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A Lyapunov Functional Approach to Robust Stability Analysis of Continuous-Time Uncertain Linear Systems in Polytopic Domains

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Abstract: In this paper, a sufficient linear matrix inequality (LMI) condition is presented for robust stability analysis of continuous-time linear time-invariant (LTI) systems in polytopic domains. The underlying idea behind the proposed approach is to introduce a family of complex functions which map the closed right-hand side of the complex plane into the inside of the closed unit circle centered at the origin. Then, the mapping properties are used to assure that all the eigenvalues of a system are located in the open left-hand side of the complex plane. Examples show the validity of the proposed condition.

Keywords: Complex functions, linear matrix inequality, linear time-invariant systems, relaxation, robust stability.

1. INTRODUCTION

Robust stability analysis of uncertain linear time-invariant systems has been a subject of recurring interest in the last decades, see, for example [1-18] and the references therein. Undoubtedly, Lyapunov stability theory is one of the most popular approaches to deal with those problems. Among them, the simplest way is to look for a quadratic Lyapunov function (QLF) [3], which leads overly conservative results in general because a constant Lyapunov matrix should be found for all uncertainty set.

To reduce the conservativeness, lots of efforts have been made in the direction of generalizing the Lyapunov functions, see e.g. [4] which proposes affine parameter-dependent QLFs (PD-QLFs) for LTI systems with affine uncertainties, and [5-10] which employ linear PD-QLFs for LTI systems with polytopic uncertainties. Recently, several important results [11-18,38-43] on robust stability for uncertain LTI systems have been proposed through the development of sophisticated convergent linear matrix inequality relaxations, sequences of LMI conditions which, as the sequences proceed, tend to necessary and sufficient conditions at the expense of increasing complexities. As a natural generalization of linear PD-QLFs, most of them are based on finding polynomial PD-QLFs depending polynomially on the uncertain parameters. Specifically, several convergent LMI conditions for the existence of polynomial PD-QLFs or homogeneous polynomial PD-QLFs have been proposed in [11] based on a systematic way to transform polynomially parameter-dependent LMIs (PD-LMIs) into finite-dimensional ones, [12] and [13] based on the complete square matrix representation of homogeneous matrix forms, and [14-16] by means of the matrix version of Pólya's theorem, introduced in [17], while a particular polynomial PD-QLF, whose parameter-dependent Lyapunov matrix is a polynomial function of the uncertain system matrices has been introduced in [18,38-40]. In addition, the use of matrix-valued sum-of-squares decompositions [19] and [20] can provide robust stability results for a large class of uncertainties.

In this paper, we pursue another possibility to assess robust stability of continuous-time LTI systems with polytopic uncertainties. The starting idea is to look for a class of Lyapunov functionals to assess the robust stability. The use of Lyapunov functionals for stability analysis of time-delay systems has been largely reported in the literature to date [21-27]. However, to the best of authors' knowledge, there are no results on the application of Lyapunov functionals for robust stability analysis yet. Specifically, the core idea of this paper stems from [26] and [27], where stability analysis of time-delay LTI systems is considered in a quadratic separation framework [28]. The philosophy behind results in [26] and [27] is to employ a Taylor series approximation of the delay operator and to consider the Taylor remainder a new uncertainty type approximation. Inspired by the idea in [26] and [27], we employ a class of complex functions which map the closed right-hand side of the complex plane into the inside of the closed unit circle centered at the origin. Then, the robust stability can be analyzed by checking whether the functions map all the eigenvalues of an uncertain system

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matrix into the numbers located outside of the closed unit circle for the whole uncertainty domain. By means of Finsler's lemma [29], the test is cast as a sufficient robust stability condition which amounts to solving LMIs. As mentioned before, this approach can be interpreted as searching for a class of Lyapunov functionals. Examples are given to demonstrate the validity of the proposed approach. Finally, the distinguished features and merits of the proposed approach are summarized as follows: (1) the developed method offers a different insight into the robust stability analysis of continuous-time LTI systems. Specifically, we use a family of complex functions and its mapping properties to check the Hurwitz stability. This technique can be viewed as a generalization of the traditional PD-QLF approach in that, with slight modifications of the proposed conditions, we can contemplate both the PD-QLF approaches and the proposed one based on Lyapunov functionals in a unified framework; (2) our robust stability condition can produce less conservative results in comparison with the approaches using the common QLF and PD-QLFs, as demonstrated in examples later. In addition, it is expected that the approach can be effectively combined with some other relaxation techniques (for instance, homogeneous polynomial PD-LMI techniques [16]) to further reduce the conservatism; (3) the results of this paper are potentially relevant to some applications in systems and control area. For example, our work admits interesting extensions to simultaneously handle both uncertainties and time-delay in a unified fashion. In addition, present research seems to eventually be extended to cope with the robust controller synthesis problems. The directions discussed above will be the subject of future research.

Notation: A^T and A^*: transpose and transpose conjugate of A, respectively; $A>0$ ($A<0$ and $A\geq 0$): symmetric positive definite (respectively, negative definite and positive semi-definite) matrix A; $A\otimes B$: Kronecker's product of matrices A and B; He$\{A\}$: a shorthand notion for $A+A^*$; $A_\perp$: matrices whose columns span the right null space of matrix A; I_n and $0_{n\times m}$: $n\times n$ identity matrix and $n\times m$ zero matrix, respectively; 0_n: null vector of size n; $\mathbb{C}$: complex plane; $\mathbb{R}_0^n := \mathbb{R}^n \setminus 0_n$;

$\mathbb{C}_{\geq 0} := \{\lambda\in\mathbb{C} : \mathrm{Re}(\lambda)\geq 0\}$;

$\mathbb{C}_{<0} := \{\lambda\in\mathbb{C} : \mathrm{Re}(\lambda)< 0\}$;

$\mathbb{R}_{\geq 0} := \{\lambda\in\mathbb{R} : \lambda\geq 0\}$;

$\mathbb{R}_{<0} := \{\lambda\in\mathbb{R} : \lambda< 0\}$;

$\mathbb{N}$ and $\mathbb{N}_0$: sets of positive integer and non-negative integer, respectively.

