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Leader-Follower Consensus for a Class of Nonlinear Multi-Agent Systems

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Abstract: This paper deals with the leader-follower consensus problem for a class of nonlinear multi-agent systems. All agents have identical nonlinear dynamics in the strict feedback form with Lipschitz growth condition. Both full state consensus protocol and dynamic output consensus protocol are provided. It is shown that under a connected undirected information communication topology, the proposed protocols can solve the leader-follower consensus problem. Two consensus protocol design procedures are presented and a numerical example is given to illustrate the proposed protocols.

Keywords: Leader-follower consensus, Lipschitz growth condition, multi-agent systems, nonlinear dynamics.

1. INTRODUCTION

Multi-agent systems have been received extensive study in last decade (see [5,10,11,14]), due to their broad applications in many real systems, such as satellite formation flying, unmanned air vehicles cooperating and air traffic control. In multi-agent systems, the control protocols are designed for each agent based on the local information obtained from its neighbors.

Consensus problem is one of the most important problems in multi-agent systems, which involves the convergence of all agents to a common value in state space. [14] proposes a general framework of average consensus problem for first order multi-agent systems with fixed or switching communication topology and communication time-delays by a Lyapunov-based approach. [15,18] study the consensus of second order multi-agent systems, and give necessary and sufficient conditions for consensus with undirected information exchange topologies.

Recently, consensus problem has been studied for multi-agent systems in general forms. In [23], a consensus protocol is proposed based on the relative state between neighbors and the consensus is achievable on condition that the linear dynamic models are completely controllable and the communication topology is frequently connected. [12] studies consensusability for linear time-invariant multi-agent systems with fixed topology by a static output feedback consensus protocol. [19] constructs a dynamic output feedback that achieves synchronization with uniformly connected communication graph. A dynamic output feedback compensator with a low gain is designed in [20], while a reduced-order consensus control is given in [21]. [2,10] propose a unified way to deal with the consensus of multi-agent systems, which transforms the consensus problem into the stability problem of a set of matrices.

In the consensus problem, leader-follower consensus, which has been studied by several researchers, is an important topic. For example, [4,5] consider consensus problem with an active leader and propose a neighbor-based controller together with a neighbor-based state-estimation rule for each autonomous agent. In [11], a distributed observer-type consensus protocol based on relative output measurements is proposed for a linear time-invariant multi-agent system on the assumption that the communication topology has a direct spanning tree and the root agent of such tree has access to the leader.

It should be pointed out that almost all physical systems are nonlinear in nature. Due to the complexity of the nonlinear systems, there is no particular control procedure which can be applied to all nonlinear systems. There are several effective ways to design controllers for some nonlinear systems that can be transformed to specific nonlinear forms. One of the most important forms is the strict feedback form, which represents many physical systems, such as two-link planar robot, aircraft wing rock control system and induction motor system [8]. Therefore, it is meaningful to study the consensus of nonlinear multi-agent systems in the strict feedback form.

In this paper, we consider the leader-follower consensus problem for a class of nonlinear multi-agent systems in the strict feedback form. Here, each individual agent including the leader has identical nonlinear dynamics, and each follower can get the measurement information from the leader if there exists an edge between them in the communication topology. We also assume that the leader's input information is known to all follower agents. When the full state is available, we construct a local state consensus protocol; while the full state is not available and only the measurement output can be used, we provide an observer-based dynamic output consensus protocol. To

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better illustrate our protocols, we present the design procedures of consensus protocols step by step.

Compared with the existing results, our main contributions are composed of two aspects.

1) Motivated by the consensus problem for general linear system in [10,11,23], and the output feedback stabilization for a class of nonlinear systems with linear growth condition in [1,9,16], we study the leader-follower consensus problem for the nonlinear systems in the strict feedback form with Lipschitz growth condition. Both full state and dynamic output consensus protocols are studied in this paper. To the authors' knowledge, there is no related result about the consensus problem for nonlinear systems in strict feedback form.
2) Due to the existence of nonlinear term in agent dynamics, the existing consensus protocols are not applicable to our system. By introducing a gain parameter in consensus protocols, we transform our consensus problem of nonlinear multi-agent systems into a stabilization problem, which can be seen as a nonlinear extension of the method in [2,10,11]. The design of our protocols is a combination of stabilization technique of nonlinear systems and consensus protocol of multi-agent systems.

The rest of the paper is organized as follows. In Section 2, some preliminaries, including basic algebraic graph theory and useful lemmas, are provided. In Section 3, the formulation of leader-follower consensus problem and some basic assumptions are given. Full state consensus protocol and dynamic output consensus protocol designs are given in Sections 4 and 5. In Section 6, an illustrative numerical example is given to illustrate our protocols. Conclusions are given in Section 7.

2. PRELIMINARIES

Firstly, we provide some basic graph theory that is used in our paper.

An undirected graph $\mathcal{G}$ consists of a vertex set $\mathcal{V}=\{1,2,\cdots,N\}$ and a set of unordered pair $\mathcal{E}=\{(i,j):i,j\in V\}$, which are called the edges of $\mathcal{G}$. Two vertices are called adjacent, if there exists an edge between them. A graph is simple if there is no self-loops or repeated edges. A path on $\mathcal{G}$ between i_1 and i_l is a sequence of edges of the form (i_k,i_{k+1}), $k=1,\cdots,l-1$. If there exists a path between any two vertices of $\mathcal{G}$, then $\mathcal{G}$ is said to be connected, otherwise disconnected. The weighted adjacent matrix of $\mathcal{G}$ is denoted by $\mathcal{A}=[a_{ij}]\in\mathbb{R}^{N\times N}$, where $a_{ii}=0$ and $a_{ij}=a_{ji}\geq 0$, and $a_{ij}>0$ if and only if there exists an edge between vertex i and j. The degree of $\mathcal{G}$ is a diagonal matrix $\mathcal{D}=\operatorname{diag}(d_1,\cdots,d_N)$, where $d_i=\sum_{j=1}^{N}a_{ij}$ for $i=1,\cdots,N$. The Laplacian of $\mathcal{G}$ is defined as $\mathcal{L}=\mathcal{D}-\mathcal{A}$, which is symmetric. A subgraph $\mathcal{H}$ of $\mathcal{G}$ ia an induced subgraph if two vertices of $\mathcal{H}$ are adjacent in $\mathcal{H}$ only if they are adjacent in $\mathcal{G}$. An induced subgraph $\mathcal{H}$ of $\mathcal{G}$ is called a component of $\mathcal{G}$ if it is maximal, subjected to be connected.

