\right)$$
$$l=0,1,2,\ldots,\quad k=0,1,2,\ldots \quad (6)$$
$$\tau_{10}-\frac{2\pi}{\omega_c}\leq 0,\quad \tau_{20}-\frac{2\pi}{\omega_c}\leq 0.$$

This signifies that τ_{i0} is the smallest positive τ_{il}, $\tau_{i0}=\min(\tau_{il}),\ i=1,2,\ l=0,1,2,\ldots,(\tau_{il}>0)$. Notice that for a fixed ω_c the distinct points of (6) generate a grid in $\{\tau\}\in\mathbb{R}^{2+}$ space with equal grid size, $2\pi/\omega_c$ in both dimensions.

Definition 1 (kernel curves): Assume that the set of $(\tau_{10},\tau_{20})|_{\omega_c}$ is determined exhaustively in $\{\tau\}=(\tau_1,\tau_2)$ space for all possible ω_c values satisfying (1) and (6). These curves as a group are called the ''kernel curves'' of system described by the characteristic equation (1). We denote these curves by $\wp_0(\tau_1,\tau_2)$.

Definition 2 (offspring curves): The trajectories of (τ_1,τ_2) grid points in (6) for $l=0,1,2,\ldots,\ k=0,1,2,\ldots$

corresponding to the kernel are called the "offspring curves" or "offspring" in short. They are represented by $\wp_{lk}(\tau_1,\tau_2)$ where l and k identify the l^{th} and k^{th} generation offspring of the kernel τ_{10} and τ_{20}, respectively, according to (6). Let's denote the complete set of kernel and offspring by

$$\wp(\tau_1,\tau_2)=\wp_0(\tau_1,\tau_2)\cup\sum_{l=0}^{n}\sum_{\substack{k=0\\ l+k>0}}^{n}\wp_{lk}(\tau_1,\tau_2). \quad (7)$$

The kernel and the offspring constitute the complete (and exhaustive) distribution of (τ_1,τ_2) points for which the characteristic equation $CE(s,\tau_1,\tau_2)$ has root sets with at least one imaginary pair. Outside the set of curves $\wp(\tau_1,\tau_2)$ there cannot be a point, which results in an imaginary characteristic root of (1). Thus in mathematical formalism, the complete imaginary root set of (1)

$$\begin{aligned}&\Omega[s\,|\,C(s,\tau_1,\tau_2)=0,\ (\tau_1,\tau_2)\in\mathbb{R}^{2+}]\cap C^0\\ &\equiv\Omega[s\,|\,C(s,\tau_1,\tau_2)=0,\ (\tau_1,\tau_2)\in\wp(\tau_1,\tau_2)]\cap C^0\end{aligned} \quad (8)$$

is generated only by a small number of contours in $(\tau_1,\tau_2)\in\mathbb{R}^2$. These are the only locations in the (τ_1,τ_2) space where the system (1) could transit from stable to unstable posture (or vice versa). These contours $\wp(\tau_1,\tau_2)$ must be determined exhaustively. Since $\wp(\tau_1,\tau_2)$ is completely generated from the kernel via (6), it is sufficient to determine the kernel itself exhaustively. The determination of the complete set of kernel and offspring is, mathematically speaking, a very challenging problem. In order to achieve this we deploy a transformation called the Rekasius substitution [11].

$$e^{-\tau_i s}=\frac{1-T_i s}{1+T_i s}\qquad T_i\in\mathbb{R},\quad i=1,2. \quad (9)$$

It is important to note that, this substitution is an exact expression of $e^{-\tau_i s}$ for purely imaginary roots $s=\pm j\omega$. Moreover, transformation (9) is different from the first-order Pade' approximation of $e^{-\tau_i s}\approx(1-0.5\tau_i s)/(1+0.5\tau_i s)$. Transformation (9) can also be written as:

$$e^{-\tau_i v^\alpha}=\frac{1-T_i v^\alpha}{1+T_i v^\alpha}\qquad T_i\in\mathbb{R},\quad i=1,2. \quad (10)$$

By examining the amplitude and phase of (9), the relationship between T and τ can be obtained as follows:

$$\tau_i=\frac{2}{\omega_c}\left[\tan^{-1}(\omega_c T_i)+r\pi\right]\qquad r=0,1,\ldots. \quad (11)$$

This equation describes an asymmetric mapping in which one T_i is mapped into countably infinite τ_i which are distributed with a periodicity of $2\pi/\omega_c$.