2. STABILITY ANALYSIS

First of all, let us consider the continuous-time LTI system

$$\dot{x}(t) = Ax(t), \tag{1}$$

where $A\in\mathbb{R}^{n\times n}$ and $x(t)\in\mathbb{R}^n$ is the state. Define complex functions

$$\begin{cases} \delta_0(\lambda,\tau) := e^{-\lambda\tau} \\ \delta_k(\lambda,\tau) := \dfrac{k!}{(-\lambda\tau)^k}\left\{ e^{-\lambda\tau} - \displaystyle\sum_{i=1}^{k-1}\frac{1}{i!}(-\lambda\tau)^i \right\}, \quad k\in\mathbb{N} \end{cases} \tag{2}$$

with $\lambda\in\mathbb{C}$ and $\tau\in\mathbb{R}_{\geq 0}$. The following lemmas play important roles in the development of the main results.

Lemma 1: Let $\tau\in\mathbb{R}_{>0}$ be given. If $\lambda\in\mathbb{C}_{\geq 0}$, then $\delta_k(\lambda,\tau)^*\delta_k(\lambda,\tau)\leq 1$ holds for all $k\in\mathbb{N}_0$.

Proof: The proof follows similar lines to the proof of Lemma 1 in [27], and thus is omitted here for the sake of space.

Lemma 2: Let $\tau\in\mathbb{R}_{>0}$ and $\lambda\in\mathbb{C}$ be given. If

$$\delta_k(\lambda,\tau)^*\delta_k(\lambda,\tau) > 1$$

holds for some $k\in\mathbb{N}_0$, then $\lambda\in\mathbb{C}_{<0}$.

Proof: The proof is completed by contraposition of Lemma 1.

Lemma 3: Let $\tau\in\mathbb{R}_{>0}$ and $\lambda\in\mathbb{C}$ be given. If

$$\sum_{i=0}^{k} a_i(\lambda^*\lambda)^i\{1-\delta_i(\lambda,\tau)^*\delta_i(\lambda,\tau)\} < 0 \tag{3}$$

holds for some $k\in\mathbb{N}_0$ and $a_i>0$, then $\lambda\in\mathbb{C}_{<0}$.

Proof: If (3) is fulfilled, then since $a_i(\lambda^*\lambda)^i>0$, it is guaranteed that $1-\delta_i(\lambda,\tau)^*\delta_i(\lambda,\tau)<0$ holds for some $i\in\{0,1,\ldots,k\}$. By Lemma 2, this ensures $\lambda\in\mathbb{C}_{<0}$.

To provide an interpretation in view of Lyapunov functionals, we will need the following result.

Lemma 4: Let $\tau\in\mathbb{R}_{\geq 0}$ be given. System (1) is asymptotically stable if and only if there exists a symmetric matrix $X\in\mathbb{R}^{n\times n}$ such that $X>0$ and $\dot{V}(x(t)) < 0$, $\forall x(t)\in\mathbb{R}_0^n$, $t\in[\tau,\infty)$ hold along the solution to (1), where

$$V(x(t)) := \int_{t-\tau}^{t} x(\theta)^T X x(\theta)d\theta .$$

Proof: The proof easily follows from the fact that A is Hurwitz stable iff $e^{A\tau}$ is Schur stable and using the discrete-time Lyapunov theory.

Remark 1: Suppose $\lambda\in\mathbb{C}$ and $v\in\mathbb{C}^n$ satisfy $Av=\lambda v$, i.e., λ and v are, respectively, an eigenvalue and the corresponding eigenvector of A. If $\dot{V}(x(t))<0$, $\forall x(t)\in\mathbb{R}_0^n$, $t\in[\tau,\infty)$ holds along (1), then we have

$$\begin{aligned}\dot{V}(x(t)) &= x(t)^T Xx(t) - x(t-\tau)^T Xx(t-\tau) \\ &= x(t)^T(X - e^{-A^T\tau}Xe^{-A\tau})x(t) \\ &< 0, \quad \forall x(t)\in\mathbb{R}_0^n.\end{aligned}$$

Setting $x(t) = v$ in the above inequality leads to

$$v^*Xv(1-e^{-\lambda^*\tau}e^{-\lambda\tau}) = v^*Xv(1-\delta_0(\lambda,\tau)^*\delta_0(\lambda,\tau)) < 0.$$

Since $v^*Xv>0$, this implies $1-\delta_0(\lambda,\tau)^*\delta_0(\lambda,\tau)<0$, which, together with Lemma 2, implies $\lambda\in\mathbb{C}_{<0}$. In this respective, the existence test of Lyapunov functional $V(x(t))$ in Lemma 4 corresponds to checking $1-e^{-\lambda^* \tau}e^{-\lambda\tau} < 0$ and hence the asymptotic stability of (1), while the conventional Lyapunov inequality $A^T P + PA < 0$ with $P > 0$ corresponds to testing $\lambda^* + \lambda < 0$, which also guarantees $\lambda \in \mathbb{C}_{<0}$.

Remark 2: The discrete-time counterpart of the Lyapunov functional approach has already been investigated by some researchers. For instance, a class of Lyapunov functionals that consist of an augmented state vector has proven to be effective in reducing the conservatism of the quadratic Lyapunov function approach for discrete-time nonlinear systems [36-38]. On the other hand, its continuous-time version has been also pursued in [18,39,40] for LTI systems. They employed a class of quadratic Lyapunov functions associated with higher-order time-derivatives of the state.

As the next step, let us assume that A is invertible, i.e., the number 0 is not an eigenvalue of A, and define

$$\begin{cases} \Delta_0(A,\tau) := e^{-A\tau} \\ \Delta_k(A,\tau) := \dfrac{k!}{(-\tau)^k} A^{-k}\left\{e^{-A\tau} - \sum_{i=0}^{k-1}\dfrac{1}{i!}(-A\tau)^i\right\}, \quad k \in \mathbb{N} \end{cases} \tag{4}$$

with $\tau \in \mathbb{R}_{\geq 0}$. By L' Hospital's rule, it is easy to see that

$$\lim_{\tau\to 0}\Delta_k(A,\tau) = \lim_{\tau\to 0}\Delta_{k-1}(A,\tau) = \cdots = \lim_{\tau\to 0}\Delta_0(A,\tau) = I_n$$

holds. Therefore, we can define signals $x_i(t)$, $i \in \{0,1,\ldots,k\}$ that satisfy

$$x_i(t-\tau) := \Delta_i(A,\tau)x_i(t), \ \forall (t,i) \in [\tau,\infty)\times\{0,1,\ldots,k\} \tag{5}$$

and consider the following functionals:

$$V_i(x_i(t)) := \int_{t-\tau}^{t} x_i(\theta)^T X_i x_i(\theta)d\theta, \quad i \in \{0,1,\ldots,k\}, \tag{6}$$

where $X_i \in \mathbb{R}^{n\times n}$ are positive definite matrices to be determined.