Consider a graph $\bar{\mathcal{G}}$ associated with the system consisting of N agents and a leader. Regarding the N agents as the vertices in $\mathcal{V}$, the relationships between agents can be described by a simple and undirected graph $\mathcal{G}$. (i,j) is an edge of $\mathcal{G}$ if and only if agents i and j are neighbors. $\bar{\mathcal{G}}$ contains $\mathcal{G}$ and a leader with edges between some agents and leader. The connection weight matrix is denoted by $\mathcal{B}=\operatorname{diag}(b_1,\cdots,b_N)$, and $b_i\geq 0$. $b_i>0$ if and only if agent i is connected to the leader.

$\bar{\mathcal{G}}$ is connected if at least one agent in each component is connected with the leader. Denote $\hat{\mathcal{L}}=\mathcal{L}+\mathcal{B}$. A useful lemma about $\hat{\mathcal{L}}$ is given as follows.

Lemma 1 [4]**:** If graph $\bar{\mathcal{G}}$ is connected, then the symmetric matrix $\hat{\mathcal{L}}$ associated with $\bar{\mathcal{G}}$ is positive definite.

The following lemmas are useful for our protocol design.

Lemma 2 [23]**:** Consider a finite set of linear systems

$$\dot{x}_k=Ax_k+\lambda_k Bu_k,$$

where $x_k\in\mathbb{R}^n$, $u_k\in\mathbb{R}^m$, (A,B) is completely controllable, $\operatorname{rank}(B)=m$, and $\lambda_k>0$, $k=1,\cdots,N$. Then, there exists a feedback gain matrix K which simultaneously assigns the poles of N systems as negative as possible. Precisely, for any $M>0$, there exists a feedback gain matrix K satisfying

$$\operatorname{Re}\sigma(A+\lambda_k BK)<-M,\quad k=1,\cdots,N.$$

By the duality principle, we have

Lemma 3: Consider a finite set of linear systems

$$\dot{x}_k=Ax_k,\quad y_k=Cx_k,$$

where $x_k\in\mathbb{R}^n$, $y\in\mathbb{R}^m$, (C,A) is completely observable, $\operatorname{rank}(C)=m$, and $\lambda_k>0$, $k=1,\cdots,N$. Then, for any $M>0$, there exists a matrix F satisfying

$$\operatorname{Re}\sigma(A+\lambda_k FC)<-M,\quad k=1,\cdots,N.$$

3. PROBLEM FORMULATION

Consider a group of $N+1$ agents with identical single-input single-output nonlinear dynamics, which are indexed by $0,1,\cdots,N$. In our multi-agent system, the agent indexed by 0 is referred as the leader and agents indexed by $1,\cdots,N$ are called the followers. The dynamics of agent k is described by

$$\begin{aligned}\dot{z}_k&=F(z_k)+G(z_k)u_k,\\ y_k&=H(z_k),\end{aligned}\tag{1}$$

where $z_k=[z_{k,1},\cdots,z_{k,n}]^T\in\mathbb{R}^n$ is the state of agent k,

$u_k \in \mathbb{R}$ is the control input, and $y_k \in \mathbb{R}$ is the measurement output, $k = 0,1,\cdots,N$. The functions F, G and H are assumed to be sufficiently smooth of their arguments satisfying

$$F(0) = 0, H(0) = 0.$$

The communication topology of N follower agents and one leader agent is denoted by $\bar{\mathcal{G}}$. Assume that graph $\bar{\mathcal{G}}$ is connected, that is, at least one agent in each component of $\mathcal{G}$ is connected with the leader. The agents that are connected to the leader can receive the information from the leader agent. It is assumed that all the followers know the input of the leader agent, and the leader agent receives no information from any follower agent.

Definition 1: The leader-follower consensus problem of the nonlinear multi-agent system (1) can be solved by a consensus protocol u_k if, and only if under the protocol u_k, for any initial condition $z_k(0)$, $k = 0,1,\cdots,N$, the state $z_k(t)$ of the follower agent k asymptotically approaches the state $z_0(t)$ of the leader agent, as $t \to \infty$. That is,

$$\lim_{t\to\infty} \| z_k(t) - z_0(t) \| = 0, \ k = 1,\cdots,N.$$

Due to the complexity of nonlinear systems, we suppose that there exists a global diffeomorphism $x = \phi(z)$ such that nonlinear dynamics of agent k in (1) is transformed into the following strict feedback form

$$\begin{aligned} \dot{x}_{k,1} &= x_{k,2} + f_1(x_{k,1}), \\ \dot{x}_{k,2} &= x_{k,3} + f_2(x_{k,1}, x_{k,2}), \\ &\vdots \\ \dot{x}_{k,n} &= u_k + f_n(x_{k,1},\cdots,x_{k,n}), \\ y_k &= x_{k,1}, \end{aligned} \tag{2}$$

where $f_1,\cdots,f_n$ are sufficiently smooth of their arguments.

Geometric conditions under which agent dynamics (1) can be transformed into strict feedback form (2) can be found in [8]. For simplicity, we study the leader-follower consensus problem of strict feedback system (2) directly. We rewrite system (2) as

$$\begin{aligned} \dot{x}_k &= Ax_k + f(x_k) + Bu_k, \\ y_k &= Cx_k, \quad k = 0,1,\cdots,N, \end{aligned} \tag{3}$$

where A, B, C are given as

$$A = \begin{bmatrix} 0 & I_{n-1} \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0_{(n-1)\times 1} \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0_{1\times(n-1)} \end{bmatrix}$$

and $f = [f_1,\cdots,f_n]^T$.

We also need the following assumption.