Since (10) is exact when $v=\sqrt[\alpha]{\omega}e^{j\pi/2\alpha}$, it is convenient to use it for solving $s=v^\alpha$ roots of (2). By inserting (10) into (2), we have:

$$\begin{aligned}&CE(v,\tau_1,\tau_2)\\ &=\sum_{\ell=0}^{n}p_\ell(v)\left(\frac{1-T_1v^\alpha}{1+T_1v^\alpha}\right)^\ell+\sum_{i=1}^{m}q_i(v)\left(\frac{1-T_2v^\alpha}{1+T_2v^\alpha}\right)^i.\end{aligned} \quad (12)$$

By multiplying equation (12) by $(1+T_1v^\alpha)^n(1+T_1v^\alpha)^m$ the polynomial form of the characteristic equation is reached:

$$\begin{aligned}h(v,T_1,T_2)&=(1+T_1v^\alpha)^n(1+T_2v^\alpha)^m CE(v,\tau_1,\tau_2)\\ &=\sum_{\ell=0}^{n}p_\ell(v)(1-T_1v^\alpha)^\ell(1+T_1v^\alpha)^{n-\ell}(1+T_2v^\alpha)^m\\ &\quad+\sum_{i=1}^{m}q_i(v)(1-T_2v^\alpha)^i(1+T_1v^\alpha)^n(1+T_2v^\alpha)^{m-i}.\end{aligned} \quad (13)$$

This expression is a polynomial in v of which the coefficients are parametric functions of T_1 and T_2. As is observed, characteristic equation (2), which had transcendental terms, has been converted into algebraic equation (13). To find the crossing frequencies set in equation (1), $v=\sqrt[\alpha]{\omega}e^{j\pi/2\alpha}$ should be inserted into relation (13) and then the real and imaginary parts of the resulting equation should be separated as follows:

$$h(v,T_1,T_2)\big|_{v=\sqrt[\alpha]{\omega}e^{j\pi/2\alpha}}=h_{\Re}(\omega,T_1,T_2)+jh_{\Im}(\omega,T_1,T_2). \quad (14)$$

In the above relations, $h_{\Re}=\Re(h)$, $h_{\Im}=\Im(h)$. For ω to be a zero of (14), $h_{\Re}$ and $h_{\Im}$ should be concurrently zero for some (T_1,T_2). Let us investigate those (T_1,T_2) solutions from

$$h_{\Re}=\sum_{i=0}^{m}a_i(\omega,T_1)T_2^i=0 \quad (15)$$

and

$$h_{\Im}=\sum_{i=0}^{m}b_i(\omega,T_1)T_2^i=0. \quad (16)$$

Not that all a_i and b_i are real polynomials in T_1. $h_{\Re}$ and $h_{\Im}$, which have positive degrees in terms of T_2, are assumed to have no common factors. Such common factors, if they exist, can be separately studied.

Definition 3: The *resultant* R_{T_2} with respect to ω and T_1 is the resultant of $h_{\Re}$ and $h_{\Im}$ by eliminating T_2 [12]. The resultant of R_{T_2} and $\partial R_{T_2}/\partial T_1$ with respect to ω is called the *discriminant* of R_{T_2} by eliminating T_1 [13,14].

Theorem 1: Minimum and maximum positive real roots of the discriminant of resultant of $h_{\Re}$ and $h_{\Im}$ with respect to ω, that correspond to $(T_1,T_2)\in\mathbb{R}^2$ solutions in (14) yield the exact lower and upper bounds of the crossing frequency set.

Proof: For the delay-dependent case, finite lower bound $\underline{\omega}$ and upper bound $\overline{\omega}$ of Ω are known to exist [1]. To find the global maximum $\overline{\omega}$ and the global minimum $\underline{\omega}$, we start studying the extrema of ω via $\partial\omega/\partial\tau_1 = 0$, which is identical to studying

$$\frac{\partial\omega}{\partial\tau_1} = \frac{\partial\omega}{\partial T_1}\frac{\partial T}{\partial\tau_1} = 0\,, \tag{17}$$

where $\partial T_1/\partial\tau_1 = 0.5(1+\omega^2 T_1^2) \neq 0$ as per (11). Since $\partial T_1/\partial\tau_1 \neq 0$, we can study $\partial\omega/\partial\tau_1 = 0$ alternatively on $\partial\omega/\partial T_1 = 0$. At this point, the differentiability of ω with respect to T_1 is essential as established above, and holds for the regular points of $R_{T_2}(h_{\Re}, h_{\Im})$. Under this condition, we can write

