International Journal of Control, Automation, and Systems (2013) 11(3):636-642
DOI 10.1007/s12555-012-0471-9
ISSN:1598-6446 eISSN:2005-4092
http://www.springer.com/12555

Two-DOF Lifted LMI Conditions for Robust $\mathcal{D}$-Stability of Polynomial Matrix Polytopes

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Abstract: This technical note investigates the problem of checking robust $\mathcal{D}$-stability of polytopes of polynomial matrices. Lifted linear matrix inequality (LMI) conditions with two-DOF (two degree of freedom) positive integers (τ, κ) are derived to possess more flexible tradeoff between the conservatism and computational complexity. In the process of formulating the LMIs, the relevant region $\mathcal{D}$ is represented by a quadratic constraint in the complex plane. The matrix, composing the quadratic form with the vector of a variable, is called the region matrix. Then a variable substitution approach is put forward for the lifted LMI version by extending the dimensions of the region matrix and the Lyapunov matrix. The effectiveness and advantages of the proposed method have been illustrated by numerical examples.

Keywords: $\mathcal{D}$-stability, linear matrix inequality, polynomial matrices, polytopic uncertainty, relaxation.

1. INTRODUCTION

Polynomial matrices play a key role in modern system and control theory, since they are capable of representing more naturally the system dynamics in many cases [1,2]. Unfortunately, checking stability of uncertain polynomial matrices are mostly NP-hard [3]. Tractable LMI methods for these problems are often based on the concept of quadratic stability, for example [4-8]. As quadratic stability results have remarkable conservatism, it has become an active topic to improve the various methods and reduce the conservatism. The parameter-dependent LMI methods have been explored in recent years [9-11], which are less conservative for robust stability analysis. Moreover, [10] provides convergent LMIs based on homogeneous polynomially method utilizing the generalized Pólya's theorem. These convergent LMIs can asymptotically eliminate the conservatism. Other methods guaranteeing the convergence discussed in [7] such as the sum of squares (SOS) based LMIs can also make the conservatism vanished. It seems that the major problem about the conservatism has been thoroughly solved, however, practically, it is not true due to the huge computation burden. The number of the LMI conditions grows exponentially for the homogeneous polynomially method [10]. And for the SOS technic, the dimension of the LMIs also suffers from the exponential expansion [7]. Therefore, for solving the practical problem, it is significant to give a suitable compromise between the conservatism and computation burden. It is interesting to consider the so-called lifted LMI conditions investigated in [11-13,15] which are obtained by employing appropriate higher degree multipliers. Recently, via combining the "lifted LMI" method with Finsler's projection framework, [14] derives a series of sufficient LMI conditions for the robust $\mathcal{D}$-stability analysis of polynomial matrices polytopes. The conditions are tied to one-DOF positive integer variable κ. Increasing the κ will simultaneously lead to the reduction of conservatism and the rise of computational complexity.

Despite the plentiful improvements, there is room for further investigation. First, although some results in [14] cover wide varieties of LMI conditions, those can be further extended. And as κ rises, the number of decision variables increases fast and aggravates the computation burden rapidly. Thus it is necessary to find LMI conditions that gradually enlarge the amount of decision variables. Second, though complicated, the LMI conditions in [14] have strong similarities in form with the original ones ($\kappa=1$ case). However, complicated theoretic deductions and vast calculations are required to tackle the lifted LMI versions, which bring difficulty for further improvement. Hence it is an open problem to explore more essential relationships and look for a simple approach to reach qualified LMI conditions.

Motivated by the discussions above, this note derives new LMI conditions for the robust $\mathcal{D}$-stability of polynomial matrices polytopes. The main contributions of this work are as follows. Firstly, our LMI conditions are more general, which can cover those in [14]. Moreover, two-DOF positive integer variables (τ, κ) are contained in the LMI conditions, which have more

Manuscript received October 31, 2012; revised February 4, 2013; accepted February 27, 2013. Recommended by Editorial Board member Young Soo Suh under the direction of Editor Ju Hyun Park.

This work was supported by the National Natural Science Foundation of China (Grant Nos. 61004017 and 60974103) and the National 863 Project of China (Grant No. 2011AA7034056).

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flexible tradeoff between the conservatism and computational complexity. Secondly, an ingenious variable substitution approach is put forward to expand equivalently the dimensions of region matrix and Lyapunov matrix at the same time. Then novel lifted LMI conditions are obtained in a comprehensible and intuitively pleasing way.

The rest of the note is organized as follows. Section 2 presents preliminaries, including fundamental knowledge of polynomial matrix polytope, description of special matrices, definition and explanation of high order LMI region in complex plane and four useful lemmas. Section 3 gives the main results. Section 4 provides examples and simulation results. Finally, Section 5 concludes the technical note.