Lemma 5: Let $\tau \in \mathbb{R}_{>0}$ be given. System (1) is asymptotically stable if and only if there exists symmetric matrices $X_i \in \mathbb{R}^{n\times n}$ such that $X_i > 0$ and $\sum_{i=0}^{k}\dot{V}_i(x(t)) < 0$ holds along $x_i(t) \in \mathbb{R}_0^n$, $(t,i) \in [\tau,\infty)\times\{0,1,\ldots,k\}$ that satisfy (5).

Proof: The proof of the necessity part straightforwardly follows from Lemma 4. To prove sufficiency, let λ and v denote an eigenvalue and the corresponding eigenvector of A, respectively, i.e., $Av = \lambda v$. We can write $\sum_{i=0}^{k}\dot{V}_i(x(t)) < 0, \ \forall x_i(t) \in \mathbb{R}_0^n$ as follows:

$$\sum_{i=0}^{k}\dot{V}_i(x(t)) = \sum_{i=0}^{k} x_i(t)^T\{X_i - \Delta_i(A,\tau)^T X_i \Delta_i(A,\tau)\}x_i(t) < 0, \quad \forall x_i(t) \in \mathbb{R}_0^n.$$

Setting $x_i(t) = \lambda^i v$, $\forall i \in \{0,1,\ldots,k\}$ in the above inequality yields

$$\sum_{i=0}^{k}(\lambda^*\lambda)^k(v^* X_i v)\{1-\delta_i(\lambda,\tau)^*\delta_i(\lambda,\tau)\} < 0.$$

Since $(\lambda^*\lambda)^k v^* X_i v > 0$, by Lemma 3, $\lambda \in \mathbb{C}_{<0}$ holds, and hence, (1) is asymptotically stable.

Remark 3: The condition of Lemma 4 is recovered by setting $k = 0$ in Lemma 5.

Based on Lemma 3 or 5, we are now ready to state the main theorem in this work, a necessary and sufficient LMI condition for asymptotic stability of (1).

Theorem 1: Let $\tau \in \mathbb{R}_{>0}$ be given. System (1) is asymptotically stable if and only if there exist symmetric matrices $X_i \in \mathbb{R}^{n\times n}$, a matrix $Z \in \mathbb{R}^{(k+2)n\times kn}$, and a positive integer $k \in \mathbb{N}$ such that the following LMIs hold:

$$\bar{X}_{k+1} > 0, \quad \mathcal{Q}(\bar{X}_{k+1}) + \mathrm{He}\{Z\mathcal{C}_k(A)(\mathcal{T}_{k+1}\otimes I_n)\} < 0, \tag{7}$$

where

$$\begin{cases} \bar{X}_{k+1} := \mathrm{diag}(X_0, X_1, \ldots, X_k) \in \mathbb{R}^{(k+1)n\times(k+1)n} \\ \mathcal{L}_k := [I_k \quad 0_{k\times 1}] \in \mathbb{R}^{k\times(k+1)} \\ \mathcal{R}_k := [0_{k\times 1} \quad I_k] \in \mathbb{R}^{k\times(k+1)} \\ \mathcal{T}_{k+1} = \mathcal{L}_{k+1} + \mathcal{R}_{k+1}\mathrm{diag}(\tau, \tau/2, \ldots, \tau/(k+2)) \\ \qquad \in \mathbb{R}^{(k+1)\times(k+2)} \\ \mathcal{Q}(\bar{X}_{k+1}) := \begin{bmatrix}\mathcal{T}_{k+1}\otimes I_n \\ \mathcal{L}_{k+1}\otimes I_n\end{bmatrix}^T \left(\begin{bmatrix}1 & 0 \\ 0 & -1\end{bmatrix}\otimes \bar{X}_{k+1}\right) \\ \qquad \times \begin{bmatrix}\mathcal{T}_{k+1}\otimes I_n \\ \mathcal{L}_{k+1}\otimes I_n\end{bmatrix} \in \mathbb{R}^{(k+2)n\times(k+2)n} \\ \mathcal{C}_k(A) := \mathcal{L}_k \otimes A - \mathcal{R}_k \otimes I_n \in \mathbb{R}^{kn\times(k+1)n}. \end{cases} \tag{8}$$

Proof: (Sufficiency) Let λ and v denote an eigenvalue and the corresponding eigenvector of A, respectively, i.e., $Av = \lambda v$. Moreover, let use assume that LMIs in (7) hold and define vector-valued complex functions $d_{k+2}(\lambda) \in \mathbb{C}^{k+2}$ and $l_{k+1}(\lambda) \in \mathbb{C}^{k+1}$ as

$$d_{k+2}(\lambda) := [\delta_0(\lambda,\tau) \quad \lambda\delta_1(\lambda,\tau) \quad \cdots \quad \lambda^{k+1}\delta_{k+1}(\lambda,\tau)]^T \in \mathbb{C}^{k+2},$$

$$l_{k+1}(\lambda) := [1 \quad \lambda \quad \cdots \quad \lambda^k]^T \in \mathbb{C}^{k+1},$$

where $\delta_k(\lambda,\tau)$ is defined in (2). Then, by direct calculations and using the relations

$$\lambda^{k-1} = [1 \quad \tau/k]\begin{bmatrix}\lambda^{k-1}\delta_{k-1}(\lambda,\tau) \\ \lambda^k\delta_k(\lambda,\tau)\end{bmatrix},$$

$$\mathcal{T}_{k+1}d_{k+2}(\lambda) = l_{k+1}(\lambda),$$

it is easy to show that

$$(\mathcal{T}_{k+1}\otimes I_n)(d_{k+2}(\lambda)\otimes v) = l_{k+1}(\lambda)\otimes v$$

and

$$\begin{aligned}\mathcal{C}_k(A)(\mathcal{T}_{k+1}\otimes I_n)(d_{k+2}(\lambda)\otimes v) &= \mathcal{C}_k(A)(l_{k+1}(\lambda)\otimes v) \\ &= l_k(\lambda)(Av - \lambda v) \\ &= 0_{kn\times 1}\end{aligned}$$

hold. Therefore, it follows from (7) that

$$\begin{aligned}
&(d_{k+2}(\lambda)\otimes v)^*[\mathcal{Q}(\bar{X}_{k+1})+\mathrm{He}\{Z\mathcal{C}_k(A)(\mathcal{T}_{k+1}\otimes I_n)\}] \\
&\times(d_{k+2}(\lambda)\otimes v) \\
&\quad=(d_{k+2}(\lambda)\otimes v)^*\mathcal{Q}(\bar{X}_{k+1})(d_{k+2}(\lambda)\otimes v) \\
&\quad=l_{k+1}(\lambda)^*\bar{X}_{k+1}l_{k+1}(\lambda)-d_{k+1}(\lambda)^*\bar{X}_{k+1}d_{k+1}(\lambda) \\
&\quad=\sum_{i=0}^{k}(\lambda^*\lambda)^i v^* X_i v\{1-\delta_i(\lambda,\tau)^*\delta_i(\lambda,\tau)\} \\
&\quad<0.
\end{aligned}$$

Since $(\lambda^*\lambda)^i v^* X_i v>0$, the asymptotic stability of (1) is confirmed by Lemma 3.