$$\frac{\partial R_{T_2}\left(h_{\Re},h_{\Im}\right)}{\partial T_1} + \frac{\partial\omega}{\partial T_1}\frac{\partial R_{T_2}\left(h_{\Re},h_{\Im}\right)}{\partial\omega} = 0\,, \tag{18}$$

which leads to $\partial R_{T_2}(h_{\Re}, h_{\Im})/\partial T_1 = 0$, assuming that $\partial R_{T_2}(h_{\Re}, h_{\Im})/\partial\omega = 0$, Thus, one has two equations, R_{T_2} and $\partial R_{T_2}/\partial T_1$, and they should be simultaneously zero. This requires to study the zeros of the resultant of these two equations, particularly by eliminating T_1. The resultant of R_{T_2} and $\partial R_{T_2}/\partial T_1$ becomes only a function of ω.

$$Z(\omega) = R_{T_1}\left(R_{T_2}, \partial R_{T_2}/\partial T_1\right), \tag{19}$$

which is the discriminant by Definition 3. The minimum and maximum positive real zeros of $Z(\omega)$ that correspond to $(T_1, T_2) \in \mathbb{R}^2$ solutions in (14) are the exact lower and upper bounds of the crossing frequency set, respectively.

In the sequel, The methodology ACFS is presented step by step. Notice that ACFS methodology only requires frequency sweeping from the precise lower bound $\underline{\omega}$ to the precise upper bound $\bar{\omega}$ that ACFS identifies via Theorem 1. For each $\omega \in [\underline{\omega}, \bar{\omega}]$ with an appropriately chosen step size, perform the following steps:

1) Solve the polynomial equation $R_{T_2}(h_{\Re}, h_{\Im})$ for $T_1 \in \mathbb{R}$ values.
2) For each $T_1 \in \mathbb{R}$ found from above, if $T_2 \in \mathbb{R}$ values exist satisfying $h_{\Re} = 0$ and $h_{\Im} = 0$ then proceed to the next step, otherwise increase ω by an amount of the step size, and restart from the step above.
3) Via (6) and (11), calculate the delay values (τ_1, τ_2) corresponding to $(T_1, T_2) \in \mathbb{R}^2$ pairs, and restart from step 1 increasing ω by an amount of the step size.

3.2. Direction of crossing

The root sensitivities associated with each purely imaginary characteristic root crossing $j\omega$ with respect to one of the time delay, τ_i, is defined as

$$S_{\tau_i}^{s}\Big|_{s=j\omega_c} = \left.\frac{ds}{d\tau_i}\right|_{s=j\omega_c} = -\left.\frac{\partial C/\partial\tau_i}{\partial C/\partial s}\right|_{s=j\omega_c}. \tag{20}$$

And the corresponding root tendency with respect to one of the delays is given as:

$$\begin{aligned} Root\ Tendency = RT\big|_{s=j\omega_c}^{\tau_i} &= sgn\left(\Re\left(S_{\tau_i}^{s}\Big|_{s=j\omega_c}\right)\right) \\ &= sgn\left(\Re\left(\frac{ds}{dv}\times\frac{dv}{d\tau}\right)\right)_{\substack{s=j\omega_c \\ v=v_c=\sqrt[\alpha]{\omega}\,e^{\pm j\frac{\pi}{2\alpha}}}} \\ &= sgn\left(\tan\left(\frac{\pi}{2\alpha}\right)\Re\left(\frac{dv}{d\tau}\right) - \Im\left(\frac{dv}{d\tau}\right)\right)_{\substack{v=v_c \\ \tau=\tau_i}}. \end{aligned} \tag{21}$$

In case the result is 0, a higher order analysis is needed, since this might be the case where the root just touches the imaginary axis and returns to its original half-plane.

Theorem 2: Take an imaginary root, $s = j\omega$ (or $v = \sqrt[\alpha]{\omega}\,e^{j\pi/2\alpha}$) caused by any one of the infinitely many grid points in (τ_1, τ_2) defined by $(\tau_{10} + 2\pi l/\omega_c,\ +2\pi k/\omega_c)$, $l = 1, 2, \ldots$, $k = 1, 2, \ldots$. The root tendency $RT\big|_{s=j\omega_c}^{\tau_1}$ (or $RT\big|_{s=j\omega_c}^{\tau_2}$) remains invariant so long as the grid points on different *offspring*. are selected keeping τ_{2k} (or τ_{1l}) fixed.