Assumption 1: It is assumed that the functions $f_1,\cdots,f_n$ satisfy the following Lipschitz growth condition

$$\begin{aligned} &| f_i(x_{k,1},\cdots,x_{k,i}) - f_i(\bar{x}_{k,1},\cdots,\bar{x}_{k,i}) | \\ &\quad \le L(| x_{k,1} - \bar{x}_{k,1} | + \cdots + | x_{k,i} - \bar{x}_{k,i} |) \end{aligned}$$

for all $x_{k,1},\cdots,x_{k,i},\bar{x}_{k,1},\cdots,\bar{x}_{k,i} \in \mathbb{R}$, where L is some known positive real constant, $i = 1,\cdots,n$, and $k = 0,1,\cdots,N$.

Remark 1: Note that in (2), the nonlinear term depends not only on the measurement output, but also on unmeasurable states. In [13], counterexamples were given indicating that it is usually impossible to globally stabilize system (2) via output feedback without introducing extra growth conditions on the unmeasurable states. Since then, much research work has been focused on the output feedback control of system (2) under various linear growth conditions. [3,7] studied the high-gain observer design. [22] provided a linear state feedback control law achieving globally exponential stabilization. When the constant L is unknown, different dynamic output feedback controls were constructed in [1,9,16]. So it is reasonable and meaningful for us to introduce the Lipschitz growth condition in Assumption 1.

In the following sections, we construct two consensus protocols for our leader-follower consensus problem.

First, we design a full state consensus protocol for follower agent k based on the relative state of other agents with respect to agent k, which can be described by

$$v_k = \sum_{j=1}^{N} a_{kj}(x_k - x_j) + b_k(x_k - x_0). \tag{4}$$

Then, we design an observer-based dynamic output consensus protocol for follower agent k based on the relative measurement of other agents with respect to agent k, which can be described by

$$\bar{v}_k = \sum_{j=1}^{N} a_{kj}(y_k - y_j) + b_k(y_k - y_0). \tag{5}$$

4. THE DESIGN OF FULL STATE CONSENSUS PROTOCOL

In this section, we consider the design of full state consensus protocol. Based on the relative states (4) for agent k, we propose the following full state protocol for agent k

$$\begin{aligned} u_k &= K\left(\sum_{j=1}^{N} a_{kj}(x_k - x_j) + b_k(x_k - x_0) \right) + u_0, \\ k &= 1,\cdots,N, \end{aligned} \tag{6}$$

where $K = \kappa^{n+1} K_0 D_\kappa^{-1}$, $D_\kappa = \mathrm{diag}(\kappa,\cdots,\kappa^n)$ with $\kappa \ge 1$ being a gain parameter, and K_0 is a gain matrix to be determined.

Let $e_k = x_k - x_0$, and $e = [e_1^T,\cdots,e_N^T]^T$. Then, the consensus error dynamics can be written in the compact form

$$\dot{e} = (I_N \otimes A + \hat{\mathcal{L}} \otimes BK)e + \Delta, \tag{7}$$

where $\hat{\mathcal{L}}$ is the matrix associated to $\bar{\mathcal{G}}$, which has been defined in Section 2, and

$$\Delta = \begin{bmatrix} f(x_1) - f(x_0) \\ \vdots \\ f(x_N) - f(x_0) \end{bmatrix}.$$

It is easy to see that the leader-follower consensus problem is solved if and only if the state of (7) converges to zero.

Theorem 1: Suppose that Assumption 1 holds and the communication topology $\bar{\mathcal{G}}$ is connected. Then, there exits a sufficiently large $\kappa^* \geq 1$, and a matrix K_0, such that, for $\kappa \geq \kappa^*$, the consensus protocol (6) solves the leader-follower consensus problem of nonlinear multi-agent system (2).

Proof: Define $\varepsilon_k = D_\kappa^{-1} e_k$, $k = 1,\cdots,N$, and $\varepsilon = [\varepsilon_1^T, \cdots, \varepsilon_N^T]^T$. A simple calculation gives

$$\dot{\varepsilon} = \kappa(I_N \otimes A + \hat{\mathcal{L}} \otimes BK_0)\varepsilon + (I_N \otimes D_\kappa^{-1})\Delta. \tag{8}$$

From Lemma 1 and the fact that $\bar{\mathcal{G}}$ is connected, $\hat{\mathcal{L}}$ is a positive definite matrix. Then there exists an orthogonal matrix T such that

$$T^{-1}\hat{\mathcal{L}}T = U = \mathrm{diag}(\lambda_1,\cdots,\lambda_N),$$

where $\lambda_1,\cdots,\lambda_N$ are the eigenvalues of $\hat{\mathcal{L}}$.

Let $\bar{T} = T \otimes I_n$, we have

$$\bar{T}^{-1}(I_N \otimes A + \hat{\mathcal{L}} \otimes BK_0)\bar{T} = I_N \otimes A + U \otimes BK_0.$$

Since all λ_k are positive, from Lemma 2, there exists a matrix K_0 such that $A + \lambda_k BK_0, k = 1,\cdots,N$ are simultaneous Hurwitz. Therefore, $I_N \otimes A + \hat{\mathcal{L}} \otimes BK_0$ is Hurwitz. By Lyapunov Theorem for linear system, there exists a positive definite matrix P satisfying

$$P(I_N \otimes A + \hat{\mathcal{L}} \otimes BK_0) + (I_N \otimes A + \hat{\mathcal{L}} \otimes BK_0)^T P = -2I.$$

Consider a Lyapunov function

$V(\varepsilon) = \varepsilon^T P \varepsilon,$

and its derivative along system (8) is given as

$$\dot{V} = -2\kappa\varepsilon^T\varepsilon + 2\varepsilon^T P(I_N \otimes D_\kappa^{-1})\Delta. \tag{9}$$

Since (note that $\kappa \geq 1$)

$$\begin{aligned} \|(I_N \otimes D_\kappa^{-1})\Delta\|^2 &= \textstyle\sum_{k=1}^N \|D_\kappa^{-1}(f(x_k) - f(x_0))\|^2 \\ &\leq \textstyle\sum_{k=1}^N \sum_{i=1}^n [\frac{1}{\kappa^i} L(|e_{k,1}| + \cdots + |e_{k,i}|)]^2 \\ &\leq \textstyle\sum_{k=1}^N \sum_{i=1}^n L^2(|\varepsilon_{k,1}| + \cdots + |\varepsilon_{k,i}|)^2 \leq n^2 L^2 \varepsilon^T \varepsilon, \end{aligned}$$

we have

$$\dot{V} \leq -2\kappa\varepsilon^T\varepsilon + 2nL\|P\|\varepsilon^T\varepsilon = -(2\kappa - 2nL\|P\|)\varepsilon^T\varepsilon.$$

If we choose $\kappa^* = \max\{1, 2n\|P\|L\}$, then, for $\kappa \geq \kappa^*$, we get

$$\dot{V} \leq -\kappa\varepsilon^T\varepsilon.$$

Thus, the state ε exponentially converges to the origin, that is, consensus errors e_k also converge to the origin, for all $k = 1,\cdots,N$. So full state consensus protocol (6) solves the leader-follower consensus problem.