Notation: A^T: transpose of A; A^*: transpose conjugate of A; A^{-*}: shorthand notion for $(A^{-1})^*$; $A>0 (A<0)$: positive definite (negative definite) matrix A; $A\otimes B$: Kronecker's product of matrices A and B; $\mathbf{He}\{A\}$: a shorthand notion for $A+A^*$; $\mathbf{Qu}\{P,A\}$: a shorthand notion for quadric form A^*PA; $A_\perp$: any matrices whose columns form bases of the right null-space of matrix A; $I_n, 0_n, 0_{n\times m}$: $n\times n$ identity matrix, $n\times n$ zero matrix, and $n\times m$ zero matrix, respectively.

2. PRELIMINARIES

2.1. Some fundamental knowledge of polynomial matrix polytope

Let polynomial matrix $\mathcal{A}(s,\alpha)$ belong to the polytope set $\left\{\sum_{i=1}^{N}\alpha_i\mathcal{A}_i(s) \mid \sum_{i=1}^{N}\alpha_i=1,\ \alpha_i\ge 0,\ i=1,2,\cdots N\right\}$, where $\mathcal{A}_i(s)=A_0+A_1 s+\cdots+A_r s^r$, $i=1,2,\cdots,N$, are N real square polynomial matrices of dimension n and degree $r\ge 1$. Without any ambiguity, the coefficient matrices A_k, $k=1,2\cdots r$ regarding to various polynomial matrices $\mathcal{A}_i(s)$, are not differed in label throughout the paper for the sake of simplicity. We assume that every leading coefficient matrix A_r is nonsingular. Then $\mathcal{A}(s,\alpha)$ always remains admissible since it cannot feature infinite eigenvalue under this nonsingular condition [2].

For any $\mathcal{A}(s)$, there always exists a pair of matrices (V,J) where $J\in\mathbb{C}^{rn\times rn}$, $V\in\mathbb{C}^{n\times rn}$. The pair contains all the information on the eigenvalues with associated multiplicities and eigenvectors from the respective Jordan chains. As the $\mathcal{A}(s)$ is real polynomial matrix, its eigenvalues occur in conjugate pairs. Then there exists matrix J_R which is the real Jordan canonical form having the same eigenvalues with matrix J. Correspondingly, there exists real pair matrices (V_R,J_R) where $J_R\in\mathbb{R}^{rn\times rn}$, $V_R\in\mathbb{R}^{n\times rn}$. They satisfy

$$A_0V_R+A_1V_RJ_R+\cdots+A_rV_RJ_R^r=0_{n\times rn}, \tag{1}$$

$$\operatorname{rank}(\mathbf{J}_{\mathbf{R}0})=rn, \tag{2}$$

where $\mathbf{J}_{\mathbf{R}k}\in\mathbb{R}^{(r+k)n\times rn}$ and

$$\mathbf{J}_{\mathbf{R}k}=\begin{bmatrix}V_R^T & (V_RJ_R)^T & \cdots & (V_RJ_R^{r+k-1})^T\end{bmatrix}^T. \tag{3}$$

For more details on the matrix polynomials, readers can refer to [16].

2.2. Description of some matrices for this note

Let x, y, z, k be positive integer numbers. Define

$$\Pi_k^{(x,y,z)}\triangleq\begin{bmatrix} I_x\otimes\begin{bmatrix}I_y & 0_{y\times1} & \cdots & 0_{y\times1}\end{bmatrix}\otimes I_z\\ I_x\otimes\begin{bmatrix}0_{y\times1} & I_y & \cdots & 0_{y\times1}\end{bmatrix}\otimes I_z\\ \vdots\\ I_x\otimes\begin{bmatrix}0_{y\times1} & 0_{y\times1} & \cdots & I_y\end{bmatrix}\otimes I_z\end{bmatrix}, \tag{4}$$
$$\in\mathbb{R}^{xyzk\times xz(k+y-1)}$$

$$\begin{cases} C_k^{(r,n)}\triangleq\begin{bmatrix} A_0 & A_1 & \cdots & A_r & 0_n & \cdots & 0_n\\ 0_n & A_0 & A_1 & \cdots & A_r & \ddots & \vdots\\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0_n\\ 0_n & \cdots & 0_n & A_0 & A_1 & \cdots & A_r\end{bmatrix}\\ \qquad\in\mathbb{R}^{kn\times(k+r)n},\ \text{for } k\ge1\\ C_0^{(r,n)}\triangleq 0_{rn},\ \text{for } k=0.\end{cases} \tag{5}$$

For scalar number s, define

$$\pi_k^{(x)}(s)\triangleq\begin{bmatrix}I_x & sI_x & \cdots & s^{k-1}I_x\end{bmatrix}^T,\quad \pi_1^{(x)}(s)\triangleq I_x \tag{6}$$

and for matrix A,

$$\pi_k^{(x)}(A)\triangleq\begin{bmatrix}I_x\otimes I_n & I_x\otimes A^T & \cdots & I_x\otimes(A^{k-1})^T\end{bmatrix}^T. \tag{7}$$

We exploit the following three relations

$$\pi_x^{(1)}(s)\otimes\pi_{y+1}^{(d)}(s)=\Pi_x^{(1,y+1,d)}\pi_{x+y}^{(d)}(s), \tag{8}$$

$$(I_{xd}\otimes\mathbf{J}_{\mathbf{R}k-1})\pi_x^{(d)}(J_R)=\Pi_x^{(d,r+k-1,n)}(I_d\otimes\mathbf{J}_{\mathbf{R}x+k-2}), \tag{9}$$

$$(I_2\otimes\Pi_k^{(x,y,z)})\Pi_2^{(1,k+y-1,xz)}=\Pi_2^{(1,k,xyz)}\Pi_{k+1}^{(x,y,z)}, \tag{10}$$

which are useful to obtain our results and can be proved after long but elementary algebraic calculations.