Proof: Characteristic equation (1) can be written in terms of the auxiliary parameter v as (2), the simple roots of (2) are continuously differentiable with respect to $\{\tau\} \in \mathbb{R}^{2+}$ [15]. Then one can find $dv/d\tau$ for simple roots of (2) as follows:

$$\left.\frac{dv}{d\tau_i}\right|_{s=j\omega_c} = -\left.\frac{\partial C/\partial\tau_i}{\partial C/\partial v}\right|_{s=j\omega_c}. \tag{22}$$

Since τ_1 and τ_2 are interchangeable we prove the theorem for τ_1 and claim that it holds for τ_2 also. First, we keep τ_2 fixed and look at the root tendency given by (21) with respect to τ_1 at grid points corresponding to ω_c, i.e., $(\tau_{10} + 2\pi l/\omega_c)$, $l = 0, 1, 2, \ldots$. It is given as

$$\begin{aligned} RT\big|_{s=j\omega_c}^{\tau_1} &= sgn\left(\Re\left[\left(\tan\left(\frac{\pi}{2\alpha}\right)+j\right)\frac{dv}{d\tau_1}\right]\right)\Bigg|_{\substack{v=v_c \\ \tau=\tau_1}} \\ &= sgn\left(\Re\left(-\frac{\partial C_v/\partial v}{\left(\tan\left(\frac{\pi}{2\alpha}\right)+j\right)\partial C_v/\partial\tau_1}\right)^{-1}\right) \\ &= sgn\left(\Re\left(\frac{\displaystyle\sum_{\ell=0}^{n}\frac{p_\ell(v)}{dv}e^{-\ell\tau_1 v^\alpha} + \sum_{\ell=0}^{n}\left(\frac{q_i(v)}{dv} - i\tau_2 v^{\alpha-1}q_i(v)\right)e^{-i\tau_2 v^\alpha}}{\left(\tan\left(\frac{\pi}{2\alpha}\right)+j\right)\displaystyle\sum_{\ell=0}^{n}p_\ell(v)\,\ell v^\alpha e^{-\ell\tau_1 v^\alpha}} - \frac{\alpha\tau_1}{\left(\tan\left(\frac{\pi}{2\alpha}\right)+j\right)v}\right)^{-1}\right) \end{aligned}$$

$$= sgn\left(\Re\left(\frac{\sum_{\ell=0}^{n}\frac{p_\ell(v)}{dv}e^{-\ell\tau_1 v^\alpha}}{\left(\tan\left(\frac{\pi}{2\alpha}\right)+j\right)\sum_{\ell=0}^{n}p_\ell(v)\ell v^\alpha e^{-\ell\tau_1 v^\alpha}} + \frac{\sum_{i=1}^{m}\left(\frac{q_i(v)}{dv}-i\tau_2 v^{\alpha-1}q_i(v)\right)e^{-i\tau_2 v^\alpha}}{\left(\tan\left(\frac{\pi}{2\alpha}\right)+j\right)\sum_{\ell=0}^{n}p_\ell(v)\ell v^\alpha e^{-\ell\tau_1 v^\alpha}}\right)\Bigg|_{\substack{v=v_c\\ \tau=\tau_1}}\right) \quad (23)$$

Since $e^{-\ell\tau_1 v^\alpha}\big|_{v=v_c}$, $\tau_1=\tau_{10}+2\pi l/\omega_c$, $l=1,2,\ldots$ expressions remain the same for all the l values, we can state that $e^{-\ell\tau_1 v^\alpha}\big|_{v=v_c}$ is dependent only on τ_{10} and independent of l, or the actual value τ_1 itself. Therefore, the root tendency is invariant for all τ_{l1}, $l = 1,2,\ldots$. In other words, the imaginary root always crosses either to C^+ (for $RT = +1$) or to C^- (for $RT = -1$), when one of the delays is kept fixed, independent of the actual values of the second delay.

This theorem helps identifying certain sections of $\wp_0$ and the 'offspring curves' to be marked as stabilizing transitions along the τ_1 (or τ_2) axis or vice versa.