From the proof, a full state leader-follower consensus protocol in the form (6) can be constructed in the following steps.

Algorithm 1 (Design of State Consensus Protocol)**:**

1) Solve the eigenvalues λ_k, $k = 1,\cdots,N$ of $\hat{\mathcal{L}}$ associated to $\bar{\mathcal{G}}$. The positiveness of all λ_k's can be guaranteed by Lemma 1 on the assumption that $\bar{\mathcal{G}}$ is connected.

2) Construct a feedback control gain K_0, such that $A + \lambda_k BK_0$ are simultaneous Hurwitz for $k = 1,\cdots,N$. The existence of K_0 is guaranteed by Lemma 2.

3) Find a positive definite matrix P, such that

$$P(I_N \otimes A + \hat{\mathcal{L}} \otimes BK_0) + (I_N \otimes A + \hat{\mathcal{L}} \otimes BK_0)^T P = -2I$$

4) Choose $\kappa \geq \kappa^*$, where $\kappa^* = \max\{1, 2n\|P\|L\}$.

5) Let $D_\kappa = \mathrm{diag}(\kappa,\cdots,\kappa^n)$, and $K = \kappa^{n+1}K_0 D_\kappa^{-1}$. Then the consensus protocol (6) is obtained.

5. THE DESIGN OF DYNAMIC OUTPUT CONSENSUS PROTOCOL

In this section, we consider the consensus problem when the full state is not available. A high-gain observer is constructed for each agent to estimate the state.

Consider the nonlinear multi-agent system (2) and the relative measurement (5), we construct the following high-gain observer based consensus protocol

$$\begin{aligned} \dot{\hat{x}}_k &= A\hat{x}_k + Bu_k + f(\hat{x}_k) \\ &\quad + G\left[\sum_{j=1}^N a_{kj} C(\hat{x}_k - \hat{x}_j) + Cb_k(\hat{x}_k - \hat{x}_0) - \bar{v}_k\right], \end{aligned} \tag{10}$$

$$u_k = K\left[\sum_{j=1}^N a_{kj}(\hat{x}_k - \hat{x}_j) + b_k(\hat{x}_k - \hat{x}_0)\right] + u_0, \tag{11}$$

for $k = 1,\cdots,N$, where $G = D_\kappa G_0$, $K = \kappa^{n+1}K_0 D_\kappa^{-1}$, $D_\kappa = \mathrm{diag}(\kappa,\cdots,\kappa^n)$ with $\kappa \geq 1$ being a gain parameter, and G_0, K_0 are gain matrices to be determined. The term $\sum_{j=1}^N a_{kj} C(\hat{x}_k - \hat{x}_j)$ denotes the information exchanges between neighbors, and $\hat{x}_0$ is the state of the observer for the leader, which is defined as

$$\dot{\hat{x}}_0 = A\hat{x}_0 + Bu_0 + f(\hat{x}_0) + G(C\hat{x}_0 - y_0). \tag{12}$$

Let $\tilde{x}_k = \hat{x}_k - x_k$, for $k = 0,\cdots,N$. Then, for $k = 1, \cdots, N$,

$$\dot{\tilde{x}}_k = A\tilde{x}_k + [f(\hat{x}_k) - f(x_k)] \\ + GC\Big[\sum_{j=1}^{N} a_{kj}(\tilde{x}_k - \tilde{x}_j) + b_k(\tilde{x}_k - \tilde{x}_0)\Big],$$

and when $k = 0$

$$\dot{\tilde{x}}_0 = A\tilde{x}_0 + GC\tilde{x}_0 + [f(\hat{x}_0) - f(x_0)].$$

Let $\chi_k = D_\kappa^{-1}\tilde{x}_k,\ k = 1,\cdots,N,\ \chi = [\chi_1^T,\cdots,\chi_N^T]^T$ and $\chi_0 = D_\kappa^{-1}\tilde{x}_0$, then, the observer error dynamics can be rewritten in the following compact form

$$\dot{\chi} = \kappa(I_N \otimes A + \hat{L} \otimes G_0C)\chi + \Delta_1 + \Delta_2, \tag{13}$$

$$\dot{\chi}_0 = \kappa(A + G_0C)\chi_0 + D_\kappa^{-1}[f(\hat{x}_0) - f(x_0)], \tag{14}$$

where

$$\Delta_1 = \begin{bmatrix} \kappa b_1 G_0 C\chi_0 \\ \vdots \\ \kappa b_N G_0 C\chi_0 \end{bmatrix}, \quad \Delta_2 = \begin{bmatrix} D_\kappa^{-1}[f(\hat{x}_1) - f(x_1)] \\ \vdots \\ D_\kappa^{-1}[f(\hat{x}_N) - f(x_N)] \end{bmatrix}.$$