2.3. High order LMI region in complex plane

Definition 1 (q-order LMI region)**:** For integers $q\ge1$, $d\ge1$ and $(q+1)d$ dimensional Hermit matrix $H^{(q)}$, define q-order LMI region $\mathcal{D}$ by

$$D\left(H^{(q)}\right)\triangleq\{s\in\mathbb{C}\mid f_D(s)<0\}, \tag{11}$$

where $f_D\triangleq\mathbf{Qu}\{H^{(q)},\pi_{q+1}^{(d)}(s)\}$. $H^{(q)}$ is called region matrix. When $q=1$, the region becomes the well-known LMI region [5]. When $d=1$, the region becomes polynomial region discussed in [17].

In order to deal with the lifted LMIs by variable substitution, one important step in this note is equivalently lifting the problems into higher order LMI regions.

2.4. Some lemmas for this note

Lemma 1 [18]**:** For a vector $x\in\mathbb{C}^n$ and two matrices $Q=Q^*\in\mathbb{C}^{n\times n}$ and $R\in\mathbb{R}^{m\times n}$ such that $\operatorname{rank}(R)<n$, following statements are equivalent:

1) $x^*Qx<0, \forall x\in\{x\in\mathbb{C}^n \mid x\neq 0, Rx=0_{m\times 1}\}$;
2) $\mathbf{Qu}\{Q,R_\perp\}<0$;
3) $\exists M\in\mathbb{R}^{n\times m}$, such that $Q+\mathbf{He}\{MR\}<0$.

Lemma 2: Let $\tau\geq 1$ and choose an arbitrary $\tau\times\tau$ matrix $Q_\tau>0$. For $m\geq 1$ and $(m+1)d$ dimensional Hermit matrix $H^{(m)}$, let $(\tau+m)d$ dimensional Hermit matrix be

$$\tilde{H}_\tau^{(m)} \triangleq \mathbf{Qu}\left\{Q_\tau\otimes H^{(m)}, \Pi_\tau^{(1,m+1,d)}\right\}. \tag{12}$$

Then $D(H^{(m)})=D(\tilde{H}_\tau^{(m)})$, i.e., the original m-order region $D(H^{(m)})$ is equal to new $(m+\tau-1)$-order region $D(\tilde{H}_\tau^{(m)})$.

Proof: Using the properties $(A\otimes B)(C\otimes D)=(AC)\otimes(BD)$ and identity (8), it can be verified that

$$\begin{aligned} f_{D(\tilde{H}_\tau^{(m)})}(s) &= \mathbf{Qu}\{\tilde{H}_\tau^{(m)}, \pi_{\tau+m}^{(d)}(s)\} \\ &= \mathbf{Qu}\{Q_\tau\otimes H^{(m)}, \Pi_\tau^{(1,m+1,d)}\pi_{\tau+m}^{(d)}(s)\} \\ &= \mathbf{Qu}\{Q_\tau\otimes H^{(m)}, \pi_\tau^{(1)}(s)\otimes\pi_{m+1}^{(d)}(s)\} \\ &= \mathbf{Qu}\{Q_\tau, \pi_\tau^{(1)}(s)\}\otimes f_{D(H^{(m)})}(s). \end{aligned}$$

Moreover, $Q_\tau>0$ implies $\mathbf{Qu}\{Q_\tau,\pi_\tau^{(1)}(s)\}>0$. Notice that for square matrix A and B, the eigenvalues of $A\otimes B$ are $\lambda_{Ai}\lambda_{Bj}$, where λ_{Ai} and λ_{Bj} are eigenvalues of A and B respectively. Therefore, $f_{D(\tilde{H}_\tau^{(m)})}(s)<0 \Leftrightarrow f_{D(H^{(m)})}(s)<0$, which implies $D(H^{(m)})=D(\tilde{H}_\tau^{(m)})$.

Lemma 3: Let

$$B_1=\{X \mid X\in\mathbb{R}^{rn\times rn}, X>0\},$$
$$B_2=B_1\bigcap\left\{\mathbf{Qu}\{Y,\mathbf{J}_{\mathbf{R}k-1}\} \mid Y\in\mathbb{R}^{(r+k-1)n\times(r+k-1)n}\right\}.$$

Then $B_1=B_2$.