(Necessity): Let us assume that (1) is asymptotically stable. This guarantees that A is invertible and, according to Lemma 4, there exists a symmetric matrix $\hat{X}\in\mathbb{R}^{n\times n}$ such that $\hat{X}>0$ and $\hat{X}-e^{-A^T\tau}\hat{X}e^{-A\tau}<0$ hold. For future reference, let $\mathcal{D}_{k+2}(A)$ denote

$$\mathcal{D}_{k+2}(A):=[\mathcal{D}_{L,k+2}\quad \mathcal{D}_{R,k+2}]\in\mathbb{R}^{(k+2)n\times 2n},$$

where

$$\mathcal{D}_{L,k+2}:=\begin{bmatrix}\Delta_0(A,\tau)\\ \vdots\\ A^k\Delta_k(A,\tau)\\ A^{k+1}\Delta_{k+1}(A,\tau)\end{bmatrix}\in\mathbb{R}^{(k+2)n\times n},$$

$$\mathcal{D}_{R,k+2}:=\begin{bmatrix}g(0)I_n\\ \vdots\\ g(k)I_n\\ g(k+1)I_n\end{bmatrix}\in\mathbb{R}^{(k+2)n\times n},$$

$g(i):=i!/(-\tau)^i$, and $\Delta_k(A,\tau)$ is defined in (4). Then, by direct calculation, it can be seen that

$$\mathcal{D}_{k+2}(A)^T\mathcal{Q}(\bar{X}_{k+1})\mathcal{D}_{k+2}(A)=\textstyle\sum_{i=0}^{k}\Omega_i(X_i)$$

with $\Omega_i(X_i)$ denoting

$$\Omega_i(X_i):=\begin{bmatrix}(A^i)^T X_i A^i-\Delta_i(A,\tau)^T(A^i)^T X_i A^i\Delta_i(A,\tau) & -g(i)\Delta_i(A,\tau)^T(A^i)^T X_i\\ -g(i)X_i A^i\Delta_i(A,\tau) & -g(i)^2X_i\end{bmatrix}. \quad (9)$$

Let $X_0=\hat{X}$, $X_1=X_2=\cdots=X_{k-1}=\varepsilon I_n$ with sufficiently small $\varepsilon>0$ and $X_k=\rho g(k)^{-2}I_n$, where ρ is a positive real number. Then, it is straightforward to show that

$$\begin{aligned}
&\mathcal{D}_{k+2}(A)^T\mathcal{Q}(\bar{X}_{k+1})\mathcal{D}_{k+2}(A)\\
&\quad\cong\Omega_0(\hat{X})+\Omega_k(\rho g(i)^{-2}I_n)\\
&\quad=\begin{bmatrix}\hat{X}-e^{-A^T}\hat{X}e^{-A}+\rho g(k)^{-2}(A^TA)^k-\rho\mathcal{I}_k^T\mathcal{I}_k & -e^{-A^T\tau}\hat{X}-\rho\mathcal{I}_k^T\\ -\hat{X}e^{-A\tau}-\rho\mathcal{I}_k & -\hat{X}-\rho I_n\end{bmatrix},
\end{aligned}$$

where $\mathcal{I}_k:=e^{-A\tau}-\sum_{i=0}^{k-1}(1/i!)(-A\tau)^i$. Noticing that

$$\begin{aligned}
&\lim_{k\to\infty}\mathcal{D}_{k+2}(A)^T\mathcal{Q}(\bar{X}_{k+1})\mathcal{D}_{k+2}(A)\\
&\quad=\begin{bmatrix}\hat{X}-e^{-A^T\tau}\hat{X}e^{-A\tau} & -e^{-A^T\tau}\hat{X}\\ -\hat{X}e^{-A\tau} & -\hat{X}-\rho I_n\end{bmatrix},
\end{aligned}$$

one can deduce the existence of sufficiently large real number $\rho\in\mathbb{R}_{>0}$ and integer $\hat{k}\in\mathbb{N}$ satisfying

$$\mathcal{D}_{k+2}(A)^T\mathcal{Q}(\bar{X}_{k+1})\mathcal{D}_{k+2}(A)<0$$

for all $k\geq\hat{k}$. Finally, it can be seen by simple algebraic manipulations that $\mathcal{D}_{k+2}(A)=\{\mathcal{C}_k(A)(\mathcal{T}_{k+1}\otimes I_n)\}_\perp$, and hence, the conclusion of the theorem is provided by Finsler's lemma [29].

Remark 4: Let us suppose that LMIs in (7) hold and $Z\in\mathbb{R}^{(k+2)n\times kn}$ has a block partitioned matrix form with $k+2$ row partitions and k column partitions. By direct calculation, it can be seen that the block (1,1) of LMI $\mathcal{Q}(\bar{X}_{k+1})+\mathrm{He}\{Z\mathcal{C}_k(A)(\mathcal{T}_{k+1}\otimes I_n)\}<0$ in (7) ensures $\mathrm{He}\{Z_{11}A\}<0$, where $Z_{11}\in\mathbb{R}^{n\times n}$ is block (1,1) of Z, which guarantees that A is invertible. On the other hand, it follows from the condition

$$\mathcal{Q}(\bar{X}_{k+1})+\mathrm{He}\{Z\mathcal{C}_k(A)(\mathcal{T}_{k+1}\otimes I_n)\}<0$$

in (7) that

$$\begin{aligned}
&x(t)^T\mathcal{D}_{L,k+2}^T[\mathcal{Q}(\bar{X}_{k+1})+\mathrm{He}\{Z\mathcal{C}_k(A)(\mathcal{T}_{k+1}\otimes I_n)\}]\\
&\quad\times\mathcal{D}_{L,k+2}x(t)\\
&=\sum_{i=0}^{k}\dot{V}_i(x(t))<0,\quad \forall x_i(t)\in\mathbb{R}_0^n,
\end{aligned}$$

where $x_i(t)$ and $V_i(x(t))$ are defined in (5) and (6), respectively. By Lemma 5, this implies that (1) is asymptotically stable. In this respect, the feasibility test of LMIs in Theorem 1 can be alternatively interpreted as checking the existence of a Lyapunov functional $\tilde{V}(x(t)):=\sum_{i=0}^{k}V_i(x(t))$ such that $d\tilde{V}(x(t))/dt<0$ along $x_i(t)\in\mathbb{R}_0^n$, $(t,i)\in[\tau,\infty)\times\{0,1,\ldots,k\}$.