Lemma 4: Suppose that Assumption 1 holds and the communication topology $\bar{\mathcal{G}}$ is connected. Then, there exist a gain matrix G_0, and a positive real number $\kappa_1^* \geq 1$, such that, for any $\kappa \geq \kappa_1^*$, the observer error dynamics composed of (13) and (14) is exponentially stable. Precisely, there exists a Lyapunov function $V_1(\chi,\chi_0) = \chi^T P_1\chi + \kappa\chi_0^T P_2\chi_0$, such that the derivative of $V_1(\chi,\chi_0)$ along dynamics composed of (13) and (14) satisfies

$$\dot{V}_1(\chi,\chi_0) \leq -\kappa\chi^T\chi - \kappa^2\chi_0^T\chi_0.$$

Proof: Following the same way in the proof of Theorem 1, the stability of matrix $I_N \otimes A + \hat{\mathcal{L}} \otimes G_0C$ is equivalent to the simultaneous stability of $A + \lambda_k G_0C$, $k = 1,\cdots,N$. Due to the positiveness of all λ_k's and Lemma 3, there exists a matrix G_0 such that $A + G_0C$ and $A + \lambda_k G_0C,\ k = 1,\cdots,N$ are simultaneous Hurwitz. By Lyapunov Theorem for linear system, there exist positive definite matrices P_1, P_2 satisfying

$$P_1(I_N \otimes A + \hat{\mathcal{L}} \otimes G_0C) + (I_N \otimes A + \hat{\mathcal{L}} \otimes G_0C)^T P_1 = -3I,$$
$$P_2(A + G_0C) + (A + G_0C)^T P_2 = -2I.$$

Let $V_1(\chi,\chi_0) = \chi^T P_1\chi + \kappa\chi_0^T P_2\chi_0$, and the derivative of $V_1(\chi,\chi_0)$ along systems (13) and (14) is given by

$$\dot{V}_1(\chi,\chi_0) = -3\kappa\chi^T\chi - 2\kappa^T\chi_0^T\chi_0 + 2\chi^T P_1(\Delta_1 + \Delta_2) \\ + 2\kappa\chi_0^T P_2 D_\kappa^{-1}[f(\hat{x}_0) - f(x_0)].$$

Since

$$2\chi^T P_1\Delta_1 \leq \kappa\chi^T\chi + \bar{b}\kappa \| P_1 \|^2 \| G_0C \|^2\ \chi_0^T\chi_0,$$
$$2\chi^T P_1\Delta_2 \leq 2\|\chi\|\|P_1\|\|\Delta_2\|,$$

where $\bar{b} = b_1^2 + \cdots + b_N^2$, and (note that $\kappa \geq 1$)

$$\|\Delta_2\|^2 = \sum_{k=1}^{N} \| D_\kappa^{-1}(f(\hat{x}_k) - f(x_k))\|^2$$
$$\leq \sum_{k=1}^{N}\sum_{i=1}^{n}[\tfrac{1}{\kappa^i}L(|\tilde{x}_{k,1}| + \cdots + |\tilde{x}_{k,i}|)]^2$$
$$\leq \sum_{k=1}^{N}\sum_{i=1}^{n} L^2(|\chi_{k,1}| + \cdots + |\chi_{k,i}|)^2$$
$$\leq n^2L^2\chi^T\chi.$$
$$2\kappa\chi_0^T P_2 D_\kappa^{-1}[f(\hat{x}_0) - f(x_0)]$$
$$\leq 2\kappa\|\chi_0\|\|P_2\|\|D_\kappa^{-1}[f(\hat{x}_0) - f(x_0)]\|$$
$$\leq 2\kappa\|\chi_0\|\|P_2\|\|[\sum_{i=1}^{n}(\tfrac{1}{\kappa^i}L(|\tilde{x}_{0,1}| + \cdots + |\tilde{x}_{0,i}|))^2]^{1/2}$$
$$\leq 2\kappa\|\chi_0\|\|P_2\|\|\sum_{i=1}^{n} L(|\mathcal{X}_{0,1}| + \cdots + |\mathcal{X}_{0,i}|)$$
$$\leq 2\kappa nL\|P_2\|\chi_0^T\chi_0,$$

we get

$$\dot{V}_1(\chi,\chi_0) \leq -(2\kappa - 2nL\|P_1\|)\chi^T\chi \\ - \kappa(2\kappa - \bar{b}\|P_1\|^2\|G_0C\|^2 \\ - 2nL\|P_2\|)\chi_0^T\chi_0.$$

If we choose $\kappa_1^* = \max\{1, 2nL\|P_1\|, \bar{b}\|P_1\|^2\|G_0C\|^2 + 2nL\|P_2\|\}$, then

$$\dot{V}_1(\chi,\chi_0) \leq -\kappa\chi^T\chi - \kappa^2\chi_0^T\chi_0$$

holds for $\kappa \geq \kappa_1^*$.

To study the leader-follower consensus error, let $e_k = x_k - x_0$, then

$$\dot{e}_k = Ae_k + [f(x_k) - f(x_0)] \\ + BK\Big(\sum_{j=1}^{N} a_{kj}(e_k - e_j) + b_ke_k\Big) \\ + BK\Big(\sum_{j=1}^{N} a_{kj}(\tilde{x}_k - \tilde{x}_j) + b_k(\tilde{x}_k - \tilde{x}_0)\Big).$$

Letting $e = [e_1^T,\cdots,e_N^T]^T$, the above consensus error dynamics can be rewritten in the following compact form

$$\dot{e} = (I_N \otimes A + \hat{\mathcal{L}} \otimes BK)e + (\hat{\mathcal{L}} \otimes BKD_\kappa)\chi + \Delta_3 + \Delta_4, \tag{15}$$

where

$$\Delta_3 = \begin{bmatrix} f(x_1) - f(x_0) \\ \vdots \\ f(x_N) - f(x_0) \end{bmatrix}, \quad \Delta_4 = -\begin{bmatrix} BKb_1D_\kappa\chi_0 \\ \vdots \\ BKb_ND_\kappa\chi_0 \end{bmatrix}.$$

Letting $\varepsilon_k = D_\kappa^{-1}e_k$, and $\varepsilon = [\varepsilon_1^T,\cdots,\varepsilon_N^T]^T$, we get

$$\dot{\varepsilon} = \kappa(I_N \otimes A + \hat{\mathcal{L}} \otimes BK_0)\varepsilon \\ + (I_N \otimes D_\kappa^{-1})((\hat{\mathcal{L}} \otimes BKD_\kappa)\chi + \Delta_3 + \Delta_4). \tag{16}$$

Lemma 5: Suppose that Assumption 1 holds and the communication topology $\bar{\mathcal{G}}$ is connected. Then, there exist a positive matrix P, a gain matrix K_0, and a real number $\kappa_2^* \geq 1$, such that, the derivative of Lyapunov function $V_2(\varepsilon)=\varepsilon^T P\varepsilon$ along consensus error dynamics (16) satisfying

$$\dot{V}_2(\varepsilon) \leq -\kappa\varepsilon^T\varepsilon + \kappa^2\lambda(\chi^T\chi + \chi_0^T\chi_0) \quad (17)$$

for $\kappa \geq \kappa_2^*$ and some positive number $\lambda > 0$.