Proof: On one hand, it is obvious that $B_1\supseteq B_2$. On the other hand, $\forall X\in B_1$, let

$$Y=\begin{bmatrix}\mathbf{Qu}\{X,\mathbf{J}_{\mathbf{R}0}^{-1}\} & 0_{rn\times(k-1)n} \\ 0_{(k-1)n\times rn} & 0_{(k-1)n\times(k-1)n}\end{bmatrix}.$$

Then $X=\mathbf{Qu}\{Y,\mathbf{J}_{\mathbf{R}k-1}\}$, i.e., $X\in B_2$. Therefore $B_1\subseteq B_2$. Finally, we get $B_1=B_2$.

Remark 1: 1) Lemma 1 enables LMI with matrix $R_\perp$ to be equivalently reformulated as an projection form with matrix R. Lemma 4 implies $\mathbf{J}_{\mathbf{R}k}=(C_k^{(r,n)})_\perp$. It is an important step to combine with this two lemmas in the formulation of LMIs, which is also used in [14].

2) Lemma 2 provides a method to extend equivalently the region matrix to higher dimensional matrix. Lemma 3 implies that "$\exists P\in B_1$ such that $\cdots$" is equivalent to "$\exists P\in\mathbb{R}^{(r+k-1)n\times(r+k-1)n}$, $\mathbf{J}_{\mathbf{R}k-1}P\mathbf{J}_{\mathbf{R}k-1}>0$ such that$\cdots$". These two lemmas enable us to lift the LMIs by variable substitution, which is a key step to obtain the main results.

3. MAIN RESULTS

We begin with the LMI conditions for $\mathcal{D}$-stability of deterministic scalar matrix A. The result and the proof are similar with those in [5]. The only difference here is rest with the more general q-order LMI region.

Theorem 1: Given $q\geq 1$ with q-order region $\mathcal{D}$ described in (11), the $n\times n$ real matrix A is $\mathcal{D}$-stable if and only if $\exists P\in\mathbb{R}^{n\times n}$ such that

$$P>0, \tag{13}$$
$$M_D(A,P)\triangleq\mathbf{Qu}\{H^{(q)}\otimes P, \pi_{q+1}^{(d)}(A)\}<0, \tag{14}$$

where $H^{(q)}$ is the matrix of q-order region $\mathcal{D}$ and $\pi_{q+1}^{(d)}(A)$ is defined in (7).

Proof: (Sufficiency): Suppose $\exists P\in\mathbb{R}^{n\times n}$ such that (13) and (14) hold. Let λ be any eigenvalue of A and $v\neq 0$ be the corresponding eigenvector. By calculation, we have

$$\begin{aligned} \mathbf{Qu}\{M_D(A,P), I_d\otimes v\} &= \mathbf{Qu}\{H^{(q)}\otimes P, \pi_{q+1}^{(d)}(\lambda)\otimes v\} \\ &= \mathbf{Qu}\{H^{(q)}, \pi_{q+1}^{(d)}(\lambda)\}\otimes(v^*Pv) \\ &= f_D(\lambda)\otimes(v^*Pv). \end{aligned}$$

Since $M_D(A,P)<0$ and $P>0$, we have $f_D(\lambda)<0$. That implies A is $\mathcal{D}$-stable.

Necessity: Suppose A is $\mathcal{D}$-stable. We notice the following three facts: 1) For any $n\times n$ nonsingular matrix T, there holds the identity

$$\begin{aligned} &M_D(T^{-1}AT,P) \\ &= \mathbf{Qu}\{H^{(q)}\otimes P, \pi_{q+1}^{(d)}(T^{-1}AT)\} \\ &= \mathbf{Qu}\{H^{(q)}\otimes(T^{-*}PT^{-1}), \pi_{q+1}^{(d)}(A)(I_{(q+1)d}\otimes T)\} \\ &= \mathbf{Qu}\{M_D(A,T^{-*}PT^{-1}), I_{(q+1)d}\otimes T\}. \end{aligned} \tag{15}$$

2) There always exist nonsingular matrices $\{T_k\}_{k=1,2,\ldots}$ such that $\lim_{k\to+\infty} T_k^{-1}AT_k=\Lambda$, where $\Lambda=diag\{\lambda_1,\lambda_2\ldots,\lambda_n\}$ which has the same eigenvalues with A.

3) $H^{(q)}$ can be regarded as a block matrix $H^{(q)}=\{H_{ij}\}_{0\leq i,j\leq q}$, where H_{ij} are $d\times d$ subblocks. Then we have

$$\begin{aligned} M_D(\Lambda,I_n) &= \mathbf{Qu}\{H^{(q)}\otimes I_n, \pi_{q+1}^{(d)}(\Lambda)\} \\ &= \sum_{0\leq i,j\leq q} H_{ij}\otimes((\Lambda^i)^*\Lambda^j) \\ &= \mathbf{Qu}\left\{\sum_{0\leq i,j\leq q}((\Lambda^i)^*\Lambda^j)\otimes H_{ij}), U\right\} \\ &= \mathbf{Qu}\{diag\{f_D(\lambda_1), f_D(\lambda_2),\cdots,f_D(\lambda_n)\}, U\}, \end{aligned} \tag{16}$$