Once we detect completely $\wp_0(\tau_1,\tau_2)$, $\wp_{lk}(\tau_1,\tau_2)$ 'offspring curves' and the invariant root sensitivities, we can determine all possible stability regions in the parametric space of time delays $\{\tau\}$ using the well-known D-Subdivision methodology [16]. This implies the exhaustiveness of our methodology because it covers the complete set of stability regions in the entire semi-infinite time delay space entirely.

4. ILLUSTRATIVE EXAMPLES

We present an example case, which display all the features discussed in the text.

Example: Consider the following combined integrating system [17]

$$G(s)=\frac{K}{s}\left(1-e^{-\tau_1 s}\right)e^{-\tau_2 s}. \quad (24)$$

This is a combined integrating process companying with time delays, the whole character of which is stable instead of being unstable or integrating. Such kind of process exists extensively in steel, petrochemical, grain processing, tobacco, and mineral mining industry but few people pay much attention to it. Now most of combined integrating systems are taken as first order plus time delay process and controlled by routine control strategy such as PI controller. Assume that a fractional order PI controller (FOPI) is used and the controller transfer function $C(s)$ is

$$C(s)=k_p+\frac{k_i}{s^\mu} \qquad (0<\mu<2). \quad (25)$$

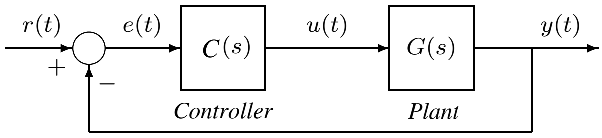

Fig. 2. A unity feedback control system.

The control structure of the system is shown in Fig. 2. It is easy to verify that the transfer function of the closed-loop system (for $K=1,\ k_p=0.1,\ k_i=0.2,\ \mu=0.9$) can be defined as follows:

$$H(s)=\frac{(0.1s^{0.9}+0.2)(1-e^{-\tau_1 s})e^{-\tau_2 s}}{s^{1.9}+(0.1s^{0.9}+0.2)(1-e^{-\tau_1 s})e^{-\tau_2 s}}. \quad (26)$$

The characteristic equation of the closed loop transfer function is

$$CE(s,\tau_1,\tau_2)=s^{1.9}+(0.1s^{0.9}+0.2)(1-e^{-\tau_1 s})e^{-\tau_2 s}. \quad (27)$$

The shaded zone represents the stable region.

Our objective in this example is to find all the stability map for $CE(s,\tau_1,\tau_2)$ based on the method described in this article. Using auxiliary variable $v=s^{1/10}$ into (27) the characteristic equation is obtained as:

$$CE(v,\tau_1,\tau_2)=v^{19}+(0.1v^9+0.2)(1-e^{-\tau_1 v^{10}})e^{-\tau_2 v^{10}}. \quad (28)$$

By applying the criterion expressed in the previous section, we can eliminate exponential term from (28) as follows:

$$\begin{aligned}h(v,T_1,T_2)&=v^{19}(1+T_1v^{10})(1+T_2v^{10})\\&\quad+2T_1v^{10}(0.1v^9+0.2)(1-T_2v^{10}).\end{aligned} \quad (29)$$

By inserting expression $v=\sqrt[10]{\omega}\,e^{j\pi/20}=\sqrt[10]{\omega}(\cos(\pi/20)+j\sin(\pi/20))$ in the above equation and equating the real and imaginary parts of the obtained relation to zero, we get:

$$\begin{aligned}h_\Re=&\left[\left(\cos\left(\frac{\pi}{20}\right)\omega^{3.9}+0.2\sin\left(\frac{\pi}{20}\right)\omega^{2.9}+0.4\omega^2\right)T_1\right.\\&\left.-\sin\left(\frac{\pi}{20}\right)\omega^{2.9}\right]T_2-\sin\left(\frac{\pi}{20}\right)\omega^{2.9}T_1\\&-0.2\cos\left(\frac{\pi}{20}\right)\omega^{1.9}T_1-\cos\left(\frac{\pi}{20}\right)\omega^{1.9}=0\end{aligned} \quad (30)$$

and

$$\begin{aligned}h_\Re=&\left[\left(-\sin\left(\frac{\pi}{20}\right)\omega^{3.9}+0.2\cos\left(\frac{\pi}{20}\right)\omega^{2.9}\right)T_1\right.\\&\left.-\cos\left(\frac{\pi}{20}\right)\omega^{2.9}\right]T_2+\left(-\cos\left(\frac{\pi}{20}\right)\omega^{2.9}\right.\\&\left.+0.2\sin\left(\frac{\pi}{20}\right)\omega^{1.9}+0.4\omega\right)T_1+\sin\left(\frac{\pi}{20}\right)\omega^{1.9}=0.\end{aligned} \quad (31)$$