Proof: From the proof of Theorem 1, there exist a positive definite matrix P and a gain matrix K_0 satisfying

$$P(I_N \otimes A + \hat{\mathcal{L}} \otimes BK_0) + (I_N \otimes A + \hat{\mathcal{L}} \otimes BK_0)^T P = -2I.$$

Consider a Lyapunov function $V_2(\varepsilon) = \varepsilon^T P\varepsilon$, then, the derivative of $V_2(\varepsilon)$ along system (16) is given as

$$\begin{aligned}\dot{V}_2(\varepsilon) &= -2\kappa\varepsilon^T\varepsilon \\ &\quad + 2\varepsilon^T P(I_N \otimes D_\kappa^{-1})((\hat{\mathcal{L}} \otimes BKD_\kappa)\chi + \Delta_3 + \Delta_4) \\ &\leq -(2\kappa - 2nL\|P\| - 2)\varepsilon^T\varepsilon \\ &\quad + \kappa^2\|P\|(\|\hat{\mathcal{L}}\| + \bar{b})\|BK_0\|^2(\chi^T\chi + \chi_0^T\chi_0). \end{aligned} \quad (18)$$

Choosing $\kappa_2^* = 2nL\|P\| + 2$ and $\lambda = \|P\|(\|\hat{\mathcal{L}}\| + \bar{b})\|BK_0\|^2$ gives

$$\dot{V}_2(\varepsilon) \leq -\kappa\varepsilon^T\varepsilon + \kappa^2\lambda(\chi^T\chi + \chi_0^T\chi_0)$$

holding for $\kappa \geq \kappa_2^*$.

Combining Lemmas 4 and 5 gives the following result.

Theorem 2: Suppose that Assumption 1 holds and the communication topology $\bar{\mathcal{G}}$ is connected. Then, there exist a positive real number $\kappa^* \geq 1$, gain matrices G_0 and K_0, such that the observer-type consensus protocol composed of (10), (11), and (12) solves the leader-follower consensus problem of nonlinear multi-agent system (2). Moreover, for each agent k, where $k = 0, 1, \cdots, N$, the state $\hat{x}_k$ of the observer will converge to the state x_k.

Proof: Let $V = V_2(\varepsilon) + 2\kappa\lambda V_1(\chi, \chi_0)$, and $\kappa^* = \max\{\kappa_1^*, \kappa_2^*\}$, then combining Lemmas 4 and 5, we have

$$\dot{V} \leq -\kappa\varepsilon^T\varepsilon - \kappa^2\lambda(\chi^T\chi + \chi_0^T\chi_0). \quad (19)$$

Therefore, ε, χ, χ_0 will asymptotically converge to zero, i.e., will asymptotically converge to zero. So, for $k = 1, \cdots, N$, x_k will asymptotically converge to x_0, and $\hat{x}_k$ will asymptotically converge to x_k. The conclusion is obtained.

Similarly, a dynamic output leader-follower consensus protocol composed of (10), (11) and (12) can be constructed in the following steps.

Algorithm 2 (Design of Dynamic Output Consensus Protocol)**:**

1) Solve the eigenvalues $\lambda_k, k = 1, \cdots, N$ of $\hat{\mathcal{L}}$ associated to $\bar{\mathcal{G}}$. The positiveness of all λ_k's can be guaranteed by Lemma 1 on the assumption that $\bar{\mathcal{G}}$ is connected.

2) Construct a simultaneous feedback control gain K_0, such that $A + \lambda_i BK_0$ are simultaneous Hurwitz for $k = 1, \cdots, N$. The existence of K_0 is guaranteed by Lemma 2.

3) Construct a simultaneous observer gain G_0, such that $A + G_0C$, $A + \lambda_k G_0 C$ are simultaneous Hurwitz for $k = 1, \cdots, N$. The existence of K_0 is guaranteed by Lemma 3.

4) Find positive definite matrices P, P_1, P_2, such that

$$P(I_N \otimes A + \hat{\mathcal{L}} \otimes BK_0) + (I_N \otimes A + \hat{\mathcal{L}} \otimes BK_0)^T P = -2I,$$
$$P_1(I_N \otimes A + \hat{\mathcal{L}} \otimes G_0C) + (I_N \otimes A + \hat{\mathcal{L}} \otimes G_0C)^T P_1 = -3I,$$
$$P_2(A + G_0C) + (A + G_0C)^T P_2 = -2I.$$

5) Choose $\kappa \geq \kappa^*$, with

$\tilde{x}_k$,

$$\begin{aligned}\kappa^* = \max\{&2nL\|P\| + 2, 2nL\|P_1\|, \\ &\bar{b}\|P_1\|^2\|G_0C\|^2 - 2nL\|P_2\|\}.\end{aligned}$$

where $\bar{b} = b_1^2 + \cdots + b_N^2$.

6) Let $D_\kappa = \mathrm{diag}(\kappa, \cdots, \kappa^n)$, $K = \kappa^{n+1}K_0D_\kappa^{-1}$, and $G = D_\kappa G_0$. Then the observer-based consensus protocol composed of (10), (11) and (12) is obtained.