where U is some permutation matrix. For the $\mathcal{D}$-stable matrix A, we have $diag\{f_D(\lambda_1), f_D(\lambda_2),\cdots f_D(\lambda_n)\}<0$. Thus $M_D(\Lambda,I_n)<0$ according to (16). Besides, the continuity of M_D implies that $\lim_{k\to+\infty} M_D(T_k^{-1}AT_k,I_n)=M_D(\Lambda,I_n)$. Consequently, there exists a sufficiently large number $\tilde{k}$ such that $M_D(T_{\tilde{k}}^{-1}AT_{\tilde{k}}, I_n) < 0$. According to identity (15), we have $M_D(A, \tilde{P}) < 0$ where $\tilde{P} = T_{\tilde{k}}^{-*}T_{\tilde{k}}^{-1}$. As the real part of a complex positive matrix is a real positive matrix, we can choose P the real part of $\tilde{P}$. Then $P \in \mathbb{R}^{n\times n}$ satisfies both the LMIs (13) and (14).

We call the matrix P in Theorem 1 the Lyapunov matrix. Theorem 1 provides the LMI criterion for the $\mathcal{D}$-stability of scalar matrix. Noticing that the polynomial matrix $\mathcal{A}(s)$ has the same eigenvalues with numerical matrix J_R, the theorem can be utilized to check the $\mathcal{D}$-stability of $\mathcal{A}(s)$ as well. For further exploration of the lifted LMIs, the following two extensions will be made at the same time.

1) Choose $\tau \ge 1$ with a $\tau\times\tau$ positive matrix Q_τ and extend the original region matrix $H^{(m)}$ to $\tilde{H}_\tau^{(m)}$ by (12).

2) Choose $\kappa \ge 1$ and replace the Lyapunov matrix P in Theorem 1 by $\mathbf{Qu}\{P, \mathbf{J}_{\mathbf{R}\kappa-1}\}$.

After the corresponding adjustment, we have the following theorem:

Theorem 2: Let $m \ge 1$. Given m-order region $\mathcal{D}$ with region matrix $H^{(m)}$, the following statements are equivalent:

1) the polynomial matrix $\mathcal{A}(s)$ is $\mathcal{D}$-stable.

2) there exists $P_\kappa \in \mathbb{R}^{(r+\kappa-1)n\times(r+\kappa-1)n}$ such that

$$\mathbf{Qu}\{P_\kappa, \mathbf{J}_{\mathbf{R}\kappa-1}\} > 0, \tag{17}$$

$$\mathbf{Qu}\{\tilde{H}_\tau^{(m)} \otimes P_\kappa, \Pi_{m+\tau}^{(d,r+\kappa-1,n)}(I_d \otimes \mathbf{J}_{\mathbf{R}m+\tau+\kappa-2})\} < 0. \tag{18}$$

3) there exist $Z_{\tau,\kappa} \in \mathbb{R}^{d(m+r+\tau+\kappa-2)n\times d(m+\tau+\kappa-2)n}$, $Y_\kappa \in \mathbb{R}^{(r+\kappa-1)n\times(\kappa-1)n}$, $P_\kappa \in \mathbb{R}^{(r+\kappa-1)n\times(r+\kappa-1)n}$ such that

$$\Omega_{1,\kappa} \triangleq P_\kappa + \mathbf{He}\{Y_\kappa C_{\kappa-1}^{(r,n)}\} > 0, \tag{19}$$

$$\begin{aligned}\Omega_{2,\tau,\kappa} \triangleq \mathbf{Qu}\{\tilde{H}_\tau^{(m)} \otimes P_\kappa), \Pi_{m+\tau}^{(d,r+\kappa-1,n)})\} \\ + \mathbf{He}\{Z_{\tau,\kappa}(I_d \otimes C_{m+\tau+\kappa-2}^{(r,n)})\} < 0,\end{aligned} \tag{20}$$

where τ,κ and $\tilde{H}_\tau^{(m)}$ are the same with those above respectively.

Proof: As checking the $\mathcal{D}$-stability of $\mathcal{A}(s)$ and J_R are equivalent, the criterion can be the existence of $P \in \mathbb{R}^{rn\times rn}$ such that $P > 0$ and $M_D(J_R, P) < 0$, according to Theorem 1. Further, by Lemma 2 and Lemma 3, $D(H^{(m)}) = D(\tilde{H}_\tau^{(m)})$ and an alternative criterion is the existence of P_κ such that (17) and $M_{D(\tilde{H}_\tau^{(m)})}(J_R, \mathbf{Qu}\{P_\kappa, \mathbf{J}_{\mathbf{R}\kappa-1}\}) < 0$ hold. And the second LMI can be reformulated as (18) using the identity (9). For 2) and 3), the equivalence can be directly recognized according to Lemma 1 and Lemma 4.

Theorem 2 gives a lifted version for checking the $\mathcal{D}$-stability of deterministic polynomial matrix. Despite being far more complicated than the equivalent original one, the lifted LMIs can less conservatively be utilized to check the robust $\mathcal{D}$-stability of uncertain polynomial matrix $\mathcal{A}(s,\alpha)$ by the following theorem.