Theorem 2: Let $\tau\in\mathbb{R}_{>0}$ be given. If the LMI condition of Theorem 1 is fulfilled for a given positive integer $k=\hat{k}\in\mathbb{N}$, then those corresponding to any $k>\hat{k}$ are also satisfied.

Proof: It suffices to check that if the LMI condition of Theorem 1 is fulfilled for $k=\hat{k}\in\mathbb{N}$, then that corresponding to $k=\hat{k}+1$ is also satisfied. To prove this, suppose that there exist symmetric matrices $X_i\in\mathbb{R}^{n\times n}$ and a matrix $Z\in\mathbb{R}^{(k+2)n\times kn}$ such that (7) holds for $k=\hat{k}$. Since $\mathcal{D}_{\hat{k}+2}(A)=\{\mathcal{C}_{\hat{k}}(A)(\mathcal{T}_{\hat{k}+1}\otimes I_n)\}_\perp$ pre-multiplying $\mathcal{Q}(\bar{X}_{\hat{k}+1})+\mathrm{He}\{Z\mathcal{C}_{\hat{k}}(A)(\mathcal{T}_{\hat{k}+1}\otimes I_n)\}<0$ by $\mathcal{D}_{\hat{k}+2}(A)^T$ and post-multiplying by the transpose yield

$$\mathcal{D}_{\hat{k}+2}(A)^T\mathcal{Q}(\bar{X}_{\hat{k}+1})\mathcal{D}_{\hat{k}+2}(A)=\textstyle\sum_{i=0}^{k}\Omega_i(X_i)<0,$$

where $\Omega_i(\cdot)$ is defined in (9). Now, let us notice that

$$\Omega_{\hat{k}+1}(\rho g(\hat{k}+1)^{-2}I_n) = \rho\begin{bmatrix} g(\hat{k}+1)^{-2}(A^TA)^{\hat{k}+1} - \mathcal{I}_{\hat{k}+1}^T\mathcal{I}_{\hat{k}+1} & -\mathcal{I}_{\hat{k}+1}^T \\ -\mathcal{I}_{\hat{k}+1} & -I_n \end{bmatrix},$$

where $g(i) := i!/(-\tau)^i$,

$$\mathcal{I}_{\hat{k}+1} := e^{-A\tau} - \textstyle\sum_{i=0}^{\hat{k}}(1/i!)(-A\tau)^i,$$

and ρ is a positive real number. Therefore, it is clear that there exists a sufficiently small $\rho \in \mathbb{R}_{>0}$ such that

$$\mathcal{D}_{\hat{k}+2}(A)^T\mathcal{Q}(\bar{X}_{\hat{k}+1})\mathcal{D}_{\hat{k}+2}(A) + \Omega_{\hat{k}+1}(\rho g(\hat{k}+1)^{-2}I_n) = \mathcal{D}_{\hat{k}+3}(A)^T\mathcal{Q}(\bar{X}_{\hat{k}+2})\mathcal{D}_{\hat{k}+3}(A) < 0$$

holds with $\bar{X}_{\hat{k}+2} := \text{diag}(\bar{X}_{\hat{k}+1}, \rho g(\hat{k}+1)^{-2}I_n)$. We apply then Finsler's lemma [29] with

$$\mathcal{D}_{\hat{k}+3}(A) = \{\mathcal{C}_{\hat{k}+1}(A)(\mathcal{T}_{k+2}\otimes I_n)\}_\perp$$

to complete the proof.

3. ROBUST STABILITY ANALYSIS

Let us consider the continuous-time uncertain LTI system

$$\dot{x}(t) = A(\alpha)x(t), \tag{10}$$

where $x(t)\in\mathbb{R}^n$ is the state and matrix $A(\alpha)\in\mathbb{R}^{n\times n}$ is not precisely known, but constant in time and assumed to belong to the polytopic type uncertain domain, i.e.,

$$A(\alpha) \in \{\mathcal{A} : \mathcal{A} = \textstyle\sum_{i=1}^N \alpha_i A_i; \alpha\in\mathcal{P}\},$$

where $\alpha := [\alpha_1\ \alpha_2\ \cdots\ \alpha_N]^T$ is the polytope coordinate and $\mathcal{P}$ is the unit simplex given by

$$\mathcal{P} := \{\alpha\in\mathbb{R}^N : \textstyle\sum_{i=1}^N \alpha_i = 1, \alpha_i \ge 0, i = 1,2,\ldots,N\}.$$

Remark 5: As stated in [16], the mathematical description of the convex polytopic domain of the system matrix is one of the most widely adopted representation of system uncertainty due to its simplicity, generality, and easy handling, mainly in the context of the Lyapunov-based method; by means of the Lyapunov theory, it provides a systematic framework to formulate several analysis and control design problems in terms of convex LMI optimization procedures. Moreover, as pointed out in [34,35], it is one of the most general ways of capturing the structured uncertainty that may affect the system parameters. For instance, it includes the well-known interval parametric uncertainty; if some elements of A is unknown but assumed to be within some known intervals, this situation can easily be accommodated by a proper choice of the set of extreme matrices of A.

As a straightforward corollary to Theorem 1, we have the following necessary and sufficient condition for robust asymptotic stability of (10).

Corollary 1: Let $\tau\in\mathbb{R}_{>0}$ be given. Uncertain system (10) is asymptotically stable if and only if there exist parameter-dependent symmetric matrices $X_i(\alpha)\in\mathbb{R}^{n\times n}$, a parameter-dependent matrix $Z(\alpha)\in\mathbb{R}^{(k+2)n\times kn}$, and a positive integer $k\in\mathbb{N}$ such that the following parameter-dependent LMIs (PD-LMIs) hold for all $\alpha\in\mathcal{P}$:

$$\bar{X}_{k+1}(\alpha) > 0,$$
$$\mathcal{Q}(\bar{X}_{k+1}(\alpha)) + \text{He}\{Z(\alpha)\mathcal{C}_k(A(\alpha))(\mathcal{T}_{k+1}\otimes I_n)\} < 0,$$

where

$$\bar{X}_{k+1}(\alpha) := \text{diag}(X_0(\alpha), X_1(\alpha), \ldots, X_k(\alpha)) \in \mathbb{R}^{(k+1)n\times(k+1)n},$$

and $\mathcal{Q}(\cdot)$, $\mathcal{C}_k(\cdot)$, and $\mathcal{T}_{k+1}$ are defined in (8).