Remark 2: In our design of full state and dynamic output consensus protocols, we assume that the information exchange topology is a fixed undirected graph. However our consensus protocols can also be applied to the case of switching information exchange topologies, as formulated in [18]. That is, there is a piecewise constant switching signal function $\sigma(t)$: $[0, \infty) \mapsto \{1, 2, \cdots, M\} \triangleq \mathcal{U}$ with switching times $t_1 \leq t_2 \leq \cdots$, where $M \in \mathbb{Z}^+$ denotes the total number of all possible interaction undirected graphs. Then $\sigma(t) = \sigma(t_i)$ for $t \in [t_i, t_{i+1})$, and the interaction graph at time $t \in [t_i, t_{i+1})$ is denoted by $\bar{\mathcal{G}}_{\sigma(t_i)}$. On the assumption that the switching graph $\bar{\mathcal{G}}_{\sigma(t)}$ is always connected for $\sigma(t) \in \mathcal{U}$ and $t_{i+1} - t_i \geq \tau_0$, where τ_0 is some positive constant, we can prove that our protocols can solve the leader-follower consensus problem with switching information exchange by combining a basic result from switching systems. The switching topologies, which does not change the results of the paper, adds a complexity in the derivation of the equations, and therefore, it has been omitted. The interested reader is referred to [18] for details.

6. NUMERICAL EXAMPLES

To illustrate the design protocols, we consider a group of 6+1 agents with identical single-input single-output nonlinear dynamics, which are indexed by $0, 1, \cdots, 6$. In this multi-agent system, the agent indexed by 0 is referred as the leader and agents indexed by $1,\cdots,6$ are called the followers. The dynamics of agent k is described by

$$\begin{aligned}\dot{x}_{k,1} &= x_{k,2} + \tanh(x_{k,1}),\\ \dot{x}_{k,2} &= -x_{k,1} + \ln(1+x_{k,2}^2) + u_k,\\ y_k &= x_{k,1},\end{aligned} \tag{20}$$

where $k=0,1,\cdots,6$, $x_{k,1}, x_{k,2} \in \mathbb{R}$ are the states, $u_k \in \mathbb{R}$ is the control input, $y_k \in \mathbb{R}$ is the measurement output of agent k.

The communication topology graphs are shown by Fig. 1. When controls in all agents equal zero, i.e., the system is unforced, the profiles of system states and consensus errors between each follower agent and the leader agent are shown in Fig. 2. It can be seen that the state of each follower agent does not asymptotically tend to the state of the leader agent.

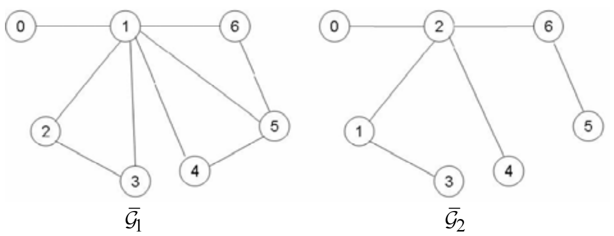
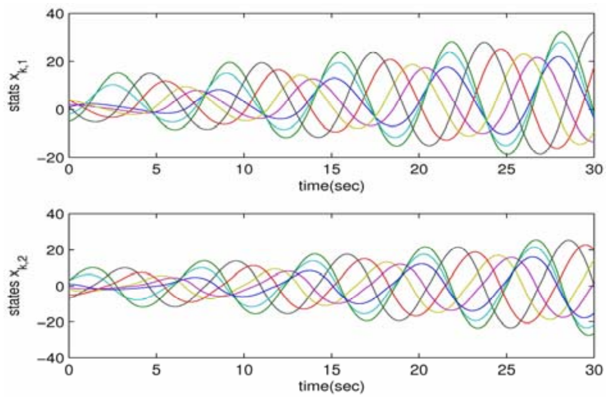
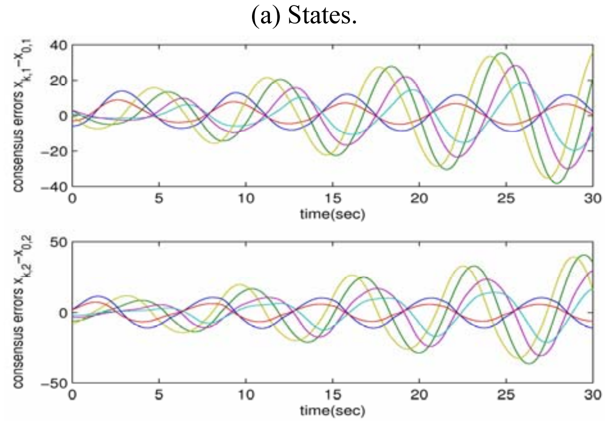


Fig. 1. Communication topologies.

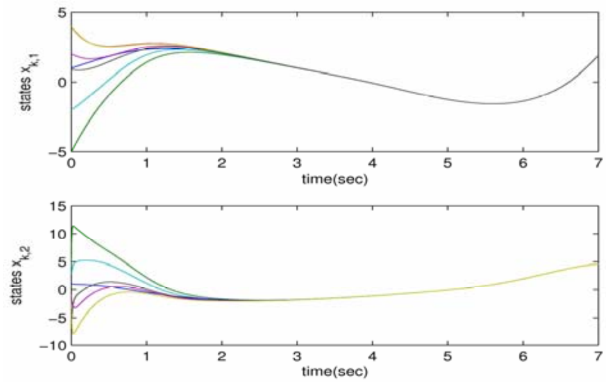


(a) States.

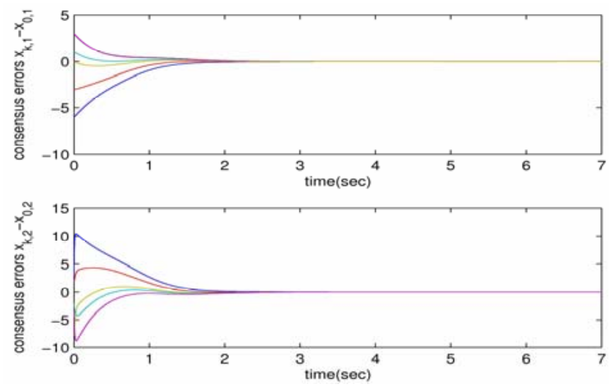


(b) Errors.

Fig. 2. Profiles of states and consensus errors with $u = 0$.