Theorem 3: Let $m \ge 1$, $\kappa \ge 1$ and $\tau \ge 1$. $\mathcal{D}$ is an m-order region the same with it in Theorem 2. Then $\mathcal{A}(s,\alpha)$ is robust $\mathcal{D}$-stable if there exist $P_{\kappa,i} \in \mathbb{R}^{(r+\kappa-1)n\times(r+\kappa-1)n}$ and common $Y_\kappa \in \mathbb{R}^{(r+\kappa-1)n\times(\kappa-1)n}$, $Z_{\tau,\kappa} \in \mathbb{R}^{d(m+r+\tau+\kappa-2)n\times d(m+\tau+\kappa-2)n}$ such that (19) and (20) hold for all $\mathcal{A}_i(s)$, $i = 1,2\cdots N$.

Proof: For any admissible $\mathcal{A}(s,\alpha)$ with $\alpha = (\alpha_1, \alpha_2,\cdots,\alpha_N)$, there exist $Z_{\tau,\kappa}$, Y_κ and $P_\kappa = \sum_{i=1}^N \alpha_i P_{\kappa,i}$ satisfying the LMI conditions (19) and (20) in Theorem 2. Therefore the uncertain $\mathcal{A}(s,\alpha)$ is robust $\mathcal{D}$-stable.

Remark 2: 1) When τ =1 and m =1 the Theorem 2 and 3 reduce to Theorem 2 and 3 in [14] respectively. Consequently, our LMIs are more general.

2) There are two-DOF of positive integers τ and κ in the LMI conditions. The total number of decision variables of LMIs in Theorem 3 are $(r+\kappa-1)N/2 + (r+\kappa-1)^2n^2N/2 + d^2n^2(m+r+\tau+\kappa-2)(m+\tau+\kappa-2) + n^2(r+\kappa-1)(\kappa-1)$. Increasing the integer τ will not make the number of decision variables increase too fast as it with κ, because the first, second and the last items in the expression above are unrelated to τ. This feature of τ enables us to make a relatively gentle increase in the number of decision variables when checking the robust $\mathcal{D}$-stability. Thus there is the benefit to reduce computational complexity compared with the LMIs of only one-DOF positive integer κ.

3) Q_τ in Theorem 2 and 3 is used to extend region matrix before formulating the LMIs. It is allowed to be any positive matrix, we can simply choose $Q_\tau = I_\tau$.

4) By our extension approach, the qualified LMI conditions for checking $\mathcal{D}$-stability of polynomial matrix in Theorem 2 are easily obtained from the original LMIs in Theorem 1.

The next theorem reveals that feasibility of the LMIs of Theorem 3 for some $(\hat{\tau},\hat{\kappa})$ implies feasibility of the corresponding LMIs for any $(\tau,\kappa) \ge (\hat{\tau},\hat{\kappa})$, where $(\tau,\kappa) \ge (\hat{\tau},\hat{\kappa})$ means $\tau \ge \hat{\tau}$, $\kappa \ge \hat{\kappa}$.

Theorem 4: If the LMIs of Theorem 3 are fulfilled for a given $(\tau,\kappa) = (\hat{\tau},\hat{\kappa})$, then the LMIs corresponding to any $(\tau,\kappa) \ge (\hat{\tau},\hat{\kappa})$ are also satisfied.

Proof: It suffices to check that (τ,κ) fulfills the conditions implies $(\tau+1,\kappa)$ and $(\tau,\kappa+1)$ also fulfill the conditions. Suppose for (τ,κ), $\Omega_{1,\kappa} > 0$ and $\Omega_{2,\tau,\kappa} < 0$ are feasible. Let

$$\begin{aligned}
&Q_{\tau+1} = \mathbf{Qu}\left\{I_2 \otimes Q_\tau, \Pi_2^{(1,\tau,1)}\right\} > 0,\\
&P_{\kappa+1} = \mathbf{Qu}\left\{I_2 \otimes P_\kappa, \Pi_2^{(1,r+\kappa-1,n)}\right\},\\
&Z_{\tau+1,\kappa} = Z_{\tau,\kappa+1} = \mathbf{Qu}\left\{I_2 \otimes Z_{\tau,\kappa}, \Pi_2^{(1,m+r+\tau+\kappa-2,nd)}\right\},\\
&Y_{\kappa+1} = \mathbf{Qu}\left\{I_2 \otimes Y_\kappa, \Pi_2^{(1,r+\kappa-1,n)}\right\}.
\end{aligned}$$

By carefully calculation combining with identity (10), we can obtain that

$$\tilde{H}_{\tau+1}^{(m)} = \mathbf{Qu}\left\{I_2 \otimes \tilde{H}_{\tau}^{(m)}, \Pi_2^{(1,\tau+m,d)}\right\},$$
$$\Omega_{1,\kappa+1} = \mathbf{Qu}\left\{I_2 \otimes \Omega_{1,\kappa}, \Pi_2^{(1,r+\kappa-1,n)}\right\} > 0,$$
$$\Omega_{2,\tau+1,\kappa} = \Omega_{2,\tau,\kappa+1} = \mathbf{Qu}\left\{I_2 \otimes \Omega_{2,\tau,\kappa}, \Pi_2^{(1,m+r+\tau+\kappa-2,nd)}\right\} < 0.$$

Therefore the LMIs are feasible for both $(\tau+1,\kappa)$ and $(\tau,\kappa+1)$.