Proof: It follows immediately from Theorem 1.

The following theorem states that, with a proper choice of $\tau\in\mathbb{R}_{>0}$, the argument of Corollary 1 remains valid even when parameter-dependent matrices $X_i(\alpha)$ are replaced by constant matrices X_i.

Theorem 3: Uncertain system (10) is asymptotically stable if and only if there exist symmetric matrices $X_i \in\mathbb{R}^{n\times n}$, a parameter-dependent matrix $Z(\alpha)\in\mathbb{R}^{(k+2)n\times kn}$, a positive real number $\tau\in\mathbb{R}_{>0}$, and a positive integer $k\in\mathbb{N}$ such that the following PD-LMIs hold for all $\alpha\in\mathcal{P}$:

$$\begin{cases} \bar{X}_{k+1} > 0, \\ \mathcal{Q}(\bar{X}_{k+1}) + \text{He}\{Z(\alpha)\mathcal{C}_k(A(\alpha))(\mathcal{T}_{k+1}\otimes I_n)\} < 0 \end{cases} \tag{11}$$

and $\bar{X}_{k+1}$, $\mathcal{Q}(\cdot)$, $\mathcal{C}_k(\cdot)$, and $\mathcal{T}_{k+1}$ are defined in (8).

Proof: The proof of the sufficiency part straightforwardly follows from Corollary 1. To prove necessity, suppose that (10) is asymptotically stable. Then, by Lyapunov theory, there exists a parameter-dependent symmetric matrix $P(\alpha)\in\mathbb{R}^{n\times n}$ such that Lyapunov equation $P(\alpha) > 0$,

$$A(\alpha)^TP(\alpha) + P(\alpha)A(\alpha) = -I_n$$

holds for all $\alpha\in\mathcal{P}$, whose solution is given analytically by

$$P(\alpha) = \int_0^\infty e^{A(\alpha)^T\theta}e^{A(\alpha)\theta}d\theta.$$

By resorting to the same reasoning adopted within the proof of Theorem 4 in [18], we conclude that there exists a sufficiently large $\hat{\tau}\in\mathbb{R}_{>0}$ independent of α such that $P(\alpha)$ can be approximated by $P_{\hat{\tau}}(\alpha) := \int_0^{\hat{\tau}} e^{A(\alpha)^T\theta}e^{A(\alpha)\theta}d\theta,$ which solves

$$P_{\hat{\tau}}(\alpha) > 0, \quad A(\alpha)^TP_{\hat{\tau}}(\alpha) + P_{\hat{\tau}}(\alpha)A(\alpha) < 0$$

for all $\alpha\in\mathcal{P}$, from which it follows that

$$A(\alpha)^TP_{\hat{\tau}}(\alpha) + P_{\hat{\tau}}(\alpha)A(\alpha) = \int_0^{\hat{\tau}}\frac{d}{d\theta}(e^{A(\alpha)^T\theta}e^{A(\alpha)\theta})d\theta$$

$$= e^{A(\alpha)^T \hat{\tau}} e^{A(\alpha)\hat{\tau}} - I_n$$
$$< 0, \quad \forall \alpha \in \mathcal{P},$$

and equivalently, $I_n - e^{-A(\alpha)^T \hat{\tau}} e^{-A(\alpha)\hat{\tau}} < 0$ holds for all $\alpha \in \mathcal{P}$. The remainder of the proof is then simple repetition of the necessity part of Theorem 1, and thus is omitted here for the sake of space.

Remark 6: It is worth pointing out that the PD-LMI condition of Theorem 3 is only sufficient when k and τ are small, but as k and τ tend to infinity, it converges to a necessary and sufficient condition.

Remark 7: In this work, we provide a generic framework that can represent the stability conditions using QLF, PD-QLFs, and the developed Lyapunov functionals in a unified fashion. For instance, recalling the definitions of $d_{k+2}(\lambda)$, $l_{k+1}(\lambda)$, and relation $\mathcal{T}_{k+1} d_{k+2}(\lambda) = l_{k+1}(\lambda)$ in the proof of Theorem 1, one can easily prove that condition (11) with $\bar{X}_{k+1} > 0$ replaced by $P_k > 0$, where $P_k \in \mathbb{R}^{kn\times kn}$, and $\mathcal{Q}(\bar{X}_{k+1})$ replaced by

$$(\mathcal{T}_{k+1} \otimes I_n)^T \begin{bmatrix} \mathcal{L}_k \otimes I_n \\ \mathcal{R}_k \otimes I_n \end{bmatrix}^T \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes P_k \right) \times \begin{bmatrix} \mathcal{L}_k \otimes I_n \\ \mathcal{R}_k \otimes I_n \end{bmatrix} (\mathcal{T}_{k+1} \otimes I_n) \in \mathbb{R}^{(k+2)n\times(k+2)n}$$

corresponds to (but is not the same as) Lemma 6 in [18], which is based on a special class of PD-QLFs using the higher-order time-derivatives of the state.

The conditions of Theorem 3 and Corollary 1 are numerically intractable (infinite-dimensional) PD-LMIs, which are known to be an NP-hard problem [33]. To obtain finite-dimensional LMIs, particular choices for $X_i(\alpha)$ and $Z(\alpha)$ can be imposed in Theorem 3 and Corollary 1. For instance, by imposing linear dependence in α on $X_i(\alpha)$ and $Z(\alpha)$ in Corollary 1, we arrive at the following sufficient LMI condition.