Using homomorphism resultant algorithm [12] eliminate T_2 from $h_\Re$ and $h_\Im$,

$$R_{T_2}\left(h_{\Re},h_{\Im}\right)=\omega^3\left[\omega^{3.8}-0.04\omega^{1.8}-0.16s\,\mathrm{in}\left(\frac{\pi}{20}\right)\omega^{0.9}\right.$$
$$\left.-0.16\right)T_1^2+\omega^{1.8}\Big]=0. \qquad (32)$$

Discriminant of the resultant of $h_{\Re}$ and $h_{\Im}$ in Theorem 1 is the resultant of R_{T_2} and $\partial R_{T_2}/\partial T_1$ with eliminating T_1,

$$Z(\omega)=R_{T_1}\left(R_{T_2},\frac{\partial R_{T_2}}{\partial T_1}\right)$$
$$=\omega\left(\omega^{3.8}-0.04\omega^{1.8}-0.16s\,\mathrm{in}\left(\frac{\pi}{20}\right)\omega^{0.9}-0.16\right)$$
$$=0. \qquad (33)$$

The real solutions of (33) for ω is

$$\omega=0 \quad \text{and} \quad \omega=0.6507872. \qquad (34)$$

The minimum and maximum positive real roots of $Z(\omega)=R_{T_1}(R_{T_2},\partial R_{T_2}/\partial T_1)$ are computed as 0 and 0.6507872. From Theorem 1, it is concluded that $[\underline{\omega},\overline{\omega}]$ is [0, 0.6507872]. One can now use this ω range and the ACFS method in previous section, in order to extract the stability maps on $\tau_1-\tau_2$; domain by sweeping the frequency from; 0; to 0.6507872. The potential stability switching curves of the system is extracted in Fig. 3, where the kernel curve is shown in red ($\wp_0$), and the offspring curves are given in blue ($\wp_{lk}$) when viewed in color. The shaded zone in Fig. 3 show the complete map of stability for the given system. The number of unstable roots in each region, *NU*, is also shown sparingly. Obviously, in the stable regions, *NU*= 0.

To get a better understanding of the properties of this system, step responses of *H*(*s*) for A, B, C and D points in Fig. 3 are also made in MATLAB to validate the stable and unstable regions. For this simulation, indicated time delay values on Fig. 3 are used and the result is depicted in Fig. 4. It is clear that the stable and unstable regions are in agreement with Fig. 4.

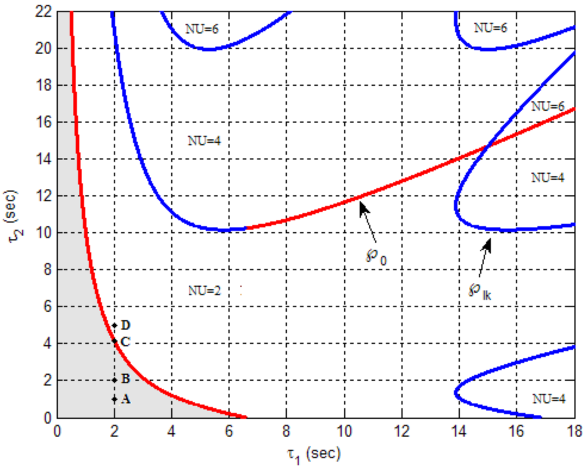

Fig. 3. Stability map in the domain of the time delays. The shaded zone represents the stable region.

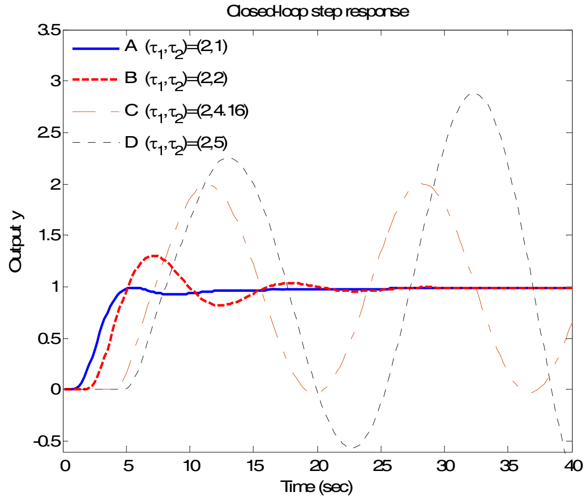

Fig. 4. Step responses of *H*(*s*) for A, B, C and D points.