In the end, we mention that particular LMI conditions for stability analysis may be extended into control designs of fractional order systems [8], fuzzy systems [13,19], uncertain systems [15,20] and synchronous machines [21]. Yet a general synthesis process based on the two-DOF method is rather complicated. Such task has not been further investigated and is beyond this paper.

4. EXAMPLES

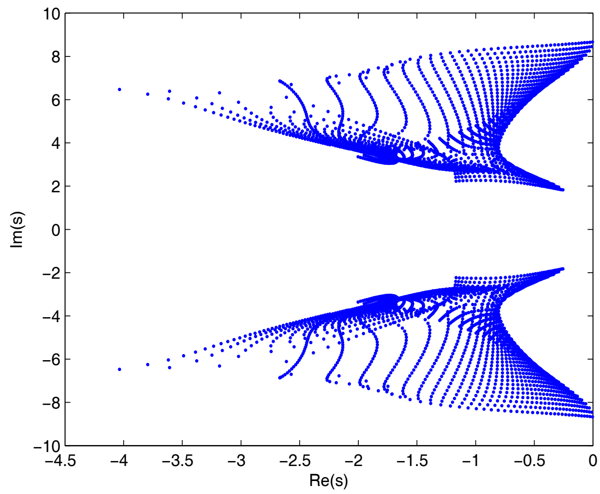
Example 1: The objective of this example is to show the effect and benefit of increasing τ in the LMIs. Here we choose $Q_\tau = I_\tau$ to form our LMI criterions. Meanwhile, when $\tau = 1$, according to the Remark 2, our LMIs reduce to the existing ones in [14]. Hence the example can also be regarded as a comparison between the two methods. This will hold for Example 2 as well. Consider the continuous-time polynomial matrix polytope of dimension $n=2$ and degree $r=2$ whose $N=3$ vertices are given by

$$A_1(s) = \begin{bmatrix} 1.9493+0.0551s+0.0237s^2 & 0.9746+0.0276s+0.0118s^2 \\ 0.5804-0.1381s+0.0118s^2 & 0.9746+0.0276s+0.0118s^2 \end{bmatrix},$$
$$A_2(s) = \begin{bmatrix} 0.2417+0.0776s+0.2353s^2 & -0.1209-0.0388s-0.1176s^2 \\ -1.6103-0.5094s-0.1176s^2 & 1.6103+0.5094s+0.1176s^2 \end{bmatrix},$$
$$A_3(s) = \begin{bmatrix} 0 & 0.1626+0.1607s+0.069s^2 \\ -1.4267-0.2986s-0.069s^2 & 1.5893+0.4593s+0.1379s^2 \end{bmatrix}.$$

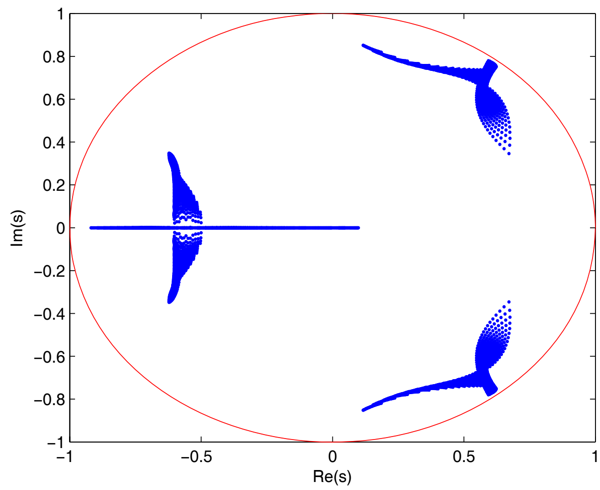
For this system, the LMI conditions of Theorem 3 for $\tau=1$, $\kappa=1$ were found infeasible, while it admitted a feasible solution for both $\tau=2$, $\kappa=1$ and $\tau=1$, $\kappa=2$. Hence either one can check the robust $\mathcal{D}$-stability of the system. The number of decision variables in LMIs for $\tau=2$, $\kappa=1$ is 62 while for $\tau=1$, $\kappa=2$ is 107. Therefore, the former LMIs have more computational efficiency. Next, we consider a discrete-time polynomial matrix polytope with

$$A_1(s) = \begin{bmatrix} 0.3+0.2s+s^2 & -0.5-1.2s-0.1s^2 \\ 0.3+1.6s+s^2 & -1.5-0.9s+1.34s^2 \end{bmatrix},$$
$$A_2(s) = \begin{bmatrix} -0.5+1.5s+2.8s^2 & -1.8-0.2s+0.5s^2 \\ 1.4+0.3s & 0.9-0.6s+1.5s^2 \end{bmatrix},$$
$$A_3(s) = \begin{bmatrix} -1-0.6s+1.2s^2 & 0.3-0.5s-0.6s^2 \\ 1-0.1s & -0.3+0.3s+1.2s^2 \end{bmatrix}.$$