Theorem 4: Let $k \in \mathbb{N}$ and $\tau \in \mathbb{R}_{>0}$ be given. Uncertain system (10) is asymptotically stable if there exist symmetric matrices $X_{l,i} \in \mathbb{R}^{n\times n}$ and matrices $Z_i \in \mathbb{R}^{(k+2)n\times kn}$ such that the following LMIs hold for all $i \in \{1,2,\ldots,N-1\}$ and $j \in \{i+1, i+2,\ldots,N\}$:

$$\bar{X}_{k+1,i} > 0,$$
$$\mathcal{Q}(\bar{X}_{k+1,i}) + \text{He}\{Z_i \mathcal{C}_k(A_i)(\mathcal{T}_{k+1} \otimes I_n)\} < 0,$$
$$\mathcal{Q}(\bar{X}_{k+1,i}) + \mathcal{Q}(\bar{X}_{k+1,j}) + \text{He}\{Z_i \mathcal{C}_k(A_j)(\mathcal{T}_{k+1} \otimes I_n) + Z_j \mathcal{C}_k(A_i)(\mathcal{T}_{k+1} \otimes I_n)\} \le 0,$$

where $\bar{X}_{k+1,i} := \text{diag}(X_{0,i}, X_{1,i}, \ldots, X_{k,i}) \in \mathbb{R}^{(k+1)n\times(k+1)n}$, and $\mathcal{Q}(\cdot)$, $\mathcal{C}_k(\cdot)$, and $\mathcal{T}_{k+1}$ are defined in (8).

Proof: The proof can be worked out in similar lines to the sufficiency part of Theorem 1 and by using the LMI relaxation technique developed in [8]. Thus, it is omitted here for the sake of space.

Remark 8: The complexity of the LMI problem can be estimated by the total number N_D of decision variables and the total number N_L of rows of the LMI problem. For Theorem 4,

$$N_D = n(n+1)N(k+1)/2 + (k+2)kn^2 N,$$
$$N_L = (2k+3)nN + (k+2)nN(N-1)/2.$$

Remark 9: Future work will proceed along the following avenues:

1) Providing a formal procedure or tuning guidelines to determine parameter τ that produces less conservative results.
2) If the LMI condition of Theorem 4 is fulfilled for a given positive integer $k = \hat{k} \in \mathbb{N}$, then do they always admit a feasible solution for any $k > \hat{k}$?
3) Does the conservativeness of Theorem 4 asymptotically vanish as k tends to infinity?
4) If not, then how can we obtain a convergent LMI relaxation by using the similar ideas?
5) How can the proposed approach be improved in such a way so that the computational complexity can be reduced?

4. EXAMPLES

All numerical examples in the sequel were treated with the help of MATLAB R2008a running on a PC with Intel Core i7-3770 3.4GHz CPU, 24GB RAM. The LMI problems were solved with SeDuMi 1.3 [31] combined with the user-friendly interface 1.04 [32].

Example 1: In order to illustrate the results of this paper, as well as in [12], we consider the problem of computing the robust parametric margin ρ defined as

$$\rho := \sup\{\bar{\eta} \in \mathbb{R} : A(\alpha,\eta) \text{ is Hurwitz for all } (\alpha,\eta) \in \mathcal{P}\times[0,\bar{\eta}]\},$$

where $A(\alpha,\eta) := \sum_{i=1}^{N} \alpha_i A_i(\eta)$, $A_i(\eta) := \bar{A}_0 + \eta \bar{A}_i$, and $\bar{A}_i, i \in \{0,1,\ldots,N\}$ are given as

$$\bar{A}_0 = \begin{bmatrix} -2.4 & -0.6 & -1.7 & 3.1 \\ 0.7 & -2.1 & -2.6 & -3.6 \\ 0.5 & 2.4 & -5 & -1.6 \\ -0.6 & 2.9 & -2 & -0.6 \end{bmatrix},$$
$$\bar{A}_1 = \begin{bmatrix} 1.1 & -0.6 & -0.3 & -0.1 \\ -0.8 & 0.2 & -1.1 & 2.8 \\ -1.9 & 0.8 & -1.1 & 2.8 \\ -2.4 & -3.1 & -3.7 & -0.1 \end{bmatrix},$$
$$\bar{A}_2 = \begin{bmatrix} 0.9 & 3.4 & 1.7 & 1.5 \\ -3.4 & -1.4 & 1.3 & 1.4 \\ 1.1 & 2 & -1.5 & -3.4 \\ -0.4 & 0.5 & 2.3 & 1.5 \end{bmatrix},$$

$$\bar{A}_3 = \begin{bmatrix} -1 & -1.4 & -0.7 & -0.7 \\ 2.1 & 0.6 & -0.1 & -2.1 \\ 0.4 & -1.4 & 1.3 & 0.7 \\ 1.5 & 0.9 & 0.4 & -0.5 \end{bmatrix}.$$

The stability bounds given by a bisection process together with several previous conditions and Theorem 4 in this paper are listed in Table 1, where the exact bound was found by means of gridding of the parameter space. The results reveal that Theorem 4 of this paper can offer less conservative results than the quadratic stability and PD-LF approaches (Theorem 4 in [6], Lemma 1 in [8], Theorem 1 in [9]). It is also observed from the results of Table 1 that for $k > 3$ in this example, the LMI condition of Theorem 4 becomes more conservative due to high computational burden. The results imply that the computational complexity can be an issue for large scale problems (large k, n, and N); for instance, as k increases, so does the problem size of Theorem 4 as well, and this can result in more frequent failure in achieving a feasible solution due to quicker saturation of the memory. However, the computational burden required for small-scale systems is still reasonable, and by a suitable choice of k, the system analyst can achieve a good compromise between computational complexity and conservatism. Finally, Figs. 1 and 2 illustrate the stability bounds obtained by using Theorem 4 and Theorem 4 with

Table 1. Example 1: Stability bounds of several approaches.

Method	ρ
Exact bound	2.2238
Quadratic stability [3]	1.0191
Theorem 4 in [6]	1.4973
Lemma 1 in [8]	1.4720
Theorem 1 in [9]	1.8784
Corollary 4.4 in [11] with $k = 1$	1.1228
Corollary 4.4 in [11] with $k = 2$	2.2238
Theorem 1 in [12] with $m = 2$	2.2237
Theorem 4 in [16] with $(g, d) = (1,0)$	1.8632
Theorem 4 in [16] with $(g, d) = (2,0)$	2.2238
Theorem 8 in [18] with $k = 2$	2.2238
Theorem 4 with $k = 2$. Maximum bound in interval $\tau \in (0, 0.4]$	1.8951
Theorem 4 with $k = 3$. Maximum bound in interval $\tau \in (0, 0.4]$	2.2127
Theorem 4 with $k = 4$. Maximum bound in interval $\tau \in (0, 0.4]$	2.0957
Theorem 4 with $k = 2$ and $X_{l,i} = X_l$. Maximum bound in interval $\tau \in (0, 0.4]$	1.0079
Theorem 4 with $k = 3$ and $X_{l,i} = X_l$. Maximum bound in interval $\tau \in (0, 0.4]$	2.0322
Theorem 4 with $k = 4$ and $X_{l,i} = X_l$. Maximum bound in interval $\tau \in (0, 0.4]$	2.0946
Theorem 4 with $k = 5$ and $X_{l,i} = X_l$. Maximum bound in interval $\tau \in (0, 0.4]$	2.0948
Theorem 4 with $k = 6$ and $X_{l,i} = X_l$. Maximum bound in interval $\tau \in (0, 0.4]$	2.0650
Theorem 4 with $k = 2$, $X_{l,i} = X_l$, and $Z_i = Z$. Maximum bound in interval $\tau \in (0, 0.4]$	1.0079
Theorem 4 with $k = 3$, $X_{l,i} = X_l$, and $Z_i = Z$. Maximum bound in interval $\tau \in (0, 0.4]$	1.4740
Theorem 4 with $k = 4$, $X_{l,i} = X_l$, and $Z_i = Z$. Maximum bound in interval $\tau \in (0, 0.4]$	1.7705
Theorem 4 with $k = 5$, $X_{l,i} = X_l$, and $Z_i = Z$. Maximum bound in interval $\tau \in (0, 0.4]$	1.8328
Theorem 4 with $k = 6$, $X_{l,i} = X_l$, and $Z_i = Z$. Maximum bound in interval $\tau \in (0, 0.4]$	1.8233