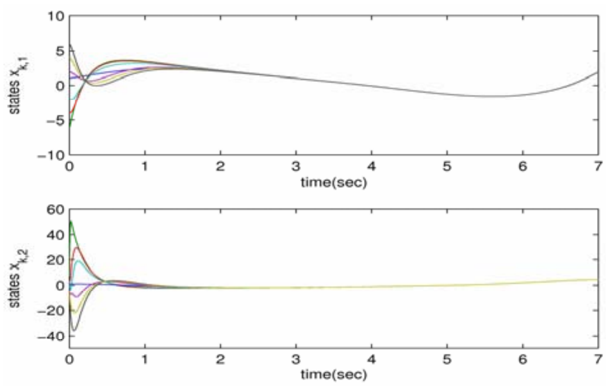


(a) States.

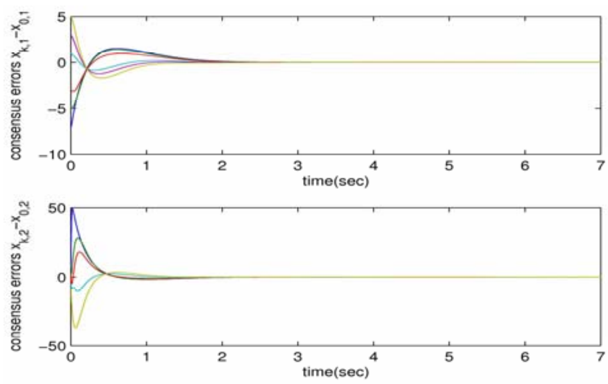


(b) Errors.

Fig. 3. Profiles of states and consensus errors with state protocol.

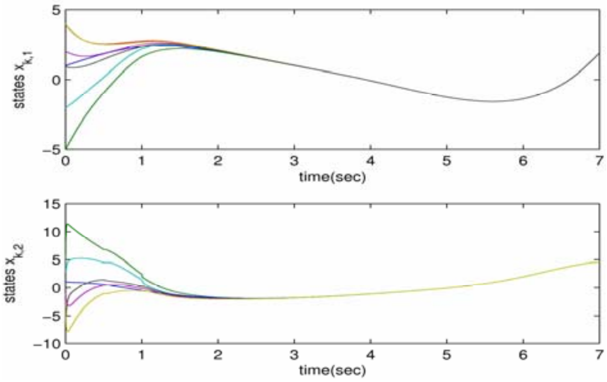


(a) States.

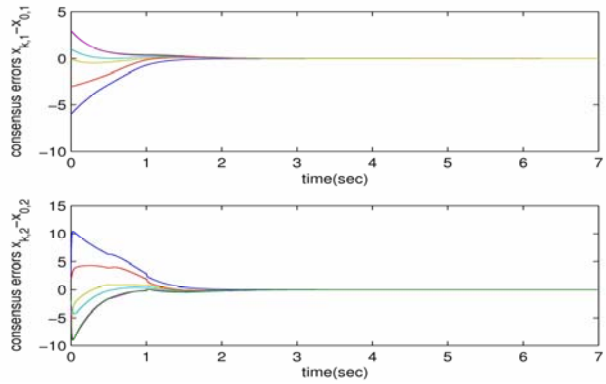


(b) Errors.

Fig. 4. Profiles of states and consensus errors with dynamic output protocol.

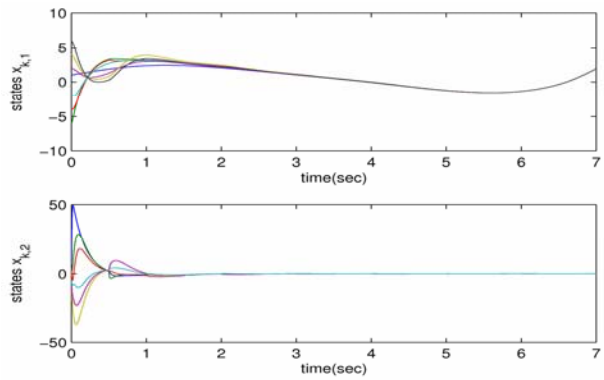


(a) States.

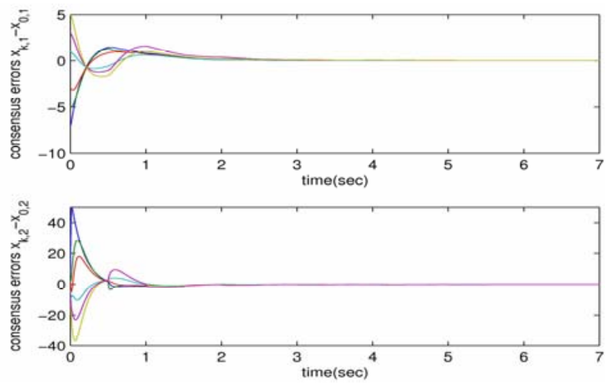


(b) Errors.

Fig. 5. Profiles of states and consensus errors with state protocol under switching topology.



(a) States.



(b) Errors.

Fig. 6. Profiles of states and consensus errors with dynamic output protocol under switching topology.

First, we design the consensus protocol for the fixed communication topology $\bar{\mathcal{G}}_1$. According to our design procedure, we can choose the following gain parameters

$$K_0 = [-2 -5], \quad G_0 = [-10 -10]^T, \quad \kappa^* = 5.5.$$

Figs. 3 and 4 depict the profiles of the states and consensus errors of the multi-agent system with full state consensus protocol and dynamic output consensus protocol, respectively. Figures show that our consensus protocols can solve the consensus problem of multi-agent system (20).

Then, we design the consensus protocol under the switching communication topologies with switching signal

$$\sigma(t) = \begin{cases} 1 & t \in [m, m+1/2) \\ 2 & t \in [m+1/2, m+1), \end{cases}$$

where $m = 0, 1, 2, \cdots$. We can still use the above consensus parameters K_0, G_0, κ^*. Figs. 5 and 6 depict the profiles of the states and consensus errors of the multi-agent system with full state consensus protocol and dynamic output consensus protocol. Figures show that our consensus protocols can also be applied to the time varying topology case.

7. CONCLUSIONS

In this paper, we study the leader-follower consensus problem for a class of nonlinear multi-agent systems. Each agent has identical nonlinear dynamics and is coupled by an undirected communication topology. We construct two consensus protocols, full state consensus protocol and dynamic output consensus protocol. On condition that the undirected communication topology is connected, our consensus protocols solve the leader-follower consensus problem for a class of nonlinear multi-agent systems.

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