The LMI conditions of Theorem 3 for $\tau=2$, $\kappa=2$ were not feasible, while for both $\tau=3$, $\kappa=2$ and $\tau=1$, $\kappa=3$ it admitted a feasible solution. The former LMIs have 171 decision variables while the latter have 200. The eigenvalues of these polytopes, obtained by a straightforward sweeping, are represented in Fig. 1 (a) and (b). The stability of the polytopes can be observed from Fig. 1 that the whole eigenvalues of them lie within the stability regions. Thus this example shows the validity and computational advantage of the proposed method.



(a)



(b)

Fig. 1. Example 1. Eigenvalues of the polynomial matrix polytopes for the continuous-time case (a) and the discrete-time case (b).

Example 2: The objective of this example is to reveal the relationship between conservatism and number of decision variables, and to show the advantage of our LMI conditions as well. Here we choose $Q_\tau = I_\tau$ to form our LMI criterions. With similar procedure adapted from [9] and [14], a thousand continuous-time polytopes of polynomial matrices and a thousand discrete-time ones for $r = 2$, $n = 2$, $N = 3$ are randomly generated as follows:

1) The polytope vertices are generated by random polynomial matrices of dimension $n = 2$ and degree $r = 2$.

2) The eigenvalues of each polytope $\mathcal{A}(s,\alpha)$ are computed by an exhaustive griding procedure and the approximative maximum of the real part (continuous-time case) or the approximative maximum of absolute values (discrete-time case) is determined.

3) With a proper value ε, $\mathcal{A}(s,\alpha)$ is replaced by $\mathcal{A}(s+\varepsilon,\alpha)$ so that the approximative maximal real part of eigenvalues in the interval [–0.02, –0.01] exists (continuous time case) or replaced by $\mathcal{A}(\varepsilon s,\alpha)$ so that the approximative maximal absolute value of eigenvalues in the interval [0.98, 0.99] exists (discrete time case).

The computational complexity of the tests directly relates with the number of decision variables, denoted by *NDV*, which can be calculated according to the Remark 2. The robust stability tests are taken using the LMI conditions in the ascending order of *NDV*. Table 1 shows the number of stable polytopes identified by Theorem 3 (labeled *CS* for continuous-time and *DS* for discrete-time polytopes). Test results verify that the increasing number of decision variables rather than κ, causes less conservative LMI criterions, see LMI test with $(\tau_1,\kappa_1) = (6,1)$ is less conservative than $(\tau_2,\kappa_2) = (1,2)$ even though $\kappa_1 < \kappa_2$. And our LMI conditions with two-DOF integers have better flexibility of tradeoff between the conservatism and computational complexity.

Table 1. Test results for polynomial matrix polytopes.

NDV	τ	κ	*CS*	*DS*
42	1	1	498	638
62	2	1	551	749
90	3	1	566	788
107	1	2	767	834
126	4	1	770	836
135	2	2	782	846
170	5	1	782	846
171	3	2	785	848
200	1	3	809	852
215	4	2	810	852
222	6	1	810	852
236	2	3	810	852
267	5	2	810	852
280	3	3	810	852
282	7	1	810	852
321	1	4	810	852
327	6	2	810	852
332	4	3	810	852
350	8	1	810	852
365	2	4	811	852
392	5	3	811	852

Example 3: The objective of this example is to show the benefit of suitable Q_τ for the LMI conditions. Consider again the discrete-time polynomial matrix polytope in Example 1. It was found in Example 1 that the LMI conditions of Theorem 3 for $\tau = 2$, $\kappa = 2$ were not feasible, where $Q_\tau = I_\tau$ was chosen. Now for the same $\tau = 2$, $\kappa = 2$, we choose $Q_\tau = \begin{bmatrix} 3.7 & 0.3 \\ 0.3 & 5.5 \end{bmatrix}$. And the LMIs with Q_τ can be found a feasible solution. This shows a suitable chosen Q_τ can make the LMIs perform better robust checking.

5. CONCLUSION

In this note, sufficient and necessary LMI criterion is derived for stability of scalar matrix in general order LMI region. Then based on that, and via our approach to extend region matrix and Lyapunov matrix, sufficient LMI conditions with two-DOF integers (τ, κ) and arbitrary chosen $Q_\tau > 0$, for checking robust $\mathcal{D}$-stability of polynomial matrix polytopes are obtained. Examples have shown the advantages of our result. Finally, Example 3 gives us an interesting revelation that less conservative LMI conditions can also be obtained without more decision variables, but with some suitable $Q_\tau > 0$. How to find the suitable $Q_\tau > 0$ for a given polytope will be the subject of our future research.

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