5. CONCLUSION

In this paper, a novel methodology, based on Advanced Clustering with Frequency Sweeping (ACFS), is proposed for studying the asymptotic stability of multiple time-delay fractional order systems in the parameter space of delays. It is evident from the literature that the stability assessment of this class of dynamics remains unsolved. The method commenced by deploying the auxiliary variable $v=\sqrt[\alpha]{s}$ and Rekasius substitution for the transcendental terms in the characteristic equation, reducing it into a finite dimensional algebraic equation. Moreover, a single-variable function is derived to precisely calculate the lower and upper bounds of the crossing frequency set. These bounds are critical to the ACFS implementation. By means of ACFS, potential stability switching curves (PSSC) on any 2D delay domain are extracted exhaustively. The proposed methodology detects all the stability regions precisely, in the space of the time delays. This map, in fact, is the exact display of robustness against delay uncertainties. an example presented to highlight the proposed approach.

REFERENCES

[1] R. Sipahi and I. I. Delice, "Advanced clustering with frequency sweeping methodology for the stability analysis of multiple time-delay systems," *IEEE Trans. on Automatic Control*, vol. 56, no. 2, pp. 467-472, 2011.

[2] Y. S. Suh, "Stability of time delay systems using numerical computation of argument principles," *International Journal of Control, Automation, and Systems*, vol. 1, no. 1, pp. 127-133, 2003.

[3] K. Gu, S. I. Niculescu, and J. Chen, "On stability crossing curves for general systems with two delays," *J. Math. Anal. Appl.*, vol. 311, no. 1, pp. 231-253, 2005.

[4] S. Pakzad and M. A. Pakzad, "Stability condition for discrete systems with multiple state delays," *WSEAS Trans. on Systems and Control*, vol. 6, no. 11, pp. 417-426, 2011.

[5] N. Ozturk and A. Uraz, “An analysis stability test for a certain class of distributed parameter systems with delays,” *IEEE Trans. on Circuits and Systems*, vol. 34, no. 4, pp. 393-396, 1985.
[6] C. Bonnet and J. R. Partington, “Analysis of fractional delay systems of retarded and neutral type,” *Automatica*, vol. 38, no. 7, pp. 1133-1138, 2002.
[7] M. A. Pakzad, S. Pakzad, and M. A. Nekoui, “Stability analysis of time-delayed linear fractional-order systems,” *International Journal of Control, Automation, and Systems*, vol. 11, no. 3, pp. 519-525, 2013.
[8] S. Pakzad, M. A. Pakzad, and M. A. Nekoui, “Stability map of fractional delay systems in the parametric space of delays and coefficient,” *Proc. of American Control Conference*, Washington, DC, pp. 176-181, 2013.
[9] K. E. Walton and J. E. Marshal, “Direct method for TDS stability analysis,” *IEE Proceeding Part D*, vol. 134, pp. 101-107, 1987.
[10] M. A. Pakzad and S. Pakzad, “Stability map of fractional order time-delay systems,” *WSEAS Trans. on Systems*, vol. 10, no. 11, pp. 541-550, 2012.
[11] Z. V. Rekasius, “A stability test for systems with delays,” *Proc. Joint Autom. Control Conf., San Francisco, CA*, 1980.
[12] G. E. Collins, “The calculation of multivariate polynomial resultants,” *J. Assoc. Comput. Mach.*, vol. 18, no. 4, pp. 515-532, 1971.
[13] I. M. Gelfand, M. M. Kapranov, and A. V. Zelevinsky, *Discriminants, Resultants, and Multidimensional Determinants*, Birkhäuser, Boston, MA, 1994.
[14] D. Wang, *Elimination Methods*, Springer-Verlag, New York, 2001.
[15] W. F. Trench, *Introduction to Real Analysis*, Pearson Education, Upper Saddle River, 2003.
[16] V. B. Kolmanovskii and V. R. Nosov, *Stability of Functional Differential Equations*, Academic Press, London, 1989.
[17] D. Zheng, Z.-Y. Ren, and J.-A. Fang, “On stabilizing sets of PI controllers for multiple time delayed process,” *Journal of Donghua University* (English Edition), vol. 26, no. 1, pp. 36-39, 2009.

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