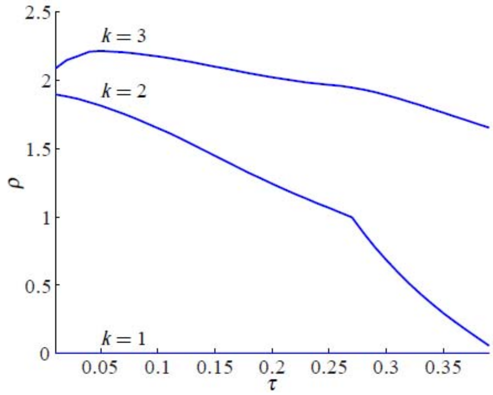

Fig. 1. Example 1. Stability bound ρ obtained by using Theorem 4 for $k \in \{1,2,3\}$ and different values of $\tau \in (0, 0.4]$.

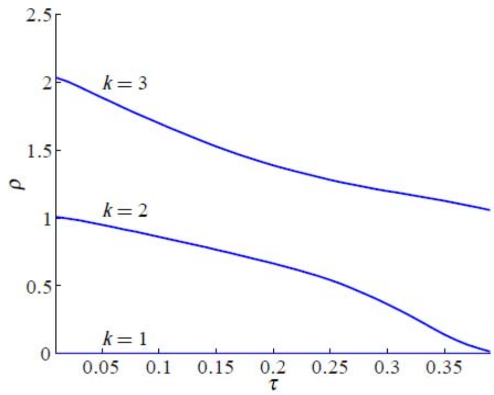

Fig. 2. Example 1. Stability bound ρ obtained by using Theorem 4 with $X_{l,i} = X_l$ for $k \in \{1,2,3\}$ and different values of $\tau \in (0, 0.4]$.

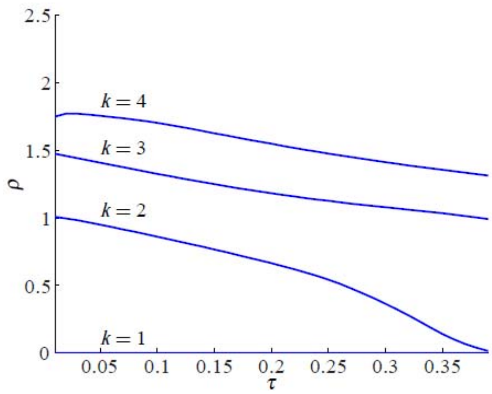

Fig. 3. Example 1. Stability bound ρ obtained by using Theorem 4 with $X_{l,i} = X_l$ and $Z_i = Z$ for $k \in \{1,2,3\}$ and different values of $\tau \in (0, 0.4]$.

Table 2. Example 2: Numerical complexity (N_D total number of decision variables; N_L total number of rows of the associated LMI problem; time in seconds) obtained using several appraoches. The computatinoal times were obtained by taking the average of ten measures with η =1.

Method	N_D	N_L	Time (s)
Theorem 4 in [6]	62	36	0.05
Lemma 1 in [8]	30	36	0.05
Theorem 1 in [9]	126	92	0.07
Corollary 4.4 in [11] with $k = 2$	1716	156	3.24
Theorem 1 in [12] with $m = 2$	750	64	0.45
Theorem 4 in [16] with $(g, d) = (2,0)$	252	104	0.09
Theorem 8 in [18] with $k = 2$	420	120	0.17
Theorem 4 with $k = 2$ and $\tau = 0.1$	474	132	0.5
Theorem 4 with $k = 3$ and $\tau = 0.1$	840	168	1.6
Theorem 4 with $k = 4$ and $\tau = 0.1$	1302	204	4.38
Theorem 4 with $k = 5$ and $\tau = 0.1$	1860	240	12.55

$X_{l,i} = X_l$, respectively, for different pairs of $(k,\tau) \in \{1,2,3\} \times (0, 0.4]$, and the results of Theorem 4 with $X_{l,i} = X_l$, $Z_i = Z$ for $(k,\tau) \in \{1,2,3,4\} \times (0, 0.4]$ are plotted in Fig. 3. The results show that the proposed condition outperforms some previous ones, but not less conservative than those in [11,12,16,18].

Example 2: This example compares Theorem 4 with existing approaches in terms of numerical complexity. Let us consider the same system as in Example 1 again. Table 2 lists the numerical complexity of several approaches in terms of N_D, the total number of decision variables, N_L the total number of rows of the associated LMI problem, the average computational time (in seconds) spent by each test to provide a feasible solution with η=1, and the average time for each test was obtained by taking the average of ten measures. From the table, it can be seen that Theorem 4 is computationally more demanding than previous conditions except for Corollary 4.4 in [11].

5. CONCLUSION

In this paper, we have suggested a systematical way to assure the robust stability via Lyapunov functionals. The approach can be interpreted as using mapping properties of a family of complex functions which map the closed right-hand side of the complex plane into the inside of the closed unit circle centered at the origin, which originally proposed in [26] and [27] for LTI time-delay systems. Using this, a sufficient LMI condition has been presented for robust stability analysis of continuous-time LTI systems subject to polytopic uncertainties. Finally examples have shown its validity.

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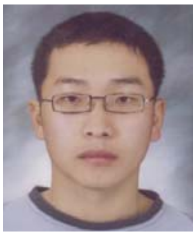

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